Journal of Mathematical Inequalities
Volume 13, Number 4 (2019), 1209–1234 doi:10.7153/jmi-2019-13-86

SHARPNESS AND GENERALIZATION OF JORDAN, BECKER–STARK AND PAPENFUSS INEQUALITIES WITH AN APPLICATION

BO ZHANG AND CHAO-PING CHEN

(*Communicated by T. Burić*)

Abstract. In this paper, we present an identity related to Jordan's inequality. More precisely, we provide a formula for determining the coefficients $b_n \equiv b_n(\theta)$ such that

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} b_n \left(\pi^\theta - (2x)^\theta\right)^n,$$

where $\theta \geqslant 2$ is a given real number. We present a generalization of Jordan's inequality. As an application, we improve the well-known Yang Le inequality. We establish sharp bounds for $\left(\tan x/x\right)^{(n)}$ for $n = 0$ and $n = 1$. Further, an interesting open problem and a conjecture regarding our present concern are posed.

1. Introduction

1.1. Jordan's inequality

The Jordan's inequality

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} < 1, \quad 0 < x \leqslant \frac{\pi}{2} \tag{1.1}$$

has important applications in many areas of pure and applied mathematics. This simple inequality has motivated a large number of research papers concerning its new proofs, various generalizations, sharpness and applications (see [10, 17, 18, 20, 21, 23, 25, 28, 30, 34, 35, 36, 37, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51] and the references cited in them).

The following sharp lower and upper bounds for the function $\frac{\sin x}{x}$ were proved in [17, 23, 30, 43, 47]:

$$\frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) \leqslant \frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi-2}{\pi^3}\left(\pi^2 - 4x^2\right), \qquad 0 < x \leqslant \frac{\pi}{2}. \tag{1.2}$$

Mathematics subject classification (2010): 26D05, 41A10.
Keywords and phrases: Jordan's inequality; Becker-Stark inequality; Papenfuss inequality; Yang Le inequality; Bell polynomials of the second kind; Faà di Bruno formula.

Zhu [48] improved (1.2) and established the following sharp lower and upper bounds:

$$\frac{2}{\pi}+\frac{1}{\pi^3}\left(\pi^2-4x^2\right)+\frac{12-\pi^2}{16\pi^5}\left(\pi^2-4x^2\right)^2\leqslant\frac{\sin x}{x}\leqslant\frac{2}{\pi}+\frac{1}{\pi^3}\left(\pi^2-4x^2\right)+\frac{\pi-3}{\pi^5}\left(\pi^2-4x^2\right)^2,\qquad 0<x\leqslant\frac{\pi}{2}. \tag{1.3}$$

Niu et al. [27] established a general result, which includes (1.2) and (1.3) as special cases.

Two analogues of the inequalities (1.2) and (1.3):

$$\frac{2}{\pi}+\frac{2}{3\pi^4}\left(\pi^3-8x^3\right)\leqslant\frac{\sin x}{x}\leqslant\frac{2}{\pi}+\frac{\pi-2}{\pi^4}\left(\pi^3-8x^3\right) \tag{1.4}$$

and

$$\frac{2}{\pi}+\frac{1}{2\pi^5}\left(\pi^4-16x^4\right)\leqslant\frac{\sin x}{x}\leqslant\frac{2}{\pi}+\frac{\pi-2}{\pi^5}\left(\pi^4-16x^4\right) \tag{1.5}$$

were established for $0<x<\pi/2$ (see [18, 20]).

Chen and Debnath [10, Theorem 2] gave an unified sharpness and generalization of inequalities (1.2)-(1.5) and proved that, for $0<x\leqslant\pi/2$,

$$\frac{2}{\pi}+\frac{2\pi^{-\theta-1}}{\theta}\left(\pi^\theta-(2x)^\theta\right)+\frac{(-\pi^2+4+4\theta)\pi^{-2\theta-1}}{4\theta^2}\left(\pi^\theta-(2x)^\theta\right)^2\leqslant\frac{\sin x}{x}\leqslant\frac{2}{\pi}+\frac{2\pi^{-\theta-1}}{\theta}\left(\pi^\theta-(2x)^\theta\right)+\frac{((\pi-2)\theta-2)\pi^{-2\theta-1}}{\theta}\left(\pi^\theta-(2x)^\theta\right)^2 \tag{1.6}$$

holds true for $\theta\geqslant 2$, and equality occurs for $x=\pi/2$.

By taking $\theta=2$ in (1.6), we obtain (1.3). By taking $\theta=3$ in (1.6), we obtain that, for $0<x\leqslant\pi/2$,

$$\frac{2}{\pi}+\frac{2}{3\pi^4}\left(\pi^3-8x^3\right)+\frac{16-\pi^2}{36\pi^7}\left(\pi^3-8x^3\right)^2\leqslant\frac{\sin x}{x}\leqslant\frac{2}{\pi}+\frac{2}{3\pi^4}\left(\pi^3-8x^3\right)+\frac{3\pi-8}{3\pi^7}\left(\pi^3-8x^3\right)^2, \tag{1.7}$$

which is sharper than (1.4). By taking $\theta=4$ in (1.6), we obtain that, for $0<x\leqslant\pi/2$,

$$\frac{2}{\pi}+\frac{1}{2\pi^5}\left(\pi^4-16x^4\right)+\frac{20-\pi^2}{64\pi^9}\left(\pi^4-16x^4\right)^2\leqslant\frac{\sin x}{x}\leqslant\frac{2}{\pi}+\frac{1}{2\pi^5}\left(\pi^4-16x^4\right)+\frac{2\pi-5}{2\pi^9}\left(\pi^4-16x^4\right)^2, \tag{1.8}$$

which is sharper than (1.5).

Chen and Debnath [10, Eq. (2.2)] presented the following approximation:

$$\frac{\sin x}{x}\approx\frac{2}{\pi}+\frac{2}{\theta\pi^{\theta+1}}\left(\pi^\theta-(2x)^\theta\right)+\frac{4\theta+4-\pi^2}{4\theta^2\pi^{2\theta+1}}\left(\pi^\theta-(2x)^\theta\right)^2+\frac{8\theta^2-(3\pi^2-12)\theta+4}{12\theta^3\pi^{3\theta+1}}\left(\pi^\theta-(2x)^\theta\right)^3,\qquad 0<x\leqslant\frac{\pi}{2}. \tag{1.9}$$

The first aim of the present paper is to develop the approximation formula (1.9) to produce a full expansion (Theorem 3.1). More precisely, we provide a formula for determining the coefficients $b_n \equiv b_n(\theta)$ such that

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} b_n\left(\pi^\theta - (2x)^\theta\right)^n.$$

The second aim of the present paper is to present a generalization of Jordan's inequality (Theorem 3.2). As an application, we improve the well-known Yang Le inequality (Theorem 5.1).

1.2. Becker-Stark and Papenfuss inequalities

It is known in the literature that, for $0 < x < \pi/2$,

$$\frac{4/\pi}{\pi - 2x} < \frac{\tan x}{x} < \frac{\pi}{\pi - 2x}. \tag{1.10}$$

The left-hand side inequality (1.10) was presented by Stečkin [31], while the right-hand side inequality (1.10) was proved by Ge [19]. This inequality is now known as Stečkin's inequality, see, e.g., [24, p. 246].

Becker and Stark [7] showed that, for $0 < x < \pi/2$,

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}. \tag{1.11}$$

The inequalities (1.11) are shaper than the inequalities (1.10). The Becker-Stark inequality (1.11) has attracted much interest of many mathematicians and has motivated a large number of research papers [6, 9, 12, 16, 26, 33, 52, 53, 54].

Recently, Chen and Elezović [11] gave a unified treatment of the inequalities (1.10) and (1.11) and proved the following result:

Let $p > 0$ be a real number. Consider the following inequalities for $0 < x < \pi/2$:

$$\frac{\pi^p}{\pi^p - (2x)^p} < \frac{\tan x}{x} < \frac{4p\pi^{p-2}}{\pi^p - (2x)^p}, \tag{1.12}$$

or alternatively

$$\frac{1}{1 - t^p} < \frac{\tan(\pi t/2)}{\pi t/2} < \left(\frac{2}{\pi}\right)^2 \frac{p}{1 - t^p} \tag{1.13}$$

for $0 < t < 1$. The left-hand side of (1.13) holds if and only if $p \geqslant \pi^2/4$, while the reversed inequality holds if and only if $0 < p \leqslant 2$. The right-hand side of (1.13) holds if and only if $p \geqslant 3$, while the reversed inequality holds if and only if $0 < p \leqslant \pi^2/4$.

The choice $p = 1$ in (1.12) yields Stečkin's inequality (1.10). The choice $p = 2$ in (1.12) yields Becker-Stark inequality (1.11). The choice $p = 3$ in (1.12) yields, for $0 < x < \pi/2$,

$$\frac{\pi^3}{\pi^3 - (2x)^3} < \frac{\tan x}{x} < \frac{12\pi}{\pi^3 - (2x)^3}. \tag{1.14}$$

Papenfuss [29] proposed the following problem: Prove that

$$x\sec^2 x-\tan x \leqslant \frac{8\pi^2x^3}{\left(\pi^2-4x^2\right)^2}, \qquad 0\leqslant x<\frac{\pi}{2}. \tag{1.15}$$

Bach [5] proved the inequality (1.15) and obtained a further result as follows:

$$x\sec^2 x-\tan x \leqslant \frac{(2\pi^4/3)x^3}{\left(\pi^2-4x^2\right)^2}, \qquad 0\leqslant x<\frac{\pi}{2}. \tag{1.16}$$

Ge [19, Theorem 1.3] presented a lower bound in (1.16) and proved that

$$\frac{64x^3}{\left(\pi^2-4x^2\right)^2}<x\sec^2 x-\tan x \leqslant \frac{(2\pi^4/3)x^3}{\left(\pi^2-4x^2\right)^2}, \qquad 0\leqslant x<\frac{\pi}{2}, \tag{1.17}$$

where the constants 64 and $2\pi^4/3$ are the best possible.

Recently, Chen and Paris [13] proved that, for $0<x<\pi/2$,

$$\frac{\frac{2\pi^4}{3}x^3+\left(\frac{8\pi^4}{15}-\frac{16\pi^2}{3}\right)x^5}{\left(\pi^2-4x^2\right)^2}<x\sec^2 x-\tan x<\frac{\frac{2\pi^4}{3}x^3+\left(\frac{256}{\pi^2}-\frac{8\pi^2}{3}\right)x^5}{\left(\pi^2-4x^2\right)^2}, \tag{1.18}$$

where $\frac{8\pi^4}{15}-\frac{16\pi^2}{3}$ and $\frac{256}{\pi^2}-\frac{8\pi^2}{3}$ are the best constants in (1.18). This answered an open problem proposed by Sun and Zhu [32].

Here, we provide a new sharp lower bound for $x\sec^2 x-\tan x$ given by Proposition 1.1.

PROPOSITION 1.1. *Let $\nu>0$ is a real number. Then, for $0<x<\pi/2$,*

$$\frac{\frac{2}{3}\pi^\nu x^3}{\pi^\nu-(2x)^\nu}<x\sec^2 x-\tan x, \tag{1.19}$$

with the best possible constant $\nu=2$, in the sense that $\nu=2$ can not be replaced by a smaller number.

Proof. We first prove (1.19) with $\nu=2$, namely,

$$\frac{\frac{2}{3}\pi^2x^3}{\pi^2-(2x)^2}<x\sec^2 x-\tan x, \qquad 0<x<\frac{\pi}{2}. \tag{1.20}$$

Direct computation yields

$$\frac{\frac{2\pi^4}{3}x^3+\left(\frac{8\pi^4}{15}-\frac{16\pi^2}{3}\right)x^5}{\left(\pi^2-4x^2\right)^2}-\frac{\frac{2}{3}\pi^2x^3}{\pi^2-(2x)^2}=\frac{8\pi^2x^5(\pi^2-5)}{15(\pi^2-4x^2)^2}>0.$$

This shows that the lower bound in (1.18) is larger than the one in (1.20). Hence, (1.20) holds true.

If we write (1.19) as

$$\frac{\ln\left(1-\frac{\frac{2}{3}x^3}{x\sec^2 x-\tan x}\right)}{\ln\frac{2x}{\pi}}<\nu,$$

we find that

$$\lim_{x\to 0^+}\frac{\ln\left(1-\frac{\frac{2}{3}x^3}{x\sec^2 x-\tan x}\right)}{\ln\frac{2x}{\pi}}=2.$$

Hence, (1.19) holds, and the constant $\nu=2$ is the best possible.

REMARK 1.1. In order to ensure that the lower bound of (1.19) is positive, we restrict $\nu>0$. We do not think about the case $\nu=0$, since

$$\lim_{\nu\to 0^+}\frac{\frac{2}{3}\pi^\nu x^3}{\pi^\nu-(2x)^\nu}=\infty.$$

REMARK 1.2. There is no strict comparison between the two lower bounds $\frac{64x^3}{(\pi^2-4x^2)^2}$ and $\frac{\frac{2}{3}\pi^2x^3}{\pi^2-4x^2}$ in (1.17) and (1.20).

The inequalities (1.17) and (1.20) can be written for $0<x<\pi/2$ as

$$\frac{\frac{64}{\pi^4}x}{\left(1-(\frac{2x}{\pi})^2\right)^2}<\left(\frac{\tan x}{x}\right)'<\frac{\frac{2}{3}x}{\left(1-(\frac{2x}{\pi})^2\right)^2} \tag{1.21}$$

and

$$\frac{\frac{2}{3}x}{1-(\frac{2x}{\pi})^2}<\left(\frac{\tan x}{x}\right)'. \tag{1.22}$$

Motivated by (1.11), (1.21) and (1.22), we establish sharp bounds for $\left(\tan x/x\right)^{(n)}$ for $n=0$ and $n=1$ (Theorems 4.1 and 4.2), which is the third aim of the present paper. Further, an interesting open problem and a conjecture regarding our present concern are posed (Section 5).

Some computations in this paper were performed using Maple software.

2. Preliminary results

In combinatorics, the Bell polynomials of the second kind (also called the partial Bell polynomials) $B_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ are defined by (see [14, p. 133] and [15])

$$B_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \sum\frac{n!}{j_1!j_2!\cdots j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \tag{2.1}$$

where the sum is taken over all sequences $j_1, j_2, j_3, \ldots, j_{n-k+1} \in \mathbb{N}_0$ such that

$$j_1 + j_2 + \cdots + j_{n-k+1} = k \quad \text{and} \quad j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n.$$

Here $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}$ denotes the set of positive integers.

The Faà di Bruno formula may be described in terms of the Bell polynomials of the second kind $B_{n,k}$ by (see [14, p. 139])

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) B_{n,k}\big(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)\big). \tag{2.2}$$

The following lemmas will be useful in our present investigation.

LEMMA 2.1. (see [8]) *Let the function* ϕ *have derivatives of all orders on* $(-\infty, \infty)$ *and* $\phi(0) = 0$. *Define the function* f *by*

$$f(x) = \begin{cases} \dfrac{\phi(x)}{x}, & x \neq 0; \\[2ex] \phi'(0), & x = 0, \end{cases}$$

then

$$f^{(n)}(x) = \begin{cases} \frac{1}{x^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \phi^{(n-k)}(x), & x \neq 0; \\[2ex] \frac{1}{n+1} \phi^{(n+1)}(0), & x = 0. \end{cases}$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \phi^{(n-k)}(x) = x^n \phi^{(n+1)}(x). \tag{2.3}$$

REMARK 2.1. It follows from (2.3) that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \phi^{(n-k)}(x) = \int_0^x t^n \phi^{(n+1)}(t) \mathrm{d}t.$$

We then find the following integral representation:

$$\left(\frac{\phi(x)}{x}\right)^{(n)} = \frac{1}{x^{n+1}} \int_0^x t^n \phi^{(n+1)}(t) \mathrm{d}t, \qquad x \neq 0. \tag{2.4}$$

Let

$$F(x) = \frac{\sin x}{x}.$$

By Lemma 2.1 (the choice $\phi(x) = \sin x$), we have,

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(x - \frac{\pi}{2}\right)^n, \tag{2.5}$$

where

$$a_n = F^{(n)}\left(\frac{\pi}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! \left(\frac{2}{\pi}\right)^{k+1} \cos\left(\frac{(n-k)\pi}{2}\right), \qquad n \in \mathbb{N}_0. \tag{2.6}$$

The choice $\phi(x) = \sin x$ in (2.4) yields

$$F^{(n)}(x) = \frac{1}{x^{n+1}} \int_0^x t^n \sin\left(\frac{(n+1)\pi}{2} + t\right) \mathrm{d}t. \tag{2.7}$$

The coefficients a_n can also be calculated by

$$a_n = F^{(n)}\left(\frac{\pi}{2}\right) = \left(\frac{2}{\pi}\right)^{n+1} \int_0^{\pi/2} t^n \sin\left(\frac{(n+1)\pi}{2} + t\right) \mathrm{d}t, \qquad n \in \mathbb{N}_0. \tag{2.8}$$

The first few coefficients a_n are

$$\begin{aligned} &a_0 = \frac{2}{\pi},\ a_1 = -\frac{4}{\pi^2},\ a_2 = -\frac{2(\pi^2-8)}{\pi^3},\ a_3 = \frac{12(\pi^2-8)}{\pi^4},\ a_4 = \frac{2(\pi^4-48\pi^2+384)}{\pi^5},\\ &a_5 = -\frac{20(\pi^4+384-48\pi^2)}{\pi^6},\ a_6 = -\frac{2(-46080-120\pi^4+5760\pi^2+\pi^6)}{\pi^7}. \end{aligned} \tag{2.9}$$

The choice $\phi(x) = \tan x$ in (2.4) yields

$$\left(\frac{\tan x}{x}\right)^{(n)} = \frac{1}{x^{n+1}} \int_0^x t^n \big(\tan t\big)^{(n+1)} \mathrm{d}t. \tag{2.10}$$

LEMMA 2.2. (see [2, 3, 4]) *Let* $-\infty < a < b < \infty$, *and let* $f,\ g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *differentiable on* (a,b). *Let* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \text{ and } [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

3. Expansion and inequality for $\sin x/x$

Theorem 3.1 develops the approximation formula (1.9) to produce a full expansion.

THEOREM 3.1. *Let* $\theta \geqslant 2$ *be a given real number. The following expansion holds:*

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} b_n \big(\pi^\theta - (2x)^\theta\big)^n, \tag{3.1}$$

with the coefficients b_n $(n \in \mathbb{N}_0)$ *given by*

$$b_n = (-1)^n \frac{a_n - \sum_{k=1}^{n-1} b_k (-1)^k 2^{\theta k} k! B_{n,k}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1}, \ldots, \theta(\theta-1)\cdots(\theta-n+k)\left(\frac{\pi}{2}\right)^{\theta-n+k-1}\right)}{n!(2\theta)^n \pi^{(\theta-1)n}}, \tag{3.2}$$

where a_n *is given in* (2.6), *an empty sum is understood to be zero.*

Proof. In view of (1.9), we can let

$$F(x)=\frac{\sin x}{x}=\frac{2}{\pi}+\sum_{j=1}^{\infty} b_j(-1)^j 2^{\theta j}\left(x^{\theta}-\left(\frac{\pi}{2}\right)^{\theta}\right)^j,$$

where b_j $(j\in\mathbb{N})$ are real numbers to be determined. Let

$$f(u)=u^j \quad \text{and} \quad u=g(x)=x^{\theta}-\left(\frac{\pi}{2}\right)^{\theta}.$$

By (2.2), we have

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{\theta}-\left(\frac{\pi}{2}\right)^{\theta}\right)^j &=\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(g(x))=\sum_{k=1}^{n} f^{(k)}(g(x))B_{n,k}\big(g'(x),g''(x),\dots,g^{(n-k+1)}(x)\big)\\ &=\sum_{k=1}^{n} j(j-1)\cdots(j-k+1)\left(x^{\theta}-\left(\frac{\pi}{2}\right)^{\theta}\right)^{j-k}\\ &\quad\times B_{n,k}\big(\theta x^{\theta-1},\theta(\theta-1)x^{\theta-2},\dots,\theta(\theta-1)\cdots(\theta-n+k)x^{\theta-n+k-1}\big).\end{aligned}$$

We then obtain, for $n\in\mathbb{N}$,

$$\begin{aligned}F^{(n)}(x)&=\sum_{j=1}^{\infty} b_j(-1)^j2^{\theta j}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{\theta}-\left(\frac{\pi}{2}\right)^{\theta}\right)^j\\ &=\sum_{j=1}^{\infty}\sum_{k=1}^{n} b_j(-1)^j2^{\theta j}j(j-1)\cdots(j-k+1)\left(x^{\theta}-\left(\frac{\pi}{2}\right)^{\theta}\right)^{j-k}\\ &\qquad\times B_{n,k}\big(\theta x^{\theta-1},\theta(\theta-1)x^{\theta-2},\dots,\theta(\theta-1)\cdots(\theta-n+k)x^{\theta-n+k-1}\big)\\ &=\left\{\sum_{k=1}^{n} b_k(-1)^k2^{\theta k}k!+\sum_{k=1}^{n} b_{k+1}(-1)^{k+1}2^{\theta(k+1)}(k+1)!\left(x^{\theta}-\left(\frac{\pi}{2}\right)^{\theta}\right)+\cdots\right\}\\ &\qquad\times B_{n,k}\big(\theta x^{\theta-1},\theta(\theta-1)x^{\theta-2},\dots,\theta(\theta-1)\cdots(\theta-n+k)x^{\theta-n+k-1}\big).\end{aligned}$$

Hence, we have

$$\begin{aligned}a_n&=F^{(n)}\left(\frac{\pi}{2}\right)\\ &=\sum_{k=1}^{n} b_k(-1)^k2^{\theta k}k!B_{n,k}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1},\dots,\theta(\theta-1)\cdots(\theta-n+k)\left(\frac{\pi}{2}\right)^{\theta-n+k-1}\right)\\ &=\sum_{k=1}^{n-1} b_k(-1)^k2^{\theta k}k!B_{n,k}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1},\dots,\theta(\theta-1)\cdots(\theta-n+k)\left(\frac{\pi}{2}\right)^{\theta-n+k-1}\right)\\ &\qquad+b_n(-1)^n2^{\theta n}n!B_{n,n}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1}\right)\\ &=\sum_{k=1}^{n-1} b_k(-1)^k2^{\theta k}k!B_{n,k}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1},\dots,\theta(\theta-1)\cdots(\theta-n+k)\left(\frac{\pi}{2}\right)^{\theta-n+k-1}\right)\\ &\qquad+b_n(-1)^n2^{\theta n}n!\theta^n\left(\frac{\pi}{2}\right)^{(\theta-1)n}.\end{aligned}$$

This yields (3.2). The proof is complete.

By using (3.2), we now give explicit numerical values of the first few b_n. Noting that

$$B_{2,1}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1},\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2}\right)=\frac{2!}{0!1!}\left(\frac{\theta\left(\frac{\pi}{2}\right)^{\theta-1}}{1!}\right)^{0}\left(\frac{\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2}}{2!}\right)^{1}$$
$$=\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2},$$

$$B_{3,1}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1},\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2},\theta(\theta-1)(\theta-2)\left(\frac{\pi}{2}\right)^{\theta-3}\right)$$
$$=\frac{3!}{0!0!1!}\left(\frac{\theta\left(\frac{\pi}{2}\right)^{\theta-1}}{1!}\right)^{0}\left(\frac{\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2}}{2!}\right)^{0}\left(\frac{\theta(\theta-1)(\theta-2)\left(\frac{\pi}{2}\right)^{\theta-3}}{3!}\right)^{1}$$
$$=\theta(\theta-1)(\theta-2)\left(\frac{\pi}{2}\right)^{\theta-3},$$

$$B_{3,2}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1},\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2}\right)=\frac{3!}{1!1!}\left(\frac{\theta\left(\frac{\pi}{2}\right)^{\theta-1}}{1!}\right)^{1}\left(\frac{\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2}}{2!}\right)^{1}$$
$$=3\theta^2(\theta-1)\left(\frac{\pi}{2}\right)^{2\theta-3},$$

we find

$$b_0=a_0=\frac{2}{\pi},$$
$$b_1=-\frac{a_1}{2\theta\pi^{\theta-1}}=-\frac{-\frac{4}{\pi^2}}{2\theta\pi^{\theta-1}}=\frac{2}{\theta\pi^{\theta+1}},$$
$$b_2=\frac{a_2+b_1 2^{\theta}B_{2,1}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1},\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2}\right)}{2(2\theta)^2\pi^{2(\theta-1)}}$$
$$=\frac{-\frac{2(\pi^2-8)}{\pi^3}+\left(\frac{2}{\theta\pi^{\theta+1}}\right)2^{\theta}\left(\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2}\right)}{2(2\theta)^2\pi^{2(\theta-1)}}=\frac{4\theta+4-\pi^2}{4\theta^2\pi^{2\theta+1}},$$

$$b_3=-\frac{1}{3!(2\theta)^3\pi^{3(\theta-1)}}$$
$$\times\left\{a_3+b_1 2^{\theta}B_{3,1}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1},\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2},\theta(\theta-1)(\theta-2)\left(\frac{\pi}{2}\right)^{\theta-3}\right)\right.$$
$$\left.-b_2 2^{2\theta+1}B_{3,2}\left(\theta\left(\frac{\pi}{2}\right)^{\theta-1},\theta(\theta-1)\left(\frac{\pi}{2}\right)^{\theta-2}\right)\right\}$$

$$= -\frac{1}{3!(2\theta)^3\pi^{3(\theta-1)}}\left\{\frac{12(\pi^2-8)}{\pi^4}+\left(\frac{2}{\theta\pi^{\theta+1}}\right)2^{\theta}\left(\theta(\theta-1)(\theta-2)\left(\frac{\pi}{2}\right)^{\theta-3}\right)\right.$$
$$\left.-\left(\frac{4\theta+4-\pi^2}{4\theta^2\pi^{2\theta+1}}\right)2^{2\theta+1}\left(3\theta^2(\theta-1)\left(\frac{\pi}{2}\right)^{2\theta-3}\right)\right\}$$
$$= \frac{8\theta^2-(3\pi^2-12)\theta+4}{12\theta^3\pi^{3\theta+1}}.$$

We note that the values of b_n (for $n = 0,1,2,3$) here are equal to the coefficients of $\left(\pi^{\theta}-(2x)^{\theta}\right)^n$ (for $n = 0,1,2,3$) in (1.9), respectively.

The formula (2.5) motivated us to establish Theorem 3.2.

THEOREM 3.2. *For* $0 < x < \pi/2$ *and* $m \in \mathbb{N}_0$,

$$\sum_{n=0}^{4m}\frac{a_n}{n!}\left(x-\frac{\pi}{2}\right)^n < \frac{\sin x}{x} < \sum_{n=0}^{4m+2}\frac{a_n}{n!}\left(x-\frac{\pi}{2}\right)^n, \tag{3.3}$$

where the coefficients a_n *(* $n \in \mathbb{N}_0$ *) are given in* (2.6).

Proof. Let $F(x) = \sin x/x$. We find by (2.7) that, for $0 < x < \pi/2$ and $n \in \mathbb{N}$,

$$(-1)^n F^{(2n-1)}(x) = \frac{1}{x^{2n}}\int_0^x t^{2n-1}\sin t\,\mathrm{d}t > 0$$

and

$$(-1)^n F^{(2n)}(x) = \frac{1}{x^{2n+1}}\int_0^x t^{2n}\cos t\,\mathrm{d}t > 0.$$

That is,

$$F^{(4m+3)}(x) > 0, \quad F^{(4m)}(x) > 0, \quad F^{(4m+1)}(x) < 0, \quad F^{(4m+2)}(x) < 0$$

for $0 < x < \pi/2$ and $m \in \mathbb{N}_0$.

By Taylor's theorem, there exists a ξ such that $0 < x < \xi < \pi/2$ and

$$F(x) - \sum_{n=0}^{4m}\frac{a_n}{n!}\left(x-\frac{\pi}{2}\right)^n = \frac{F^{(4m+1)}(\xi)}{5!}\left(x-\frac{\pi}{2}\right)^5 > 0. \tag{3.4}$$

There exists a η such that $0 < x < \eta < \pi/2$ and

$$F(x) - \sum_{n=0}^{4m+2}\frac{a_n}{n!}\left(x-\frac{\pi}{2}\right)^n = \frac{F^{(4m+3)}(\eta)}{3!}\left(x-\frac{\pi}{2}\right)^3 < 0. \tag{3.5}$$

The proof is complete.

4. Sharp Becker-Stark-type and Papenfuss-type inequalities

4.1. Sharp Becker-Stark-type inequality

By using Maple, we find that

$$\frac{\tan x}{x}\left(\pi^2-4x^2\right)^{\theta}=\pi^{2\theta}+\left(\frac{\pi^2}{12}-\theta\right)4\pi^{2p-2}x^2+\ldots.$$

This fact motivated us to establish Theorem 4.1. Theorem 4.1 presents sharp Becker-Stark-type inequality.

THEOREM 4.1. *For $0<x<\pi/2$, we have*

$$\frac{\pi^{2\theta}}{\left(\pi^2-(2x)^2\right)^{\theta}}<\frac{\tan x}{x}<\frac{\pi^{2\vartheta}}{\left(\pi^2-(2x)^2\right)^{\vartheta}}, \tag{4.1}$$

or alternatively

$$\frac{1}{\left(1-(\frac{2x}{\pi})^2\right)^{\theta}}<\frac{\tan x}{x}<\frac{1}{\left(1-(\frac{2x}{\pi})^2\right)^{\vartheta}}, \tag{4.2}$$

where the constants $\theta=\pi^2/12=0.822467\ldots$ and $\vartheta=1$ are the best possible, in the sense that $\theta=\pi^2/12$ can not be replaced by a larger number, and $\vartheta=1$ can not be replaced by a smaller number.

Proof. The inequality (4.2) can be written for $0<x<\pi/2$ as

$$\theta<\frac{\ln\left(\frac{x}{\tan x}\right)}{\ln\left(1-(\frac{2x}{\pi})^2\right)}<\vartheta.$$

For $0\leqslant x<\pi/2$, let

$$F_1(x)=\ln\left(\frac{x}{\tan x}\right),\quad F_1(0)=0\quad\text{and}\quad F_2(x)=\ln\left(1-\left(\frac{2x}{\pi}\right)^2\right),$$

and let

$$F(x)=\frac{F_1(x)}{F_2(x)}=\frac{\ln\left(\frac{x}{\tan x}\right)}{\ln\left(1-(\frac{2x}{\pi})^2\right)},\qquad 0<x<\frac{\pi}{2}.$$

Then,

$$\frac{F_1'(x)}{F_2'(x)}=\frac{\left(x-\sin x\cos x\right)(\pi^2-4x^2)}{8x^2\sin x\cos x}=:G(x).$$

Differentiation yields

$$G'(x) = \frac{I(x)}{8x^3 \sin^2 x \cos^2 x},$$

where

$$I(x) = -(4x^3 + \pi^2 x)\sin x \cos x + (8x^4 - 2\pi^2 x^2 + 2\pi^2)\cos^2 x - 2\pi^2 \cos^4 x + \pi^2 x^2 - 4x^4.$$

We are in a position to prove $I(x) > 0$ for $0 < x < \pi/2$. We consider two cases.

Case 1: $0 < x \leqslant 1.25$.

It is known that, for $x > 0$ and $n \in \mathbb{N}_0$,

$$\sum_{k=0}^{2n+1} (-1)^k \frac{x^{2k+1}}{(2k+1)!} < \sin x < \sum_{k=0}^{2n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \tag{4.3}$$

and

$$\sum_{k=0}^{2n+1} (-1)^k \frac{x^{2k}}{(2k)!} < \cos x < \sum_{k=0}^{2n} (-1)^k \frac{x^{2k}}{(2k)!}. \tag{4.4}$$

Using (4.3) and (4.4), we have, for $0 < x \leqslant 1.25$,

$$\begin{aligned}
I(x) &= -\left(2x^3 + \frac{1}{2}\pi^2 x\right)\sin(2x) - x^2(\pi^2 - 4x^2)\cos(2x) - \frac{1}{4}\pi^2\cos(4x) + \frac{1}{4}\pi^2 \\
&> -\left(2x^3 + \frac{1}{2}\pi^2 x\right)\left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \frac{4}{2835}x^9\right) \\
&\quad - x^2(\pi^2 - 4x^2)\left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{315}x^8\right) \\
&\quad - \frac{1}{4}\pi^2\left(1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \frac{512}{315}x^8 - \frac{4096}{14175}x^{10} + \frac{16384}{467775}x^{12}\right) + \frac{1}{4}\pi^2 \\
&= x^6\left\{\left(\frac{28\pi^2}{45} - \frac{16}{3}\right) - \left(\frac{32\pi^2}{105} - \frac{32}{15}\right)x^2 + \left(\frac{44\pi^2}{675} - \frac{32}{105}\right)x^4 \right. \\
&\qquad\qquad \left. - \left(\frac{4096\pi^2}{467775} - \frac{64}{2835}\right)x^6\right\} > 0.
\end{aligned}$$

Case 2: $1.25 < x < \pi/2$.

We now prove $I(x) > 0$ for $1.25 < x < \pi/2$. Replacing x by $\frac{\pi}{2} - t$ leads to equivalent inequality:

$$J(t) > 0, \qquad 0 < t < \frac{\pi}{2} - 1.25,$$

where

$$\begin{aligned}J(t) = &-\left\{4\left(\frac{t}{2}-t\right)^3+\pi^2\left(\frac{t}{2}-t\right)\right\}\cos t\sin t\\&+\left\{8\left(\frac{t}{2}-t\right)^4-2\pi^2\left(\frac{t}{2}-t\right)^2+2\pi^2\right\}\sin^2 t-2\pi^2\sin^4 t\\&+\pi^2\left(\frac{t}{2}-t\right)^2-4\left(\frac{t}{2}-t\right)^4\\=&-\left(\frac{1}{2}\pi^3-2\pi^2 t+3\pi t^2-2t^3\right)\sin(2t)+(\pi^3 t-5\pi^2 t^2+8\pi t^3-4t^4)\cos(2t)\\&-\frac{1}{4}\pi^2\cos(4t)+\frac{1}{4}\pi^2.\end{aligned}$$

Using (4.3) and (4.4), we have, for $0 < t < \frac{\pi}{2} - 1.25$,

$$\begin{aligned}J(t) > &-\left(\frac{1}{2}\pi^3-2\pi^2 t+3\pi t^2-2t^3\right)\left(2t-\frac{4}{3}t^3+\frac{4}{15}t^5\right)\\&+(\pi^3 t-5\pi^2 t^2+8\pi t^3-4t^4)(1-2t^2)-\frac{1}{4}\pi^2\left(1-8t^2+\frac{32}{3}t^4\right)+\frac{1}{4}\pi^2\\=&\,t^2\left\{\pi^2-\left(\frac{4}{3}\pi^3-2\pi\right)t+\frac{14}{3}\pi^2 t^2-\left(\frac{2}{15}\pi^3+12\pi\right)t^3\right.\\&\left.+\left(\frac{8}{15}\pi^2+\frac{16}{3}\right)t^4-\frac{4}{5}\pi t^5+\frac{8}{15}t^6\right\} > 0.\end{aligned}$$

This proves $I(x) > 0$ for all $0 < x < \pi/2$.

We then obtain $G'(x) >$ for $0 < x < \pi/2$. Therefore, the functions $G(x)$ and $F_1'(x)/F_2'(x)$ are strictly increasing on $(0, \pi/2)$. By Lemma 2.2, the function

$$f(x) = \frac{F_1(x)}{F_2(x)} = \frac{F_1(x)-F_1(0)}{F_2(x)-F_2(0)}$$

is strictly increasing on $(0, \pi/2)$. And hence, we have, $0 < x < \pi/2$,

$$\frac{\pi^2}{12} = \lim_{u\to 0^+} F(u) < F(x) = \frac{\ln\left(\frac{x}{\tan x}\right)}{\ln\left(1-(\frac{2x}{\pi})^2\right)} < \lim_{u\to \pi/2^-} F(u) = 1.$$

Hence, (4.2) holds for $0 < x < \pi/2$, and the constants $\theta = \pi^2/12$ and $\vartheta = 1$ are the best possible. The proof is complete.

4.2. Sharp Papenfuss-type inequalities

By using Maple, we find that

$$\left(\frac{\tan x}{x}\right)'(\pi^2-4x^2)^p = \frac{2}{3}\pi^{2p}x+\left(\frac{\pi^2}{5}-p\right)\frac{8\pi^{2p-2}}{5}x^3+\ldots.$$

This fact motivated us to establish Theorem 4.2. Theorem 4.2 presents sharp Papenfuss-type inequality.

THEOREM 4.2. *For $0 < x < \pi/2$, we have*

$$\frac{\frac{2}{3}\pi^{2p}x^3}{\left(\pi^2-(2x)^2\right)^p} < x\sec^2 x - \tan x < \frac{\frac{2}{3}\pi^{2q}x^3}{\left(\pi^2-(2x)^2\right)^q}, \tag{4.5}$$

or alternatively

$$\frac{\frac{2}{3}\pi^{2p}x}{\left(\pi^2-(2x)^2\right)^p} < \left(\frac{\tan x}{x}\right)' < \frac{\frac{2}{3}\pi^{2q}x}{\left(\pi^2-(2x)^2\right)^q}, \tag{4.6}$$

i.e.,

$$\frac{\frac{2}{3}x}{\left(1-(\frac{2x}{\pi})^2\right)^p} < \left(\frac{\tan x}{x}\right)' < \frac{\frac{2}{3}x}{\left(1-(\frac{2x}{\pi})^2\right)^q}, \tag{4.7}$$

where the constants $p = \pi^2/5 = 1.97392\ldots$ and $q = 2$ are the best possible, in the sense that $p = \pi^2/5$ can not be replaced by a larger number, and $q = 2$ can not be replaced by a smaller number.

Proof. The inequality (4.7) can be written for $0 < x < \pi/2$ as

$$p < \frac{\ln\left(\frac{\frac{2}{3}x}{\left(\frac{\tan x}{x}\right)'}\right)}{\ln\left(1-(\frac{2x}{\pi})^2\right)} < q.$$

For $0 \leqslant x < \pi/2$, let

$$f_1(x) = \ln\left(\frac{\frac{2}{3}x}{\left(\frac{\tan x}{x}\right)'}\right) = \ln\left(\frac{2x^3\cos^2 x}{3(x-\sin x\cos x)}\right), \quad f_1(0) = \lim_{x\to 0^+} f(x) = 0$$

and

$$f_2(x) = \ln\left(1-\left(\frac{2x}{\pi}\right)^2\right),$$

and let

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{\ln\left(\frac{\frac{2}{3}x}{\left(\frac{\tan x}{x}\right)'}\right)}{\ln\left(1-(\frac{2x}{\pi})^2\right)}, \qquad 0 < x < \frac{\pi}{2}.$$

Then,

$$\frac{f_1'(x)}{f_2'(x)} = \frac{(3\sin x\cos^2 x + 2x^2\sin x - 3x\cos x)(\pi^2-4x^2)}{8x^2\cos x(x-\sin x\cos x)} =: g(x).$$

Differentiation yields

$$g'(x) = \frac{h(x)}{4x^3\cos^2 x\big(x - \sin x\cos x\big)^2},$$

where

$$h(x) = -(6\pi^2 x + 8x^5 - 2\pi^2 x^3)\cos^3 x \sin x - x^3(3\pi^2 - 4x^2)\cos x \sin x \\ + (3\pi^2 x^2 + 8x^4)\cos^2 x + (3\pi^2 - 8x^4)\cos^4 x - 3\pi^2\cos^6 x + x^4(\pi^2 - 4x^2).$$

Motivated by the investigations in [22], we are in a position to prove $h(x) > 0$ for $0 < x < \pi/2$. We consider two cases.

Case 1: $0 < x \leqslant 1.25$.

Let

$$H(x) = \begin{cases} a, & x = 0, \\ \dfrac{h(x)}{x^{10}(\frac{\pi}{2} - x)^3}, & 0 < x \leqslant 1.25, \end{cases}$$

where a is constant determined with limit:

$$a = \lim_{x\to 0^+} \frac{h(x)}{x^{10}(\frac{\pi}{2} - x)^3} = \frac{3712\pi^2 - 35840}{1575\pi^3} = 0.01629924\ldots.$$

Using Maple we determine Taylor approximation for the function $H(x)$ by the polynomial of the ninth order:

$$P(x) = \frac{128(29\pi^2 - 280)}{1575\pi^3} + \frac{256(29\pi^2 - 280)}{525\pi^4}x \\ + \cdots + \frac{512(74687\pi^{10} - 7939920\pi^8 + 394878120\pi^6 - 10949178240\pi^4 + 157491734400\pi^2 - 799134336000)}{638512875\pi^{12}}x^9$$

which has a bound of absolute error

$$\varepsilon_1 = 0.00013674\ldots$$

for values $0 \leqslant x \leqslant 1.25$. It is true that

$$H(x) - (P(x) - \varepsilon_1) \geqslant 0$$

and

$$P(x) - \varepsilon_1 > 0$$

for $0 \leqslant x \leqslant 1.25$. Hence, for $x \in [0, 1.25]$ it is true that $H(x) > 0$ and therefore $h(x) > 0$ for $x \in (0, 1.25]$.

Case 2: $1.25 < x < \pi/2$.

We now prove $h(x) > 0$ for $1.25 < x < \pi/2$. Replacing x by $\frac{\pi}{2} - t$ leads to equivalent inequality:

$$u(t) > 0, \qquad 0 < t < \frac{\pi}{2} - 1.25,$$

where

$$u(t) = -\left\{6\pi^2\left(\frac{\pi}{2}-t\right)+8\left(\frac{\pi}{2}-t\right)^5-2\pi^2\left(\frac{\pi}{2}-t\right)^3\right\}\sin^3 t\cos t$$
$$-\left(\frac{\pi}{2}-t\right)^3\left\{3\pi^2-4\left(\frac{\pi}{2}-t\right)^2\right\}\sin t\cos t+\left\{3\pi^2\left(\frac{\pi}{2}-t\right)^2+8\left(\frac{\pi}{2}-t\right)^4\right\}\sin^2 t$$
$$+\left\{3\pi^2-8\left(\frac{\pi}{2}-t\right)^4\right\}\sin^4 t-3\pi^2\sin^6 t+\left(\frac{\pi}{2}-t\right)^4\left\{\pi^2-4\left(\frac{\pi}{2}-t\right)^2\right\}.$$

Let

$$U(t) = \begin{cases} b, & t=0, \\ \dfrac{u(t)}{t^3(\frac{\pi}{2}-t)^{10}}, & 0<t<\frac{\pi}{2}-1.25, \end{cases}$$

where b is constant determined with limit:

$$b = \lim_{t\to 0^+}\frac{u(t)}{t^3(\frac{\pi}{2}-t)^{10}} = \frac{512(\pi^2-9)}{3\pi^7} = 0.04913843\ldots.$$

Using Maple we determine Taylor approximation for the function $U(t)$ by the polynomial of the third order:

$$Q(t) = \frac{512(\pi^2-9)}{3\pi^7}+\frac{256(11\pi^2-108)}{\pi^8}t$$
$$-\frac{512(7920-810\pi^2+\pi^4)}{15\pi^9}t^2-\frac{512(167400-17970\pi^2+103\pi^4)}{45\pi^{10}}t^3$$

which has a bound of absolute error

$$\varepsilon_2 = 0.000007293\ldots$$

for values $0\leqslant t\leqslant\frac{\pi}{2}-1.25$. It is true that

$$U(t)-(Q(t)-\varepsilon_2)\geqslant 0$$

and

$$Q(t)-\varepsilon_2>0$$

for $0\leqslant t\leqslant\frac{\pi}{2}-1.25$. Hence, for $t\in[0,\frac{\pi}{2}-1.25]$ it is true that $U(t)>0$ and therefore $u(t)>0$ for $t\in(0,\frac{\pi}{2}-1.25)$.

This proves $h(x)>0$ for all $0<x<\pi/2$.

We then obtain $g'(x)>$ for $0<x<\pi/2$. Therefore, the functions $g(x)$ and $f_1'(x)/f_2'(x)$ are strictly increasing on $(0,\pi/2)$. By Lemma 2.2, the function

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x)-f_1(0)}{f_2(x)-f_2(0)}$$

is strictly increasing on $(0,\pi/2)$. And hence, we have, $0<x<\pi/2$,

$$\frac{\pi^2}{5}=\lim_{u\to 0^+} f(u)<f(x)=\frac{\ln\left(\frac{\frac{2}{3}x}{\left(\frac{\tan x}{x}\right)'}\right)}{\ln\left(1-\frac{2x}{\pi}\right)}<\lim_{u\to\pi/2^-} f(u)=2.$$

Hence, (4.7) holds for $0<x<\pi/2$, and the constants $p=\pi^2/5$ and $q=2$ are the best possible. The proof is complete.

REMARK 4.1. Integrating (4.6) from 0 to x, we obtain that, for $0<x<\pi/2$,

$$\frac{\pi^{2p}}{12(p-1)\left(\pi^2-(2x)^2\right)^{p-1}}-\frac{\pi^2}{12(p-1)}<\frac{\tan x}{x}-1$$
$$<\frac{\pi^{2q}}{12(q-1)\left(\pi^2-(2x)^2\right)^{q-1}}-\frac{\pi^2}{12(q-1)}. \tag{4.8}$$

The choice $p=\pi^2/5$ and $q=2$ in (4.8) yields

$$\frac{\pi^{2\pi^2/5}}{12(\pi^2/5-1)\left(\pi^2-(2x)^2\right)^{\pi^2/5-1}}+\frac{12(\pi^2/5-1)-\pi^2}{12(\pi^2/5-1)}<\frac{\tan x}{x}<\frac{\pi^2-(4-\frac{1}{3}\pi^2)x^2}{\pi^2-4x^2}. \tag{4.9}$$

There is no strict comparison between the two lower bounds in (1.11) and (4.9). Clearly, the upper bound in (4.9) is sharper than that in (1.11).

5. An application

It is well-known that the Yang Le inequality plays an important role in the theory of distribution of values of functions (see [41] for details). This inequality is stated below:

If $A_1>0$, $A_2>0$, $A_1+A_2\leqslant\pi$ and $0\leqslant\mu\leqslant 1$, then,

$$\cos^2\mu A_1+\cos^2\mu A_2-2\cos\mu\pi\cos\mu A_1\cos\mu A_2\geqslant\sin^2\mu\pi. \tag{5.1}$$

Debnath and Zhao [17, Theorem 1] obtained an improvement of the Yang Le inequality and proved:

Let $A_i>0\,(i=1,2,\ldots,n)$ with $\sum_{i=1}^{n}A_i\leqslant\pi$, $0\leqslant\lambda\leqslant 1$, $\theta\geqslant 2$, and let $n\geqslant 2$ be a natural number. Then

$$N(\lambda)\leqslant(n-1)\sum_{k=1}^{n}\cos^2\lambda A_k-2\cos\lambda\pi\sum_{1\leqslant i<j\leqslant n}\cos\lambda A_i\cos\lambda A_j\leqslant M(\lambda), \tag{5.2}$$

where

$$N(\lambda)=\binom{n}{2}\left(3-\lambda^2\right)^2\left(\lambda\cos\frac{\lambda\pi}{2}\right)^2\quad\text{and}\quad M(\lambda)=\binom{n}{2}\lambda^2\pi^2.$$

By using inequality (3.3), we here present an improvement of the Yang Le inequality.

THEOREM 5.1. *Let* $A_i > 0\,(i = 1,2,\ldots,n)$ *with* $\sum_{i=1}^{n} A_i \leqslant \pi,\ 0 \leqslant \lambda \leqslant 1,\ \theta \geqslant 2,$ *and let* $n \geqslant 2$ *be a natural number. Then*

$$N_m(\lambda) \leqslant (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - 2\cos\lambda\pi \sum_{1\leqslant i<j\leqslant n}\cos\lambda A_i\cos\lambda A_j \leqslant M_m(\lambda), \quad (5.3)$$

where

$$N_m(\lambda) = \binom{n}{2}\left(\sum_{j=0}^{4m}\frac{2a_j}{j!}\left(\frac{\pi}{2}\right)^{j+1}(\lambda-1)^j\right)^2\left(\lambda\cos\frac{\lambda\pi}{2}\right)^2$$

and

$$M_m(\lambda) = \binom{n}{2}\left(\sum_{j=0}^{4m+2}\frac{2a_j}{j!}\left(\frac{\pi}{2}\right)^{j+1}(\lambda-1)^j\right)^2\lambda^2.$$

Proof. Let

$$H_{ij} = \cos^2\lambda A_i + \cos^2\lambda A_j - 2\cos\lambda\pi\cos\lambda A_i\cos\lambda A_j.$$

It follows from [44] that

$$\sin^2\lambda\pi \leqslant H_{ij} \leqslant 4\sin^2\frac{\lambda}{2}\pi, \quad 1\leqslant i<j\leqslant n. \quad (5.4)$$

By summing all of the inequalities in (5.4), we obtain

$$\sum_{1\leqslant i<j\leqslant n}\sin^2\lambda\pi \leqslant \sum_{1\leqslant i<j\leqslant n}H_{ij} \leqslant \sum_{1\leqslant i<j\leqslant n}4\sin^2\frac{\lambda}{2}\pi,$$

that is,

$$4\binom{n}{2}\sin^2\frac{\lambda}{2}\pi\cos^2\frac{\lambda}{2}\pi \leqslant (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - 2\cos\lambda\pi\sum_{1\leqslant i<j\leqslant n}\cos\lambda A_i\cos\lambda A_j \leqslant 4\binom{n}{2}\sin^2\frac{\lambda}{2}\pi. \quad (5.5)$$

On the other hand, it follows from the inequality (3.3), by a direct calculation, that

$$\sum_{j=0}^{4m}\frac{a_j}{j!}\left(\frac{\pi}{2}\right)^{j+1}(\lambda-1)^j\lambda < \sin\frac{\pi\lambda}{2} < \sum_{j=0}^{4m+2}\frac{a_j}{j!}\left(\frac{\pi}{2}\right)^{j+1}(\lambda-1)^j\lambda. \quad (5.6)$$

Applying the inequality (5.6) to (5.5) leads to the desired inequality (5.3). The proof is complete.

The choice $m=1$ in (5.3) yields

$$N_1(\lambda) \leqslant (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - 2\cos\lambda\pi \sum_{1\leqslant i<j\leqslant n}\cos\lambda A_i \cos\lambda A_j \leqslant M_1(\lambda), \quad (5.7)$$

where

$$N_1(\lambda) = \binom{n}{2}\left(10-\frac{3}{4}\pi^2+\frac{1}{192}\pi^4+\left(\frac{9}{4}\pi^2-20-\frac{1}{48}\pi^4\right)\lambda+\left(20-\frac{5}{2}\pi^2+\frac{1}{32}\pi^4\right)\lambda^2 + \left(\frac{5}{4}\pi^2-10-\frac{1}{48}\pi^4\right)\lambda^3+\left(2+\frac{1}{192}\pi^4-\frac{1}{4}\pi^2\right)\lambda^4\right)^2\left(\lambda\cos\frac{\lambda\pi}{2}\right)^2$$

and

$$M_1(\lambda) = \binom{n}{2}\left(6-\frac{1}{4}\pi^2+\left(\frac{1}{2}\pi^2-6\right)\lambda+\left(-\frac{1}{4}\pi^2+2\right)\lambda^2\right)^2\lambda^2.$$

REMARK 5.1. Noting that

$$\begin{aligned}10&-\frac{3}{4}\pi^2+\frac{1}{192}\pi^4+\left(\frac{9}{4}\pi^2-20-\frac{1}{48}\pi^4\right)\lambda+\left(20-\frac{5}{2}\pi^2+\frac{1}{32}\pi^4\right)\lambda^2\\ &+\left(\frac{5}{4}\pi^2-10-\frac{1}{48}\pi^4\right)\lambda^3+\left(2+\frac{1}{192}\pi^4-\frac{1}{4}\pi^2\right)\lambda^4\\ &>10-\frac{3}{4}\pi^2+\frac{1}{192}\pi^4+\left(\frac{9}{4}\pi^2-20-\frac{1}{48}\pi^4\right)\lambda-\left(-20+\frac{5}{2}\pi^2-\frac{1}{32}\pi^4\right)\lambda^2>0\end{aligned}$$

holds for $0\leqslant\lambda\leqslant 1$, we find

$$\begin{aligned}10&-\frac{3}{4}\pi^2+\frac{1}{192}\pi^4+\left(\frac{9}{4}\pi^2-20-\frac{1}{48}\pi^4\right)\lambda+\left(20-\frac{5}{2}\pi^2+\frac{1}{32}\pi^4\right)\lambda^2\\ &+\left(\frac{5}{4}\pi^2-10-\frac{1}{48}\pi^4\right)\lambda^3+\left(2+\frac{1}{192}\pi^4-\frac{1}{4}\pi^2\right)\lambda^4-(3-\lambda^2)\\ &=\frac{(\lambda-1)^2}{192}\left(\pi^4+1344-144\pi^2+(-2\pi^4-1152+144\pi^2)\lambda+(\pi^4-48\pi^2+384)\lambda^2\right)>0\end{aligned}$$

for $0\leqslant\lambda\leqslant 1$. We then obtain

$$N(\lambda) < N_1(\lambda).$$

Hence, the lower bound in inequality (5.3) is sharper than the one in inequality (5.2).

REMARK 5.2. There is no strict comparison between the two upper bounds in inequalities (5.2) and (5.3).

6. Open problem and conjecture

6.1. Open problem

Computer experiments suggest that, for $0 < x < \pi/2$,

$$\frac{\frac{2}{3}}{\left(1-(\frac{2x}{\pi})^2\right)^{\lambda_2}} < \left(\frac{\tan x}{x}\right)'' < \frac{\frac{2}{3}}{\left(1-(\frac{2x}{\pi})^2\right)^{\mu_2}}, \tag{6.1}$$

$$\frac{\frac{16}{5}x}{\left(1-(\frac{2x}{\pi})^2\right)^{\lambda_3}} < \left(\frac{\tan x}{x}\right)''' < \frac{\frac{16}{5}x}{\left(1-(\frac{2x}{\pi})^2\right)^{\mu_3}}, \tag{6.2}$$

$$\frac{\frac{16}{5}}{\left(1-(\frac{2x}{\pi})^2\right)^{\lambda_4}} < \left(\frac{\tan x}{x}\right)^{(4)} < \frac{\frac{16}{5}}{\left(1-(\frac{2x}{\pi})^2\right)^{\mu_4}}, \tag{6.3}$$

$$\frac{\frac{272}{7}x}{\left(1-(\frac{2x}{\pi})^2\right)^{\lambda_5}} < \left(\frac{\tan x}{x}\right)^{(5)} < \frac{\frac{272}{7}x}{\left(1-(\frac{2x}{\pi})^2\right)^{\mu_5}}, \tag{6.4}$$

$$\frac{\frac{272}{7}}{\left(1-(\frac{2x}{\pi})^2\right)^{\lambda_6}} < \left(\frac{\tan x}{x}\right)^{(6)} < \frac{\frac{272}{7}}{\left(1-(\frac{2x}{\pi})^2\right)^{\mu_6}}, \tag{6.5}$$

$$\frac{\frac{7936}{9}x}{\left(1-(\frac{2x}{\pi})^2\right)^{\lambda_7}} < \left(\frac{\tan x}{x}\right)^{(7)} < \frac{\frac{7936}{9}x}{\left(1-(\frac{2x}{\pi})^2\right)^{\mu_7}}, \tag{6.6}$$

$$\frac{\frac{7936}{9}}{\left(1-(\frac{2x}{\pi})^2\right)^{\lambda_8}} < \left(\frac{\tan x}{x}\right)^{(8)} < \frac{\frac{7936}{9}}{\left(1-(\frac{2x}{\pi})^2\right)^{\mu_8}}, \tag{6.7}$$

where the constants

$$\lambda_2 = 3, \quad \mu_2 = \frac{3\pi^2}{5} = 5.92176\ldots,$$

$$\lambda_3 = 4, \quad \mu_3 = \frac{85\pi^2}{168} = 4.99354\ldots,$$

$$\lambda_4 = 5, \quad \mu_4 = \frac{85\pi^2}{56} = 14.9806\ldots,$$

$$\lambda_5 = 6, \quad \mu_5 = \frac{434\pi^2}{459} = 9.332044\ldots,$$

$$\lambda_6 = 7, \quad \mu_6 = \frac{434\pi^2}{153} = 27.9961\ldots,$$

$$\lambda_7 = 8, \quad \mu_7 = \frac{2073\pi^2}{1364} = 14.99977\ldots,$$

$$\lambda_8 = 9, \quad \mu_8 = \frac{6219\pi^2}{1364} = 44.99931\ldots$$

are the best possible.

In view of (6.1)-(6.7), we now propose the following open problem.

Open problem 6.1. (i) Let $n \geqslant 2$ be a given integer. Find the best possible constant q such that

$$\frac{\frac{T_{2n+1}}{2n+1}x}{\left(1-\left(\frac{2x}{\pi}\right)^2\right)^{2n}} < \left(\frac{\tan x}{x}\right)^{(2n-1)} < \frac{\frac{T_{2n+1}}{2n+1}x}{\left(1-\left(\frac{2x}{\pi}\right)^2\right)^{q}}, \qquad 0 < x < \frac{\pi}{2}. \tag{6.8}$$

(ii) Let $n \geqslant 1$ be a given integer. Find the best possible constant μ such that

$$\frac{\frac{T_{2n+1}}{2n+1}}{\left(1-\left(\frac{2x}{\pi}\right)^2\right)^{2n+1}} < \left(\frac{\tan x}{x}\right)^{(2n)} < \frac{\frac{T_{2n+1}}{2n+1}}{\left(1-\left(\frac{2x}{\pi}\right)^2\right)^{\mu}}, \qquad 0 < x < \frac{\pi}{2}. \tag{6.9}$$

Here T_{2n+1} are the tangent numbers.

The tangent numbers T_{2k-1} are defined by the series expansion of $\tan x$,

$$\tan x = \sum_{k=1}^{\infty} T_{2k-1}\frac{x^{2k-1}}{(2k-1)!}, \qquad |x| < \frac{\pi}{2}. \tag{6.10}$$

The tangent numbers T_{2k-1} can be calculated by

$$T_{2k-1} = \frac{2^{2k-1}(2^{2k}-1)}{k}|B_{2k}|, \qquad k \in \mathbb{N},$$

where B_n denote the Bernoulli numbers defined by the following generating function:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi.$$

The first few tangent numbers are

$$T_1 = 1, \quad T_3 = 2, \quad T_5 = 16, \quad T_7 = 272, \quad T_9 = 7936.$$

6.2. Conjecture

We here present (without proof) another sharp bounds for $\left(\tan x/x\right)^{(n)}$. The denominators of the upper and lower bounds are the same.

Computer experiments suggest that, for $0 < x < \pi/2$,

$$\frac{\frac{2}{3}\pi^6}{\left(\pi^2-(2x)^2\right)^3} < \left(\frac{\tan x}{x}\right)'' < \frac{256\pi^2}{\left(\pi^2-(2x)^2\right)^3}, \tag{6.11}$$

$$\frac{\frac{16}{5}\pi^8 x}{\left(\pi^2-(2x)^2\right)^4} < \left(\frac{\tan x}{x}\right)''' < \frac{6144\pi^2 x}{\left(\pi^2-(2x)^2\right)^4} \tag{6.12}$$

and

$$\frac{\frac{16}{5}\pi^{10}}{\left(\pi^2-(2x)^2\right)^5} < \left(\frac{\tan x}{x}\right)^{(4)} < \frac{49152\pi^4}{\left(\pi^2-(2x)^2\right)^5}, \tag{6.13}$$

where the constants

$$\frac{2}{3}\pi^6, \quad 256\pi^2, \quad \frac{16}{5}\pi^8, \quad 6144\pi^2, \quad \frac{16}{5}\pi^{10}, \quad 49152\pi^4$$

are the best possible.

In view of (1.11), (1.21) and (6.11)-(6.13), we define the function $F_n(x)$ and $G_n(x)$ by

$$F_n(x) = \left(\frac{\tan x}{x}\right)^{(2n-1)} \cdot \frac{\left(\pi^2 - (2x)^2\right)^{2n}}{x} \tag{6.14}$$

and

$$G_n(x) = \left(\frac{\tan x}{x}\right)^{(2n)} \cdot \left(\pi^2 - (2x)^2\right)^{2n+1} \tag{6.15}$$

for $0 < x < \pi/2$.

Theorem 6.1 gives the limits of $F_n(x)$ and $G_n(x)$ at $x = 0$ and $x = \pi/2$.

THEOREM 6.1. *Let $F_n(x)$ and $G_n(x)$ be defined by* (6.14) *and* (6.15). *We have*

$$\lim_{x\to 0^+} F_n(x) = \frac{T_{2n+1}}{2n+1}\pi^{4n}, \qquad \lim_{x\to \pi/2^-} F_n(x) = \frac{32}{n}(4\pi)^{2n-2}(2n)!,$$

$$\lim_{x\to 0^+} G_n(x) = \frac{T_{2n+1}}{2n+1}\pi^{4n+2}, \qquad \lim_{x\to \pi/2^-} G_n(x) = 8(4\pi)^{2n}(2n)!.$$

Proof. Write (6.10) as

$$\frac{\tan x}{x} = \sum_{k=1}^{\infty} \frac{T_{2k-1}}{(2k-1)!}x^{2k-2}, \qquad |x| < \frac{\pi}{2}.$$

We find that

$$\left(\frac{\tan x}{x}\right)^{(2n-1)} = \frac{T_{2n+1}}{2n+1}x + O(x^3) \quad \text{and} \quad \left(\frac{\tan x}{x}\right)^{(2n)} = \frac{T_{2n+1}}{2n+1} + O(x^2).$$

We then obtain

$$\lim_{x\to 0^+} F_n(x) = \pi^{4n} \lim_{x\to 0^+} \left(\frac{\tan x}{x}\right)^{(2n-1)} \cdot \frac{1}{x} = \pi^{4n} \lim_{x\to 0^+} \left\{\frac{T_{2n+1}}{2n+1} + O(x^2)\right\} = \frac{T_{2n+1}}{2n+1}\pi^{4n}$$

and

$$\lim_{x\to 0^+} G_n(x) = \pi^{4n+2} \lim_{x\to 0^+} \left(\frac{\tan x}{x}\right)^{(2n)} = \pi^{4n+2} \lim_{x\to 0^+} \left\{\frac{T_{2n+1}}{2n+1} + O(x^2)\right\} = \frac{T_{2n+1}}{2n+1}\pi^{4n+2}.$$

Using (2.10), we have

$$F_n(x) = \frac{\left(\pi^2 - (2x)^2\right)^{2n}}{x^{2n+1}} \int_0^x t^{2n-1}\left(\tan t\right)^{(2n)} \mathrm{d}t$$

and

$$G_n(x) = \frac{\left(\pi^2-(2x)^2\right)^{2n+1}}{x^{2n+1}} \int_0^x t^{2n}\left(\tan t\right)^{(2n+1)} \mathrm{d}t.$$

Further, we have

$$\begin{aligned}
\lim_{x\to\pi/2^-} F_n(x) &= \left(\frac{2}{\pi}\right)^{2n+1} \lim_{x\to\pi/2^-} \frac{\int_0^x t^{2n-1}\left(\tan t\right)^{(2n)}\mathrm{d}t}{\left(\pi^2-(2x)^2\right)^{-2n}} \\
&= \left(\frac{2}{\pi}\right)^{2n+1} \lim_{x\to\pi/2^-} \frac{x^{2n-1}\left(\tan x\right)^{(2n)}}{16nx\left(\pi^2-(2x)^2\right)^{-2n-1}} \qquad \text{(by L'Hospital's rule)} \\
&= \frac{1}{16n}\left(\frac{2}{\pi}\right)^3 \lim_{x\to\pi/2^-} \left(\tan x\right)^{(2n)}\left(\pi^2-(2x)^2\right)^{2n+1} \\
&= \frac{1}{16n}\left(\frac{2}{\pi}\right)^3 \lim_{t\to0^+} \left(\cot t\right)^{(2n)} 4^{2n+1} t^{2n+1}\left(\pi-t\right)^{2n+1} \qquad \left(\text{where } t=\frac{\pi}{2}-x\right) \\
&= \frac{(4\pi)^{2n+1}}{16n}\left(\frac{2}{\pi}\right)^3 \lim_{t\to0^+} \left(\cot t\right)^{(2n)} t^{2n+1}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to\pi/2^-} G_n(x) &= \left(\frac{2}{\pi}\right)^{2n+1} \lim_{x\to\pi/2^-} \frac{\int_0^x t^{2n}\left(\tan t\right)^{(2n+1)}\mathrm{d}t}{\left(\pi^2-(2x)^2\right)^{-2n-1}} \\
&= \left(\frac{2}{\pi}\right)^{2n+1} \lim_{x\to\pi/2^-} \frac{x^{2n}\left(\tan x\right)^{(2n+1)}}{8(2n+1)x\left(\pi^2-(2x)^2\right)^{-2n-2}} \qquad \text{(by L'Hospital's rule)} \\
&= \frac{1}{8(2n+1)}\left(\frac{2}{\pi}\right)^2 \lim_{x\to\pi/2^-} \left(\tan x\right)^{(2n+1)}\left(\pi^2-(2x)^2\right)^{2n+2} \\
&= \frac{1}{8(2n+1)}\left(\frac{2}{\pi}\right)^2 \lim_{t\to0^+} \left(-\cot t\right)^{(2n+1)} 4^{2n+2} t^{2n+2}\left(\pi-t\right)^{2n+2} \qquad \left(\text{where } t=\frac{\pi}{2}-x\right) \\
&= \frac{8(4\pi)^{2n}}{2n+1} \lim_{t\to0^+} \left(-\cot t\right)^{(2n+1)} t^{2n+2}.
\end{aligned}$$

From the power series expansion

$$\cot t = \frac{1}{t} - \sum_{j=1}^{\infty} \frac{2^{2j}|B_{2j}|}{(2j)!} t^{2j-1}, \qquad |t|<\pi, \tag{6.16}$$

we find that

$$(\cot t)^{(2n)} = \frac{(2n)!}{t^{2n+1}} + O(t) \quad \text{and} \quad (\cot t)^{(2n+1)} = -\frac{(2n+1)!}{t^{2n+2}} + O(1).$$

We then obtain

$$\lim_{x\to\pi/2^-} F_n(x) = \frac{(4\pi)^{2n+1}}{16n}\left(\frac{2}{\pi}\right)^3 \lim_{t\to 0^+}\left\{\frac{(2n)!}{t^{2n+1}} + O(t)\right\} t^{2n+1} = \frac{32}{n}(4\pi)^{2n-2}(2n)!$$

and

$$\lim_{x\to\pi/2^-} G_n(x) = \frac{8(4\pi)^{2n}}{2n+1}\lim_{t\to 0^+}\left\{\frac{(2n+1)!}{t^{2n+2}} + O(1)\right\} t^{2n+2} = 8(4\pi)^{2n}(2n)!.$$

The proof is complete.

It was proved in [7, 19] that $G_0(x)$ and $F_1(x)$ are both strictly decreasing for $0 < x < \pi/2$. Computer experiments indicate that the functions $F_n(x)$ (for $n \geqslant 2$) and $G_n(x)$ (for $n \geqslant 1$) are both strictly increasing for $0 < x < \pi/2$. We then proposed the following conjecture.

CONJECTURE 6.1. (i) For $0 < x < \pi/2$ and $n \geqslant 2$, we have

$$\frac{\frac{T_{2n+1}}{2n+1}x}{\left(1-\left(\frac{2x}{\pi}\right)^2\right)^{2n}} < \left(\frac{\tan x}{x}\right)^{(2n-1)} < \frac{\frac{(2n)!}{8n}\left(\frac{4}{\pi}\right)^{2n+2}x}{\left(1-\left(\frac{2x}{\pi}\right)^2\right)^{2n}}. \tag{6.17}$$

(ii) For $0 < x < \pi/2$ and $n \geqslant 1$, we have

$$\frac{\frac{T_{2n+1}}{2n+1}}{\left(1-\left(\frac{2x}{\pi}\right)^2\right)^{2n+1}} < \left(\frac{\tan x}{x}\right)^{(2n)} < \frac{\frac{(2n)!}{2}\left(\frac{4}{\pi}\right)^{2n+2}}{\left(1-\left(\frac{2x}{\pi}\right)^2\right)^{2n+1}}. \tag{6.18}$$

Here T_{2n+1} are the tangent numbers.

REMARK 6.1. When $n = 1$, the reversed inequality of (6.17) holds, that is to say, the Papenfuss inequality (1.21) holds. When $n = 0$, the reversed inequality of (6.18) holds, that is to say, the Becker-Stark inequality (1.11) holds.

REFERENCES

[1] M. ABRAMOWITZ AND I.A. STEGUN (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Dover, New York, 1965.

[2] G. D. ANDERSON, S.-L. QIU, M. K. VAMANAMURTHY AND M. VUORINEN, *Generalized elliptic integral and modular equations*, Pacific J. Math. **192** (2000), 1–37.

[3] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, New York, 1997.

[4] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Monotonicity of Some Functions in Calculus*; Available at http://www.math.auckland.ac.nz/Research/Reports/Series/538.pdf.

[5] G. BACH, *Trigonometric inequality*, Amer. Math. Monthly **87** (1) (1980), 62.

[6] B. BANJAC, M. MAKRAGIĆ AND B. MALEŠEVIĆ, *Some notes on a method for proving inequalities by computer*, Results. Math. **69** (2016), 161–176.

[7] M. BECKER AND E. L. STARK, *On a hierarchy of quolynomial inequalities for* $\tan x$, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **602-633** (1978), 133–138.

[8] C.-P. CHEN, *Complete monotonicity and logarithmically complete monotonicity properties for the gamma and psi functions*, J. Math. Anal. Appl. **336** (2007), 812–822.

[9] C.-P. CHEN AND W.-S. CHEUNG, *Sharp Cusa and Becker–Stark inequalities*, J. Inequal. Appl. (2011) 136, http://www.journalofinequalitiesandapplications.com/content/2011/1/136.

[10] C.-P. CHEN AND L. DEBNATH, *Sharpness and Generalization of Jordan's Inequality and its applications*, Appl. Math. Lett. **25** (2012), 594–599.

[11] C.-P. CHEN AND N. ELEZOVIĆ, *Sharp Redheffer-type and Becker-Stark-type inequalities with an application*, Math. Inequal. Appl. **21** (4) (2018), 1059–1078.

[12] C.-P. CHEN AND R.B. PARIS, *Series representations of the remainders in the expansions for certain trigonometric functions and some related inequalities, I*, Math. Inequal. Appl. **20** (2017), 1003–1016.

[13] C.-P. CHEN AND R.B. PARIS, *Series representations of the remainders in the expansions for certain trigonometric functions and some related inequalities, II*, RGMIA Res. Rep. Coll. **20** (2017), Art. 152, 17 pp. http://rgmia.org/v20.php.

[14] L. COMTET, *Advanced Combinatorics*, D. Reidel Publishing Co., Dordrecht, 1974.

[15] D. CVIJOVIĆ, *New identities for the partial Bell polynomials*, Appl. Math. Lett. **24** (2011), 1544–1547.

[16] L. DEBNATH, C. MORTICI AND L. ZHU, *Refinements of Jordan–Stečkin and Becker–Stark inequalities*, Results Math. **67** (2015), 207–215.

[17] L. DEBNATH AND C. J. ZHAO, *New strengthened Jordan's inequality and its applications*, Appl. Math. Lett. **16** (2003), 557–560.

[18] K. DENG, *On extensions of the Jordan's inequality*, J. Xiangtan Mining Inst. **10** (1995), 60–63 (in Chinese).

[19] H.-F. GE, *New Sharp Bounds for the Bernoulli Numbers and Refinement of Becker-Stark Inequalities*, J. Appl. Math. (2012) Article ID 137507, 7 pages. https://www.hindawi.com/journals/jam/2012/137507/

[20] W. D. JIANG AND H. YUN, *Sharpening of Jordan's inequality and its applications*, J. Inequal. Pure Appl. Math. **7** (3) (2006), Article 102.

[21] J. L. LI, *An identity related to Jordan's inequality*, Internat. J. Math. Math. Sci, (2006), Article ID 76782.

[22] B.J. MALEŠEVIĆ, *One method for proving inequalities by computer*, J. Inequal. Appl. (2007) Article ID 78691.

[23] A. MCD. MERCER, U. ABEL AND D. CACCIA, *A sharpening of Jordan's inequality*, Amer. Math. Monthly **93** (1986), 568–569.

[24] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer, New York, 1970.

[25] Y. NISHIZAWA, *Sharpening of Jordan's type and Shafer-Fink's type inequalities with exponential approximations*, Appl. Math. Comput. **269** (2015), 146–154.

[26] Y. NISHIZAWA, *Sharp Becker-Stark's type inequalities with power exponential functions*, J. Inequal. Appl. (2015) 402, http://www.journalofinequalitiesandapplications.com/content/2015/1/402.

[27] D.-W. NIU, Z.-H. HUO, J. CAO AND F. QI, *A general refinement of Jordan's inequality and a refinement of L. Yang's inequality*, Integral Transforms Spec. Funct.**19** (2008), 157–164.

[28] A. Y. ÖZBAN, *A new refined form of Jordan's inequality and its applications*, Appl. Math. Lett. **19** (2006), 155–160.

[29] M. C. PAPENFUSS, *Problem E2739*, Amer. Math. Monthly **85** (9) (1978), 765.

[30] F. QI, *Extension and sharpenings of Jordon's and Kober's inequalities*, Journal of Mathematics for Technology **4** (1996), 98–101 (in Chinese).

[31] S. B. STEČKIN, *Some remarks on trigonometric polynomials*, Uspekhi Matematicheskikh Nauk, vol. 10, no. 1 (63), (1955) 159–166 (in Russian).

[32] Z. SUN AND L. ZHU, *Some refinements of inequalities for circular functions*, J. Appl. Math. (2011) Article ID 869261, http://www.hindawi.com/journals/jam/2011/869261/.

[33] Z.-J. SUN AND L. ZHU, *Simple proofs of the Cusa–Huygens–type and Becker–Stark–type inequalities*, J. Math. Inequal. **7** (2013), 563–567.

[34] S. H. WU, *On generalizations and refinements of Jordan type inequality*, Octog. Math. Mag. **12** (2004), 267–272.

[35] S. H. WU, *Sharpness and generalization of Jordan's inequality and its application*, Taiwanese J. Math. **12** (2008), 325–336.

[36] S. H. WU AND L. DEBNATH, *A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality*, Appl. Math. Lett. **19** (2006), 1378–1384.

[37] S. H. WU AND L. DEBNATH, *A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality, II*, Appl. Math. Lett. **20** (2007), 532–538.

[38] S. H. WU AND L. DEBNATH, *Jordan-type inequalities for differentiable functions and their applications*, Appl. Math. Lett. **21** (2008), 803–809.

[39] S. H. WU AND L. DEBNATH, *Generalizations of a parameterized Jordan-type inequality, Janous's inequality and Tsintsifas's inequality*, Appl. Math. Lett.**22** (2009), 130–135.

[40] S. H. WU AND H. M. SRIVASTAVA, *A further refinement of a Jordan type inequality and its application*, Appl. Math. Comput. **197** (2008), 914–923.

[41] L. YANG, *Distribution of Values and New Research*, Science Press, Beijing, 1982 (in Chinese).

[42] F. YUEFENG, *Jordan's inequality*, Math. Mag. **69** (1996), 126–127.

[43] X. H. ZHANG, G. D. WANG AND Y. M. CHU, *Extensions and sharpenings of Jordan's and Kober's inequalities*, J. Inequal. Pure Appl. Math. 7 (2006) no. 2, Article 63.

[44] C. J. ZHAO, *Generalization and strengthen of Yang Le inequality*, Mathematics Practice Theory **30** (2000), 493–497 (in Chinese).

[45] C. J. ZHAO AND L. DEBNATH, *On generalizations of L. Yangs inequality*, J. Inequal. Pure Appl. Math. **3** (4) (2002), Article 56.

[46] L. ZHU, *Sharpening of Jordan's inequalities and its applications*, Math. Inequal. Appl. **9** (2006), 103–106.

[47] L. ZHU, *Sharpening Jordan's inequality and Yang Le inequality*, Appl. Math. Lett. **19** (2006), 240–243.

[48] L. ZHU, *Sharpening Jordan's inequality and Yang Le inequality, II*, Appl. Math. Lett. **19** (2006), 990–994.

[49] L. ZHU, *A general refinement of Jordan-type inequality*, Comput. Math. Appl. **55** (2008), 2498–2505.

[50] L. ZHU, *General forms of Jordan and Yang Le inequalities*, Appl. Math. Lett. **22** (2009), 236–241.

[51] L. ZHU, *An extended Jordan's inequality in exponential type*, Appl. Math. Lett. **24** (2011), 1870–1873.

[52] L. ZHU, *Sharp Becker-Stark-type inequalities for Bessel functions*, J. Inequal. Appl. 2010, Article ID 838740, http://www.journalofinequalitiesandapplications.com/content/2010/1/838740.

[53] L. ZHU, *A refinement of the Becker-Stark inequalities*, Math. Notes **93** (3–4) (2013), 421–425.

[54] L. ZHU AND J.K. HUA, *Sharpening the Becker-Stark inequalities*, J. Inequal. Appl. 2010, Article ID 931275, http://www.journalofinequalitiesandapplications.com/content/2010/1/931275.

(Received May 3, 2019)

Bo Zhang
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000
Henan Province, China
e-mail: zhangbohpu@sohu.com

Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000
Henan Province, China
(Corresponding author) *e-mail:* chenchaoping@sohu.com