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LYAPUNOV–TYPE INEQUALITIES FOR CERTAIN HIGHER–ORDER HALF–LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we will establish some new Lyapunov-type inequalities for some higher-order half-linear differential equations with anti-periodic boundary conditions. Our results not only improve the results in [9] for some cases, but also extend the results of [11].

1. Introduction

Integral inequalities are one kind of important inequalities that have received much attention in recent years, due to their wide applications in the research of qualitative and quantitative properties such as boundedness, global existence and stability of differential and integral equations (see [3]-[7], [10], [12]-[44] and the references therein).

In the feilds of integral inequalities, Lyapunov inequality, which with many of its generalizations have proved to be useful tools in oscillation theory, disconjugacy and eigenvalues problems of differential equations, was originally presented by Lyapunov in [1] as follows:

If $u(t)$ is a solution of

$$u'' + q(t)u = 0 \tag{1}$$

satisfying $u(a) = u(b) = 0$ $(a < b)$ and $u(t) \neq 0$ for $t \in (a,b)$, then

$$\int_a^b |q(t)|\mathrm{d}t > \frac{4}{b-a}.$$

In the last twenty years, a lot of efforts have been made to obtain Lyapunov-type inequalities for higher-order differential equations. In particular, Çakmak [2] considered Lyapunov-type inequality for the following even higher-order linear differential equation

$$u^{(2m)}(t) + r(t)u(t) = 0, \tag{2}$$

where $r \in C([a,b],[0,\infty))$, and he obtained the following result.

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THEOREM 1. *([2],Theorem 2) If there exists a nonzero solution $u(t)$ of Eq.(2) satisfying the following boundary conditions:*

$$u^{(2i)}(a) = u^{(2i)}(b) = 0, \quad i = 0, 1, 2, \ldots, m-1, \tag{3}$$

then

$$\int_a^b r(t)\mathrm{d}t > \frac{2^{2m}}{(b-a)^{2m-1}}. \tag{4}$$

Later, Watanabe, Yamagishi and Kametaka [8] used one Sobolev inequality to get a new Lyapunov-type inequality for Eq.(2):

$$\int_a^b r(t)\mathrm{d}t > \frac{2^{2m}}{(b-a)^{2m-1}} \cdot \frac{\pi^{2m}}{2(2^{2m}-1)\zeta(2m)},$$

where $\zeta(s) = \Sigma_{n=1}^{+\infty} \frac{1}{n^s}$ is the Riemann zeta function. Their result sharpened the result of Çakmak [2].

Recently, Wang et al. [9] considered the following $(m+1)$-order half-linear differential equation

$$(|u^{(m)}|^{p-2}u^{(m)})' + \sum_{k=0}^{m} r_k(t)|u^{(k)}|^{p-2}u^{(k)} = 0 \tag{5}$$

where $m \geqslant 1$, $r_k \in C([a,b],\mathbf{R}), k = 0, 1, 2, \ldots, m,$ $p > 1$, and they obtained the following result.

THEOREM 2. *([9],Theorem 2.1) If there exists a nonzero solution $u(t)$ of Eq.(5) satisfying the following anti-periodic boundary conditions:*

$$u^{(i)}(a) + u^{(i)}(b) = 0, \quad i = 0, 1, 2, \ldots, m, \tag{6}$$

then

$$\sum_{k=0}^{m-1} [(b-a)C_{m-k}]^{\frac{p-1}{2}} \int_a^b |r_k(t)|\mathrm{d}t + \int_a^b |r_m(t)|\mathrm{d}t > 2, \tag{7}$$

where

$$C_n = \frac{(2^{2n}-1)(b-a)^{2n-1}\zeta(2n)}{2^{2n-1}\pi^{2n}}, \quad n = 1, 2, \ldots, \tag{8}$$

and $\zeta(s) = \Sigma_{n=1}^{+\infty} \frac{1}{n^s}$ is the Riemann zeta function for Re $(s) > 1$.

On the other hand, Yang and Lo [11] obtained some Lyapunov-type inequalities for higher-order linear differential equation

$$u^{(m)} + \alpha u^{(m-1)} + \sum_{k=0}^{m-2} r_k(t)u^{(k)} = 0, \tag{9}$$

on the interval (a,b), under the following anti-periodic boundary

$$u^{(i)}(a)+u^{(i)}(b)=0, \quad i=0,1,\ldots,m-1, \tag{10}$$

where $m \geqslant 2$, $p_k \in C([a,b],\mathbf{R}), k=0,1,2,\ldots,m-2$.

We find that in Eq.(9), the coefficient α of $u^{(m-1)}$ is a constant. The natural question now is: Can one obtain Lyapunov-type inequality for Eq.(9) with the coefficient of $u^{(m-1)}$ is a function? Although Theorem 2.1 in [9] gives an affirmative answer to this question, we find the result can be improved.

In the present paper, we shall use the Sobolev inequality established in [8] and some techniques different from [9] to obtain some new Lyapunov-type inequalities for Eq.(5) with $p>2$ and the anti-periodic boundary conditions (6). Further, we will also prove a new Lyapunov-type inequality for m-order linear differential equation

$$u^{(m)}+\sum_{k=0}^{m-1} r_k(t)u^{(k)}=0 \tag{11}$$

with the anti-periodic boundary conditions (10). Our work not only improves the result in [9] for some cases but also extends the result of [11].

2. Main results

LEMMA 1. [8] *For $m \geqslant 1$, define the following Sobolev space:*

$$H_m=\{u|u^{(m)} \in L^2[a,b],\ u^{(k)}(a)+u^{(k)}(b)=0, \quad k=0,1,2,\ldots,m-1\}.$$

For any $u \in H_m$, there exists a positive constant C_m such that the Sobolev inequality

$$\Big(\sup_{a\leqslant t\leqslant b}|u(t)|\Big)^2 \leqslant C_m \int_a^b |u^{(m)}(t)|^2 \mathrm{d}t \tag{12}$$

holds, where

$$C_m=\frac{2(2^{2m}-1)(b-a)^{2m-1}\zeta(2m)}{2^{2m}\pi^{2m}}, \quad m=1,2,\ldots, \tag{13}$$

and the constants $\{C_m\}$ are sharp, $\zeta(s)=\Sigma_{n=1}^{+\infty}\frac{1}{n^s}$ is the Riemann zeta function for Re $(s)>1$.

THEOREM 3. *Assume $m \in \mathbf{N}$, $p>2$, $r_k \in C([a,b],\mathbf{R}), k=0,1,2,\ldots,m$. If $u(t)$ is a nonzero solution of Eq.(5) satisfying the anti-periodic boundary conditions (6), then*

$$\begin{aligned} &C_1^{\frac{1}{2}}(b-a)^{\frac{p-2}{2p}}\Big(\int_a^b |r_m(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}}+C_1^{\frac{p}{2}}\cdot(b-a)^{\frac{p-2}{2}}\int_a^b r_{m-1}^+(t)\mathrm{d}t \\ &+\sum_{k=0}^{m-2} C_{m-k}^{\frac{p-1}{2}}C_1^{\frac{1}{2}}(b-a)^{\frac{p-2}{2}}\int_a^b |r_k(t)|\mathrm{d}t>1, \end{aligned} \tag{14}$$

where $r_{m-1}^+(t):=\max\{r_{m-1}(t),0\}$, and C_k, $k=1,2,\ldots,m$, are defined as in (13).

Proof. Let $u(t)$ be a solution of Eq.(5) satisfying the anti-periodic boundary conditions (6). It is easy to see that $u(t)$ is an element of H_m. Multiplying Eq.(5) by $u^{(m-1)}(t)$ and integrating over $[a,b]$, yields

$$\int_a^b (|u^{(m)}(t)|^{p-2}u^{(m)}(t))'u^{(m-1)}(t)\mathrm{d}t+\sum_{k=0}^{m}\int_a^b r_k(t)|u^{(k)}(t)|^{p-2}u^{(k)}(t)u^{(m-1)}(t)\mathrm{d}t=0. \tag{15}$$

Using integration by parts to the first integral on the left-hand side of (15) and (6), we have

$$\begin{aligned}\int_a^b |u^{(m)}(t)|^p\mathrm{d}t &= \sum_{k=0}^{m}\int_a^b r_k(t)|u^{(k)}(t)|^{p-2}u^{(k)}(t)u^{(m-1)}(t)\mathrm{d}t\\ &= \int_a^b r_m(t)|u^{(m)}(t)|^{p-2}u^{(m)}(t)u^{(m-1)}(t)\mathrm{d}t\\ &\quad+\int_a^b r_{m-1}(t)|u^{(m-1)}(t)|^{p-2}u^{(m-1)}(t)u^{(m-1)}(t)\mathrm{d}t\\ &\quad+\sum_{k=0}^{m-2}\int_a^b r_k(t)|u^{(k)}(t)|^{p-2}u^{(k)}(t)u^{(m-1)}(t)\mathrm{d}t\\ &\leqslant \int_a^b |r_m(t)||u^{(m)}(t)|^{p-1}|u^{(m-1)}(t)|\mathrm{d}t+\int_a^b r_{m-1}^{+}(t)|u^{(m-1)}(t)|^p\mathrm{d}t\\ &\quad+\sum_{k=0}^{m-2}\int_a^b |r_k(t)||u^{(k)}(t)|^{p-1}|u^{(m-1)}(t)|\mathrm{d}t. \end{aligned}\tag{16}$$

Applying Hölder's inequality

$$\int_a^b |f(t)g(t)|\mathrm{d}t \leqslant \left(\int_a^b |f(t)|^{\alpha}\mathrm{d}t\right)^{\frac{1}{\alpha}}\left(\int_a^b |g(t)|^{\beta}\mathrm{d}t\right)^{\frac{1}{\beta}} \tag{17}$$

to the first integral on the right-hand side of (16) with $f(t)=|u^{(m)}(t)|^{p-1}$, $g(t)=|r_m(t)||u^{(m-1)}(t)|$, $\alpha=\frac{p}{p-1}$ and $\beta=p$, we obtain that

$$\begin{aligned}&\int_a^b |r_m(t)||u^{(m)}(t)|^{p-1}|u^{(m-1)}(t)|\mathrm{d}t\\ &\leqslant \left(\int_a^b |u^{(m)}(t)|^p\mathrm{d}t\right)^{\frac{p-1}{p}}\left(\int_a^b |r_m(t)|^p|u^{(m-1)}(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\\ &\leqslant \left(\int_a^b |u^{(m)}(t)|^p\mathrm{d}t\right)^{\frac{p-1}{p}}\sup_{a\leqslant t\leqslant b}|u^{(m-1)}(t)|\left(\int_a^b |r_m(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}. \end{aligned}\tag{18}$$

Since

$$u^{(k)}\in H_{m-k},\quad k=0,1,\dots,m-1,$$

by Lemma 1, we have

$$\sup_{a\leqslant t\leqslant b}|u^{(k)}(t)|\leqslant C_{m-k}^{\frac{1}{2}}\left(\int_a^b |u^{(m)}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}},\ \ k=0,1,\cdots,m-1. \tag{19}$$

Applying Hölder's inequality (17) to the integral on the right-hand side of (19) with $f(t) = |u^{(m)}(t)|^2$, $g(t) = 1$, $\alpha = \frac{p}{2}$ and $\beta = \frac{p}{p-2}$, we get

$$\Big(\int_a^b |u^{(m)}(t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \leqslant \Big(\int_a^b |u^{(m)}(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} (b-a)^{\frac{p-2}{2p}}, \quad k = 0, 1, \cdots, m-1. \tag{20}$$

From (18)-(20), we have

$$\begin{aligned} &\int_a^b |r_m(t)||u^{(m)}(t)|^{p-1}|u^{(m-1)}(t)|\mathrm{d}t \\ &\leqslant \Big(\int_a^b |u^{(m)}(t)|^p \mathrm{d}t\Big)^{\frac{p-1}{p}} C_1^{\frac{1}{2}} \Big(\int_a^b |u^{(m)}(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} (b-a)^{\frac{p-2}{2p}} \Big(\int_a^b |r_m(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} \\ &= \int_a^b |u^{(m)}(t)|^p \mathrm{d}t \cdot C_1^{\frac{1}{2}} (b-a)^{\frac{p-2}{2p}} \Big(\int_a^b |r_m(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}}. \end{aligned} \tag{21}$$

On the other hand, for the second integral on the right-hand side of (16), from (19) and (20), we obtain

$$\begin{aligned} &\int_a^b r_{m-1}^+(t)|u^{(m-1)}(t)|^p \mathrm{d}t \\ &\leqslant \Big(\sup_{a \leqslant t \leqslant b} |u^{(m-1)}(t)|\Big)^p \int_a^b r_{m-1}^+(t)\mathrm{d}t \\ &\leqslant C_1^{\frac{p}{2}} \int_a^b |u^{(m)}(t)|^p \mathrm{d}t \cdot (b-a)^{\frac{p-2}{2}} \int_a^b r_{m-1}^+(t)\mathrm{d}t, \end{aligned} \tag{22}$$

and for the third part on the right-hand side of (16), from (19) and (20), we have

$$\begin{aligned} &\sum_{k=0}^{m-2} \int_a^b |r_k(t)||u^{(k)}(t)|^{p-1}|u^{(m-1)}(t)|\mathrm{d}t \\ &\leqslant \sum_{k=0}^{m-2} \Big(\sup_{a \leqslant t \leqslant b} |u^{(k)}(t)|\Big)^{p-1} \sup_{a \leqslant t \leqslant b} |u^{(m-1)}(t)| \int_a^b |r_k(t)|\mathrm{d}t \\ &\leqslant \sum_{k=0}^{m-2} \Bigg[C_{m-k}^{\frac{p-1}{2}} \Big(\int_a^b |u^{(m)}(t)|^p \mathrm{d}t\Big)^{\frac{p-1}{p}} (b-a)^{\frac{(p-2)(p-1)}{2p}} \cdot \\ &\qquad C_1^{\frac{1}{2}} \Big(\int_a^b |u^{(m)}(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} (b-a)^{\frac{p-2}{2p}} \int_a^b |r_k(t)|\mathrm{d}t \Bigg] \\ &= \int_a^b |u^{(m)}(t)|^p \mathrm{d}t \sum_{k=0}^{m-2} C_{m-k}^{\frac{p-1}{2}} C_1^{\frac{1}{2}} (b-a)^{\frac{p-2}{2}} \int_a^b |r_k(t)|\mathrm{d}t. \end{aligned} \tag{23}$$

By (16) and (21)-(23), we get

$$\int_a^b |u^{(m)}(t)|^p \mathrm{d}t \leqslant \int_a^b |u^{(m)}(t)|^p \mathrm{d}t \cdot C_1^{\frac{1}{2}} (b-a)^{\frac{p-2}{2p}} \Big(\int_a^b |r_m(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} + C_1^{\frac{p}{2}} \int_a^b |u^{(m)}(t)|^p \mathrm{d}t \cdot (b-a)^{\frac{p-2}{2}} \int_a^b r_{m-1}^+(t)\mathrm{d}t + \int_a^b |u^{(m)}(t)|^p \mathrm{d}t \sum_{k=0}^{m-2} C_{m-k}^{\frac{p-1}{2}} C_1^{\frac{1}{2}} (b-a)^{\frac{p-2}{2}} \int_a^b |r_k(t)|\mathrm{d}t. \tag{24}$$

Now, we claim that $\int_a^b |u^{(m)}(t)|^p \mathrm{d}t > 0$. In fact, if the above inequality is not true, we have $\int_a^b |u^{(m)}(t)|^p \mathrm{d}t = 0$, then $u^{(m)}(t) = 0$ for $t \in [a,b]$. By the anti-periodic conditions (6), we obtain $u(t) = 0$ for $t \in [a,b]$, which contradicts to $u(t) \neq 0$, $t \in [a,b]$. Thus dividing both sides of (24) by $\int_a^b |u^{(m)}(t)|^p \mathrm{d}t$, we obtain

$$1 \leqslant C_1^{\frac{1}{2}} (b-a)^{\frac{p-2}{2p}} \Big(\int_a^b |r_m(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} + C_1^{\frac{p}{2}} \cdot (b-a)^{\frac{p-2}{2}} \int_a^b r_{m-1}^+(t)\mathrm{d}t + \sum_{k=0}^{m-2} C_{m-k}^{\frac{p-1}{2}} C_1^{\frac{1}{2}} (b-a)^{\frac{p-2}{2}} \int_a^b |r_k(t)|\mathrm{d}t. \tag{25}$$

Moreover, this inequality is strict, since $u(t)$ is not a constant. This completes the proof of Theorem 3.

REMARK 1. The inequality obtained in Theorem 3 is sharper than (7) for the case where $p > 2$ and $r_m(t) \equiv 0$. When $r_m(t) \equiv 0$, (7) reduces to

$$\frac{1}{2} \sum_{k=0}^{m-1} [(b-a)C_{m-k}]^{\frac{p-1}{2}} \int_a^b |r_k(t)|\mathrm{d}t > 1. \tag{26}$$

It is easy to see that the coefficients of $\int_a^b |r_k(t)|\mathrm{d}t$, $k = 0,1,2,\ldots,m-2$ in (26) are the same as those in (14), and the coefficient of $\int_a^b |r_{m-1}(t)|\mathrm{d}t$ in (26) is the same as the coefficient of $\int_a^b r_{m-1}^+(t)\mathrm{d}t$ in (14). So by Theorem 3, the integral of $|r_{m-1}(t)|$ on the left-hand side of (26) is replaced by the integral of $r_{m-1}^+(t)$.

For Eq.(11), with a similar argument to the proof of Theorem 3, we have the following Theorem.

THEOREM 4. *Assume $m \in \mathbf{N}$, $r_k \in C([a,b],\mathbf{R}), k = 0,1,2,\ldots,m-1$. If $u(t)$ is a nonzero solution of Eq.(11) satisfying the anti-periodic boundary conditions (10), then*

$$C_1^{\frac{1}{2}} \Big(\int_a^b |r_{m-1}(t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} + C_1 \int_a^b r_{m-2}^+(t)\mathrm{d}t + \sum_{k=0}^{m-3} C_{m-1-k}^{\frac{1}{2}} C_1^{\frac{1}{2}} \int_a^b |r_k(t)|\mathrm{d}t > 1, \tag{27}$$

where $r_{m-2}^{+}(t) := \max\{r_{m-2}(t), 0\}$, *and* $C_k,\ k = 1, 2, \ldots, m-1$, *are defined as in* (13).

Proof. Let $u(t)$ be a solution of Eq.(11) satisfying the anti-periodic boundary conditions (10). It is easy to see that $u(t)$ is an element of H_{m-1}. Multiplying Eq.(11) by $u^{(m-2)}(t)$ and integrating over $[a,b]$, yields

$$\int_a^b u^{(m)}(t)u^{(m-2)}(t)\mathrm{d}t + \sum_{k=0}^{m-1}\int_a^b r_k(t)u^{(k)}(t)u^{(m-2)}(t)\mathrm{d}t = 0. \tag{28}$$

Using integration by parts to the first integral on the left-hand side of (28) and (10), we have

$$\begin{aligned}\int_a^b (u^{(m-1)}(t))^2\mathrm{d}t &= \sum_{k=0}^{m-1}\int_a^b r_k(t)u^{(k)}(t)u^{(m-2)}(t)\mathrm{d}t \\ &= \int_a^b r_{m-1}(t)u^{(m-1)}(t)u^{(m-2)}(t)\mathrm{d}t \\ &\quad + \int_a^b r_{m-2}(t)u^{(m-2)}(t)u^{(m-2)}(t)\mathrm{d}t \\ &\quad + \sum_{k=0}^{m-3}\int_a^b r_k(t)u^{(k)}(t)u^{(m-2)}(t)\mathrm{d}t \\ &\leqslant \int_a^b |r_{m-1}(t)||u^{(m-1)}(t)||u^{(m-2)}(t)|\mathrm{d}t + \int_a^b r_{m-2}^{+}(t)|u^{(m-2)}(t)|^2\mathrm{d}t \\ &\quad + \sum_{k=0}^{m-3}\int_a^b |r_k(t)||u^{(k)}(t)||u^{(m-2)}(t)|\mathrm{d}t.\end{aligned} \tag{29}$$

Applying Hölder's inequality (17) to the first integral on the right-hand side of (29) with $f(t) = |u^{(m-1)}(t)|,\ \ g(t) = |r_{m-1}(t)||u^{(m-2)}(t)|,\ \ \alpha = 2$ and $\beta = 2$, we obtain that

$$\begin{aligned}&\int_a^b |r_{m-1}(t)||u^{(m-1)}(t)||u^{(m-2)}(t)|\mathrm{d}t \\ &\leqslant \Big(\int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_a^b |r_{m-1}(t)|^2|u^{(m-2)}(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\ &\leqslant \Big(\int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\sup_{a\leqslant t\leqslant b}|u^{(m-2)}(t)|\Big(\int_a^b |r_{m-1}(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}.\end{aligned} \tag{30}$$

Since

$$u^{(k)} \in H_{m-1-k}, \quad k = 0, 1, \ldots, m-2,$$

by Lemma 1, we have

$$\sup_{a\leqslant t\leqslant b}|u^{(k)}(t)| \leqslant C_{m-1-k}^{\frac{1}{2}}\Big(\int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}, \ \ k = 0, 1, \cdots, m-2. \tag{31}$$

From (30)-(31), we have

$$\begin{aligned}
&\int_a^b |r_{m-1}(t)||u^{(m-1)}(t)||u^{(m-2)}(t)|\mathrm{d}t \\
&\leqslant \left(\int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} C_1^{\frac{1}{2}} \left(\int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_a^b |r_{m-1}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
&= \int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t \cdot C_1^{\frac{1}{2}} \left(\int_a^b |r_{m-1}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}. \qquad (32)
\end{aligned}$$

On the other hand, for the second integral on the right-hand side of (29), from (31), we get

$$\begin{aligned}
&\int_a^b r_{m-2}^{+}(t)|u^{(m-2)}(t)|^2\mathrm{d}t \\
&\leqslant \big(\sup_{a\leqslant t\leqslant b} |u^{(m-2)}(t)|\big)^2 \int_a^b r_{m-2}^{+}(t)\mathrm{d}t \\
&\leqslant C_1 \int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t \int_a^b r_{m-2}^{+}(t)\mathrm{d}t, \qquad (33)
\end{aligned}$$

and for the third part on the right-hand side of (29), we have

$$\begin{aligned}
&\sum_{k=0}^{m-3} \int_a^b |r_k(t)||u^{(k)}(t)||u^{(m-2)}(t)|\mathrm{d}t \\
&\leqslant \sum_{k=0}^{m-3} \big(\sup_{a\leqslant t\leqslant b} |u^{(k)}(t)|\big) \sup_{a\leqslant t\leqslant b} |u^{(m-2)}(t)| \int_a^b |r_k(t)|\mathrm{d}t \\
&\leqslant \sum_{k=0}^{m-3} \Bigg[C_{m-1-k}^{\frac{1}{2}} \left(\int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \cdot \\
&\qquad C_1^{\frac{1}{2}} \left(\int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \int_a^b |r_k(t)|\mathrm{d}t\Bigg] \\
&= \int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t \sum_{k=0}^{m-3} C_{m-1-k}^{\frac{1}{2}} C_1^{\frac{1}{2}} \int_a^b |r_k(t)|\mathrm{d}t. \qquad (34)
\end{aligned}$$

By (29) and (32)-(34), we get

$$\begin{aligned}
\int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t \leqslant{}& \int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t \cdot C_1^{\frac{1}{2}} \left(\int_a^b |r_{m-1}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
&+ C_1 \int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t \int_a^b r_{m-2}^{+}(t)\mathrm{d}t \\
&+ \int_a^b |u^{(m-1)}(t)|^2\mathrm{d}t \sum_{k=0}^{m-3} C_{m-1-k}^{\frac{1}{2}} C_1^{\frac{1}{2}} \int_a^b |r_k(t)|\mathrm{d}t. \qquad (35)
\end{aligned}$$

The rest of the proof is similar to Theorem 3, and we omit it here.

3. Examples

We list the first 6 values of $\zeta(2n)$, $n=1,2,\cdots,6$, in the following table:

n	1	2	3	4	5	6
$\zeta(2n)$	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9450}$	$\frac{\pi^{10}}{93555}$	$\frac{691\pi^{12}}{638512875}$

By applying Theorem 3 and Theorem 4, we get the following inequalities:

EXAMPLE 1. Let us consider the following boundary value problem

$$(|u'''|u''')' + \sum_{k=0}^{3} r_k(t)|u^{(k)}|u^{(k)} = 0, \tag{36}$$

with the anti-periodic boundary conditions

$$u^{(i)}(a) + u^{(i)}(b) = 0, \quad i = 0,1,2,3. \tag{37}$$

If $u(t)$ is a nonzero solution of Eq.(36), then

$$\begin{aligned} &\frac{(b-a)^{\frac{2}{3}}}{2}\Big(\int_a^b |r_3(t)|^3 \mathrm{d}t\Big)^{\frac{1}{3}} + \frac{(b-a)^2}{8}\int_a^b r_2^+(t)\mathrm{d}t \\ &+ \frac{(b-a)^4}{96}\int_a^b |r_1(t)|\mathrm{d}t + \frac{(b-a)^6}{960}\int_a^b |r_0(t)|\mathrm{d}t > 1. \end{aligned} \tag{38}$$

Proof. From Theorem 3, let $m=3$ and $p=3$, we get

$$\begin{aligned} &C_1^{\frac{1}{2}}(b-a)^{\frac{1}{6}}\Big(\int_a^b |r_3(t)|^3 \mathrm{d}t\Big)^{\frac{1}{3}} + C_1^{\frac{3}{2}}\cdot(b-a)^{\frac{1}{2}}\int_a^b r_2^+(t)\mathrm{d}t \\ &+ \sum_{k=0}^{1} C_{3-k}C_1^{\frac{1}{2}}(b-a)^{\frac{1}{2}}\int_a^b |r_k(t)|\mathrm{d}t > 1. \end{aligned} \tag{39}$$

From (15), $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$, $\zeta(6)=\frac{\pi^6}{945}$ and a simple computation, we have

$$C_1 = \frac{2(2^2-1)(b-a)^{2-1}\zeta(2)}{2^2\pi^2} = \frac{b-a}{4}, \tag{40}$$

$$C_2 = \frac{2(2^4-1)(b-a)^{4-1}\zeta(4)}{2^4\pi^4} = \frac{(b-a)^3}{48}, \tag{41}$$

and

$$C_3 = \frac{2(2^6-1)(b-a)^{6-1}\zeta(6)}{2^6\pi^6} = \frac{(b-a)^5}{480}. \tag{42}$$

Thus, from (39)-(42), we obtain the result.

EXAMPLE 2. Let us consider the following boundary value problem

$$u''' + r_2(t)u'' + r_1(t)u' + r_0(t)u = 0, \tag{43}$$

with the anti-periodic boundary conditions

$$u^{(i)}(a) + u^{(i)}(b) = 0, \quad i = 0,1,2. \tag{44}$$

If $u(t)$ is a nonzero solution of Eq.(43), then

$$\frac{(b-a)^{\frac{1}{2}}}{2}\left(\int_a^b |r_2(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} + \frac{b-a}{4}\int_a^b r_1^+(t)\mathrm{d}t + \frac{(b-a)^2}{8\sqrt{3}}\int_a^b |r_0(t)|\mathrm{d}t > 1. \tag{45}$$

Proof. From Theorem 4, let $m = 3$, we have

$$C_1^{\frac{1}{2}}\left(\int_a^b |r_2(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} + C_1\int_a^b r_1^+(t)\mathrm{d}t + C_2^{\frac{1}{2}}C_1^{\frac{1}{2}}\int_a^b |r_0(t)|\mathrm{d}t > 1. \tag{46}$$

From (40)-(41) and (46), we obtain the result.

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