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AN EXTENSION BY MEANS OF ω–WEIGHTED CLASSES OF THE GENERALIZED RIEMANN–LIOUVILLE k–FRACTIONAL INTEGRAL INEQUALITIES

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Abstract. In this paper, we aim at establishing an analog of the recently published results [1] with the help of new $k-$ type fractional integral operator $R_\omega[f](t)$, which is introduced here by using the ω-weighted classes. Then we establish some new ω-weighted Pólya-Szegö type integral inequalities and ω-weighted fractional integral inequalities, which are the an analog of the recently published results [1].

1. Introduction

Present investigation is devoted to the construction of the analog of the recent results established by Agarwal *et al.* [1], which relates some new Pólya-Szegö type integral inequalities involving the generalized Riemann-Liouville k-fractional integral operator. And, these inequalities are used then to establish some fractional integral inequalities of Chebyshev type. The paper gives an extension of the results [1] by means of a ω-weighted classes and a new ω-operator that becomes in the generalized Riemann-Liouville k-fractional integral in a particular case.
In this paper, some new ω-weighted Pólya-Szegö type inequalities by making use of a new operator and then use them to establish some ω-weighted Chebyshev type integral inequalities.

The well known functional was introduced by Chebyshev [2] and during last four decades or so, several interesting and useful rediscovered for their many applications, in various inequalities have been considered by several authors [3, 4, 5, 6, 7, 8, 9, 10] and, for recent work, see Wang et al. [11] and P. Agarwal et al [1]; it is defined by

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.1}$$

where f and g are two integrable functions. If these functions are synchronous on $[a,b]$, i.e., for any $x,y \in [a,b]$

$$(f(x)-f(y))(g(x)-g(y)) \geqslant 0, \tag{1.2}$$

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then $T(f,g) \geqslant 0$.
The well known Grüss inequality [12] established

$$|T(f,g)| \leqslant \frac{(M-m)(N-m)}{4}, \tag{1.3}$$

where f and g are two integrable functions which are synchronous on $[a,b]$ and satisfy:

$$m \leqslant f(x) \leqslant M, n \leqslant g(y) \leqslant N, \quad x,y \in [a,b], \tag{1.4}$$

for some $m,M,n,N \in \mathbb{R}$.
Pólya and Szegö [13] proved the following inequality:

$$\frac{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}{\left(\int_a^b f(x)g(x)dx\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2 \tag{1.5}$$

Dragomir and Diamond [14] by using the Pólya and Szegö inequality, proved that

$$|T(f,g)| \leqslant \frac{(M-m)(N-n)}{4(b-a)^2\sqrt{MmNn}} \int_a^b f(x)dx \int_a^b g(x)dx, \tag{1.6}$$

where f and g are two positive integrable functions which are synchronous on $[a,b]$, and

$$0 < m \leqslant f(x) \leqslant M < \infty, 0 < n \leqslant g(y) \leqslant N < \infty, \quad x,y \in [a,b], \tag{1.7}$$

for some $m,M,n,N \in \mathbb{R}$.

Now, some neccesary definitions to introduce our new ω-weighted class Ω and the new operator R_ω.

DEFINITION 1.1. *Let $k > 0$, then the generalized k-gamma and k-beta functions defined by* [15]

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \tag{1.8}$$

where $(x)_{n,k}$ is the Pochhammer k-symbol defined by

$$(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k) \quad (n \geqslant 1).$$

DEFINITION 1.2. *The k-gamma function is defined by*

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, \quad \text{Re } x > 0.$$

and it has the following properties:

$$\Gamma(x) = \lim_{k\to 1}\Gamma_k(x), \quad \Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \quad \Gamma_k(x+k) = x\Gamma_k(x).$$

DEFINITION 1.3. *If $k>0$, let $f \in L^1(a,b)$, $a \geqslant 0$, then the Riemann-Liouville k-fractional integral $R^{\alpha}_{a,k}$ of order $\alpha>0$ for a real-valued continuous function $f(t)$ is defined by* ([1], [16], [17])

$$R^{\alpha}_{a,k}\{f(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau)d\tau, \quad t\in[a,b]. \tag{1.9}$$

For $k=1$, this operator becomes to the classical Riemann-Liouville fractional integral.

DEFINITION 1.4. *If $k>0$, $f\in L^1(a,b)$, $a \geqslant 0$ and $r \in \mathbb{R}\setminus\{-1\}$ then the generalized Riemann-Liouville k-fractional integral $R^{\alpha,r}_{a,k}$ of order $\alpha>0$ for a real-valued continuous function $f(t)$ is defined by* [18]

$$R^{\alpha,r}_{a,k}\{f(t)\} = \frac{(1+r)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t (t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1} f(\tau)d\tau, \quad t\in[a,b]. \tag{1.10}$$

This operator has the following properties:

$$R^{\alpha,r}_{a,k}\{R^{\beta,r}_{a,k}\{f(t)\} = R^{\alpha+\beta,r}_{a,k}\{f(t)\} = R^{\beta,r}_{a,k}\{R^{\alpha,r}_{a,k}\{f(t)\}\} \tag{1.11}$$

and

$$R^{\alpha,r}_{a,k}\{1\} = \frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(1+r)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}, \quad \alpha>0. \tag{1.12}$$

Everywhere below, a function $\omega(t,\tau)$ is said to be is of the class Ω, if $\omega_t(t,\tau) = \frac{\partial\omega(t,\tau)}{\partial t} \geqslant 0$ for any $t,\tau \in (c,d)\times(e,f)$ $(c,d,e,f\in\mathbb{R})$ and is continuous respect to varible τ in $[e,f]$.

For a functional parameter $\omega(t)\in\Omega$, everywhere in this paper, we use the following operator which we formally define on real-valued continuous function $f(t)$ given in $[a,b]$ $(a\geqslant 0)$ by:

$$R_\omega[f](t) = \frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\int_a^t \omega_t(t,\tau)f(\tau)d\tau, \quad t\in[a,b], \tag{1.13}$$

for any $r\in\mathbb{R}\setminus\{-1\}$, $k>0$ and $\alpha>0$. One can see that the parameter $\frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}$ is a bounded constant then it will be useful for the future consider the following operator without the constant, it means:

$$R'_\omega[f](t) = \int_a^t \omega_t(t,\tau)f(\tau)d\tau, \quad t\in[a,b]. \tag{1.14}$$

REMARK 1.1. *Note that in a particular case when $\omega(t,\tau) = (t^{r+1}-\tau^{r+1})^{\alpha/k}$ $(a\leqslant t\leqslant b, a\leqslant\tau\leqslant t)$, we get the generalized Riemann-Liouville k-fractional integral $R^{\alpha,r}_{a,k}$ of order $\alpha>0$ given in definition 1.4, i.e. $R_\omega[f](t) = R^{\alpha,r}_{a,k}\{f(t)\}$.*

REMARK 1.2. *If* $\omega(t,\tau)=\frac{t\tau^{\frac{x}{k}-1}(1-\tau)^{\frac{y}{k}-1}}{kf(\tau)}$ *for* $t,\tau\in[0,1]$, $\operatorname{Re}x>0$, $\operatorname{Re}y>0$, $k>0$ *and* f *is a continuous function on* $[0,1]$, *then the operator* (1.14)

$$R'_{\omega}[f](t)=\frac{1}{k}\int_0^t \tau^{\frac{x}{k}-1}(1-\tau)^{\frac{y}{k}-1}d\tau=\beta_k^{[0,t]}(x,y),$$

i.e., it becomes to the k*-beta function of* [15] *when* $t=1$. *Hence, if* $t\in[0,d)$ $(d<1)$ *we shall undertand the last integral as* $\beta_k^{[0,t]}(x,y)$ *because at one point of view is the same function defined in a small domain.*
On the other hand, if $\omega(t,\tau)=\frac{t\tau^{x-1}(1-\tau)^{y-1}}{k}$, *then*

$$R'_{\omega}[1]=\beta_k^{[0,t]}(x,y).$$

REMARK 1.3. *If* $\omega(t,\tau)=(t^{s+1}-\tau^{s+1})^{\frac{\alpha}{k}}\tau^s$ *for* $t\in[a,b]$, $t\geqslant\tau\geqslant a$, $k>0$, $\alpha>0$, $s\in\mathbb{R}\setminus\{-1\}$ *and* f *is a continuous function on* $[a,b]$, *then the operator* (1.14)

$$R_{\omega}[f](t)=\frac{(1+s)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t (t^{s+1}-\tau^{s+1})^{\frac{\alpha}{k}-1}\tau^s f(\tau)d\tau={}^s_kJ_a^{\alpha}f(t),$$

i.e., $(k;s)$*-Riemann-Liouville fractional integral of* f *of order* $\alpha>0$ *of* [18].

These remarks have demostrated that the operator R_ω and R'_ω defined in (1.13) and (1.14) respectively, it has many applications when we consider some particular cases.

2. Some ω-weighted Polya-Szego types inequalities

To continuation, we prove some ω-weighted Pólya-Szegö type integral inequalities for positive integrable functions involving the ω-weighted classes Ω and the operator R_ω.

LEMMA 2.1. *Let* f *and* g *be two positive integrable funstions on* $[a,\infty)$. *Assume that there exist four integrable functions* φ_1, φ_2, ψ_1 *and* ψ_2 *on* $[a,\infty)$ *such that:*

$$(H1)\quad 0<\varphi_1(\tau)\leqslant f(\tau)\leqslant\varphi_2(\tau),\,0<\psi_1(\tau)\leqslant g(\tau)\leqslant\psi_2(\tau)\,(\tau\in[a,t],t>a).$$

Then, for $t>a$, $k>0$, $a\geqslant 0$, $\alpha>0$, $r\in\mathbb{R}\setminus\{-1\}$ *and* $\omega\in\Omega$, *the following inequality holds:*

$$\frac{R_\omega[\psi_1\psi_2f^2](t)R_\omega[\varphi_1\varphi_2g^2](t)}{[R_\omega[(\varphi_1\psi_1+\varphi_2\psi_2)fg](t)]^2}\leqslant\frac{1}{4}. \tag{2.1}$$

Proof. From $(H1)$, for $\tau\in[a,t]$, $t>a$, we have

$$\frac{f(\tau)}{g(\tau)}\leqslant\frac{\varphi_2(\tau)}{\psi_1(\tau)}, \tag{2.2}$$

hence

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)}-\frac{f(\tau)}{g(\tau)}\right)\geqslant 0. \tag{2.3}$$

Similarly, we get

$$\frac{\varphi_1(\tau)}{\psi_2(\tau)} \leqslant \frac{f(\tau)}{g(\tau)}, \tag{2.4}$$

thus

$$\left(\frac{f(\tau)}{g(\tau)} - \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right) \geqslant 0. \tag{2.5}$$

Multiplying (2.3) and (2.5), it follows

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right) \geqslant 0,$$

i.e.

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} + \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right)\frac{f(\tau)}{g(\tau)} \geqslant \frac{f^2(\tau)}{g^2(\tau)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\tau)\psi_2(\tau)}.$$

The last inequality can be written as

$$(\varphi_1(\tau)\psi_1(\tau) + \varphi_2(\tau)\psi_2(\tau))f(\tau)g(\tau) \geqslant \psi_1(\tau)\psi_2(\tau)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\tau). \tag{2.6}$$

Consequently, , multiplying both sides of (2.6) by $\frac{(1+r)^{-\alpha/k}\omega_t(t,\tau)}{\Gamma_k(\alpha)t^r\alpha}$ $(\omega \in \Omega)$ and integrating with respect to τ from a to t, we obtain

$$R_\omega[(\varphi_1\psi_1 + \varphi_2\psi_2)fg](t) \geqslant R_\omega[\psi_1\psi_2 f^2](t) + R_\omega[\varphi_1\varphi_2 g^2](t)$$

Besides, by AM-GM inequality,i.e., $a+b \geqslant 2\sqrt{ab}$ $a,b \in \mathbb{R}^+$, we get

$$R_\omega[(\varphi_1\psi_1 + \varphi_2\psi_2)fg](t) \geqslant 2\sqrt{R_\omega[\psi_1\psi_2 f^2](t)R_\omega[\varphi_1\varphi_2 g^2](t)}$$

and it follows straightforward the statement (2.1).

Corollary 2.1. Let f and g be two positive integrable functions on $[0,\infty)$ satisfying

$$(H2) \quad 0 < m \leqslant f(\tau) \leqslant M, 0 < n \leqslant g(\tau) \leqslant N\,(\tau \in [a,t], t > a).$$

Then, for $t > a$, $k > 0$, $a \geqslant 0$, $\alpha > 0$, $r \in \mathbb{R}\setminus\{-1\}$ and $\omega \in \Omega$, we obtain

$$\frac{R_\omega[f^2](t)R_\omega[g^2](t)}{(R_\omega[fg](t))^2} \leqslant \left(\frac{\sqrt{mn}}{\sqrt{MN}} + \frac{\sqrt{MN}}{\sqrt{mn}}\right)^2. \tag{2.7}$$

LEMMA 2.2. *Let f and g be two positive integrable funstions on $[a,\infty)$. Assume that there exist four integrable functions φ_1, φ_2, ψ_1 and ψ_2 on $[a,\infty)$ satisfying $(H1)$ on $[a,\infty)$. Then, for $t > a$, $k > 0$, $a \geqslant 0$, $\alpha > 0$, $\beta > 0$, $r \in \mathbb{R}\setminus\{-1\}$ and $\omega_{1,2} \in \Omega$, the following inequality holds:*

$$\frac{R_{\omega_1}[\varphi_1\varphi_2](t)R_{\omega_2}[\psi_1\psi_2](t)R_{\omega_1}[f^2](t)R_{\omega_2}[g^2](t)}{[R_{\omega_1}[\varphi_1 f](t)R_{\omega_2}[\psi_1 g](t) + R_{\omega_1}[\varphi_2 f](t)R_{\omega_2}[\psi_2 g](t)]^2} \leqslant \frac{1}{4}. \tag{2.8}$$

Proof. By condition $(H1)$, it clear that

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\rho)} - \frac{f(\tau)}{g(\rho)}\right) \geqslant 0$$

and

$$\left(\frac{f(\tau)}{g(\rho)} - \frac{\varphi_1(\tau)}{\psi_2(\rho)}\right) \geqslant 0,$$

These inequalities imply that

$$\left(\frac{\varphi_1(\tau)}{\psi_2(\rho)} + \frac{\varphi_2(\tau)}{\psi_1(\rho)}\right)\frac{f(\tau)}{g(\rho)} \geqslant \frac{f^2(\tau)}{g^2(\rho)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\rho)\psi_2(\rho)}. \tag{2.9}$$

Besides, multiplying both sides of (2.9) by $\psi_1(\rho)\psi_2(\rho)g^2(\rho)$, we get

$$\begin{aligned}\varphi_1(\tau)f(\tau)\psi_1(\rho)g(\rho) + \varphi_2(\tau)f(\tau)\psi_2(\rho)g(\rho)&\\ \geqslant \psi_1(\rho)\psi_2(\rho)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\rho)&\end{aligned} \tag{2.10}$$

hence, multiplying both sides of (2.10) by

$$\frac{(1+r)^{-\alpha/k}\omega_{1,t}(t,\tau)}{\Gamma_k(\alpha)t^r\alpha} \cdot \frac{(1+r)^{-\beta/k}\omega_{2,t}(t,\rho)}{\Gamma_k(\beta)t^r\beta}$$

and double integrating with respect to τ and ρ from a to t, we have

$$\begin{aligned}R_{\omega_1}[\varphi_1 f](t)R_{\omega_2}[\psi_1 g](t) + R_{\omega_1}[\varphi_2 f](t)R_{\omega_2}[\psi_2 g](t)&\\ \geqslant R_{\omega_1}[f^2](t)R_{\omega_2}[\psi_1\psi_2](t) + R_{\omega_1}[\varphi_1\varphi_2](t)R_{\omega_2}[g^2](t)&\end{aligned}$$

Finally, applying the AM-GM inequality to the last inequality, we come to (2.8).

LEMMA 2.3. *Let f and g be two positive integrable functions on $[a,\infty)$. Assume that there exist four integrable functions φ_1, φ_2, ψ_1 and ψ_2 on $[a,\infty)$ satisfying $(H1)$ on $[a,\infty)$. Then, for $t > a$, $k > 0$, $a \geqslant 0$, $\alpha > 0$, $\beta > 0$, $r \in \mathbb{R}\setminus\{-1\}$ and $\omega_{1,2} \in \Omega$, the following inequality holds:*

$$R_{\omega_1}[f^2](t)R_{\omega_2}[g^2](t) \leqslant R_{\omega_1}[(\varphi_2 fg)/\psi_1](t)R_{\omega_2}[(\psi_2 fg)/\varphi_1](t). \tag{2.11}$$

Proof. By (2.2), we have for any $\omega_{1,2} \in \Omega$

$$\begin{aligned}\frac{(r+1)^{-\alpha/k}}{\Gamma_k(\alpha)t^r\alpha}\int_a^t \frac{\partial}{\partial t}\omega_1(t,\tau)f^2(t)d\tau&\\ \leqslant \frac{(r+1)^{-\alpha/k}}{\Gamma_k(\alpha)t^r\alpha}\int_a^t \frac{\partial}{\partial t}\omega_1(t,\tau)\frac{\varphi_2(\tau)}{\psi_1(\tau)}f(\tau)g(\tau)d\tau,&\end{aligned}$$

which implies

$$R_{\omega_1}[f^2](t) \leqslant R_{\omega_1}[(\varphi_2 fg)/\psi_1](t) \tag{2.12}$$

and analogously, by (2.4), we get

$$R_{\omega_2}[g^2](t) \leqslant R_{\omega_2}[(\psi_2 fg)/\varphi_1](t) \tag{2.13}$$

hence, by multiplying (2.12) and (2.13) follow (2.11).

Corollary 2.2. Let f and g be two positive integrable functions on $[0,\infty)$ satisfying $(H2)$. Then, for $t>a$, $k>0$, $a\geqslant 0$, $\alpha>0$, $\beta>0$, $r\in\mathbb{R}\setminus\{-1\}$ and $\omega_{1,2}\in\Omega$, we obtain

$$\frac{R_{\omega_1}[f^2](t)R_{\omega_2}[g^2](t)}{R_{\omega_1}[fg](t)R_{\omega_2}[fg](t)}\leqslant\frac{MN}{mn}. \tag{2.14}$$

3. ω-weighted Chebyshev type integral inequalities

In this section, some ω-weighted Chebyshev type integral inequalities are established involving the operator R_ω and using the ω-weighted Pólya-Szegö fractional integral inequality of Lemma 2.1.

THEOREM 3.1. *Let f and g be two positive integrable funstions on $[a,\infty)$. Assume that there exist four integrable functions φ_1, φ_2, ψ_1 and ψ_2 on $[a,\infty)$ satisfying $(H1)$. Then, for $t>a$, $k>0$, $a\geqslant 0$, $\alpha>0$, $\beta>0$, $r\in\mathbb{R}\setminus\{-1\}$ and $\omega_{1,2}\in\Omega$, the following inequality holds:*

$$\begin{aligned}|R_{\omega_1}[fg](t)R_{\omega_2}[1](t)+R_{\omega_2}[fg](t)R_{\omega_1}[1](t)\\ -R_{\omega_1}[f](t)R_{\omega_2}[g](t)-R_{\omega_1}[g](t)R_{\omega_2}[f](t)|\\ \leqslant 2[G_{\omega_1,\omega_2}(f,\varphi_1,\varphi_2)(t)G_{\omega_1,\omega_2}(g,\varphi_1,\varphi_2)(t)]^{1/2}\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}G_{\omega_1,\omega_2}(u,v,w)(t)=&\frac{1}{8}\frac{[R_{\omega_1}[(v+w)u](t)]^2}{R_{\omega_1}[vw](t)}R_{\omega_2}[1]+\frac{1}{8}\frac{[R_{\omega_2}[(v+w)u](t)]^2}{R_{\omega_2}[vw](t)}R_{\omega_1}[1](t)\\&-R_{\omega_1}[u](t)R_{\omega_2}[u](t)\end{aligned}$$

Proof. For $\tau,\rho\in(a,t)$ $(t>a)$, we defined $A(\tau,\rho)=(f(\tau)-f(\rho))(g(\tau)-g(\rho))$, what is the same

$$A(\tau,\rho)=f(\tau)g(\tau)+f(\rho)g(\rho)-f(\tau)g(\rho)-f(\rho)g(\tau). \tag{3.2}$$

Further, multiplying both sides of (3.2) by

$$\frac{(1+r)^{-\frac{\alpha}{k}}\omega_{1,t}(t,\tau)}{\Gamma_k(\alpha)t^r\alpha}\frac{(1+r)^{-\frac{\beta}{k}}\omega_{2,t}(t,\rho)}{\Gamma_k(\beta)t^r\beta}$$

where $\omega_{1,2} \in \Omega$ and double integrating with respect to τ and ρ from a to t, we obtain

$$\begin{aligned}
&\frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\frac{(1+r)^{-\frac{\beta}{k}}}{\Gamma_k(\beta)t^r\beta}\int_a^t\int_a^t \omega_{1,t}(t,\tau)\omega_{2,t}(t,\rho)A(\tau,\rho)d\tau d\rho \\
&= \frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\int_a^t \omega_{1,t}(t,\tau)f(\tau)g(\tau)d\tau \frac{(1+r)^{-\frac{\beta}{k}}}{\Gamma_k(\beta)t^r\beta}\int_a^t \omega_{2,t}(t,\rho)d\rho \\
&\quad + \frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\int_a^t \omega_{2,t}(t,\rho)f(\rho)g(\rho)d\rho \frac{(1+r)^{-\frac{\beta}{k}}}{\Gamma_k(\beta)t^r\beta}\int_a^t \omega_{1,t}(t,\tau)d\tau \\
&\quad - \frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\int_a^t \omega_{1,t}(t,\tau)f(\tau)d\tau \frac{(1+r)^{-\frac{\beta}{k}}}{\Gamma_k(\beta)t^r\beta}\int_a^t \omega_{2,t}(t,\rho)g(\rho)d\rho \\
&\quad - \frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\int_a^t \omega_{1,t}(t,\tau)g(\tau)d\tau - \frac{(1+r)^{-\frac{\beta}{k}}}{\Gamma_k(\beta)t^r\beta}\int_a^t \omega_{2,t}(t,\rho)f(\rho)d\rho \\
&\quad = R_{\omega_1}[fg](t)R_{\omega_2}[1](t) + R_{\omega_2}[fg](t)R_{\omega_1}[1](t) \\
&\qquad - R_{\omega_1}[f](t)R_{\omega_2}[g](t) - R_{\omega_1}[g](t)R_{\omega_2}[f](t) \qquad (3.3)
\end{aligned}$$

By using the Cauchy-Schwartz inequality for double integrals, we have

$$\begin{aligned}
&\left|\frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\frac{(1+r)^{-\frac{\beta}{k}}}{\Gamma_k(\beta)t^r\beta}\int_a^t\int_a^t \omega_{1,t}(t,\tau)\omega_{2,t}(t,\rho)A(\tau,\rho)d\tau d\rho\right| \\
&\quad \leqslant \left(\frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\frac{(1+r)^{-\frac{\beta}{k}}}{\Gamma_k(\beta)t^r\beta}\left[\int_a^t\int_a^t \omega_{1,t}(t,\tau)\omega_{2,t}(t,\rho)[f(\tau)]^2 d\tau d\rho\right.\right. \\
&\quad + \int_a^t\int_a^t \omega_{1,t}(t,\tau)\omega_{2,t}(t,\rho)[f(\rho)]^2 d\tau d\rho \\
&\quad \left.\left. -2\int_a^t\int_a^t \omega_{1,t}(t,\tau)\omega_{2,t}(t,\rho)f(\tau)f(\rho)d\tau d\rho\right]\right)^{1/2} \\
&\quad \times \left(\frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\frac{(1+r)^{-\frac{\beta}{k}}}{\Gamma_k(\beta)t^r\beta}\left[\int_a^t\int_a^t \omega_{1,t}(t,\tau)\omega_{2,t}(t,\rho)[g(\tau)]^2 d\tau d\rho\right.\right. \\
&\quad + \int_a^t\int_a^t \omega_{1,t}(t,\tau)\omega_{2,t}(t,\rho)[g(\rho)]^2 d\tau d\rho \\
&\quad \left.\left. -2\int_a^t\int_a^t \omega_{1,t}(t,\tau)\omega_{2,t}(t,\rho)g(\tau)g(\rho)d\tau d\rho\right]\right)^{1/2} \qquad (3.4)
\end{aligned}$$

hence, it follows

$$\begin{aligned}
&\left|\frac{(1+r)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)t^r\alpha}\frac{(1+r)^{-\frac{\beta}{k}}}{\Gamma_k(\beta)t^r\beta}\int_a^t\int_a^t \omega_{1,t}(t,\tau)\omega_{2,t}(t,\rho)A(\tau,\rho)d\tau d\rho\right| \\
&\leqslant 2\{1/2R_{\omega_1}[f^2](t)R_{\omega_2}[1](t) + 1/2R_{\omega_2}[f^2](t)R_{\omega_1}[1](t) - R_{\omega_1}[f](t)R_{\omega_2}[f](t)\}^{1/2} \\
&\quad \times \{1/2R_{\omega_1}[g^2](t)R_{\omega_2}[1](t) + 1/2R_{\omega_2}[g^2](t)R_{\omega_1}[1](t) - R_{\omega_1}[g](t)R_{\omega_2}[g](t)\}^{1/2} \qquad (3.5)
\end{aligned}$$

By applying Lemma 2.1, for $\psi_1(t)=\psi_2(t)=g(t)=1$, we get for any $\omega\in\Omega$

$$R_\omega[f^2](t)\leqslant\frac{1}{4}\frac{\{R_\omega[(\varphi_1+\varphi_2)f](t)\}^2}{R_\omega[\varphi_1\varphi_2](t)}$$

this implies

$$\begin{aligned}1/2R_{\omega_1}[f^2](t)R_{\omega_2}[1](t)+1/2R_{\omega_2}[f^2](t)R_{\omega_1}[1](t)-R_{\omega_1}[f](t)R_{\omega_2}[f](t)\\ \leqslant\frac{1}{8}\frac{\{R_{\omega_1}[(\varphi_1+\varphi_2)f](t)\}^2}{R_{\omega_1}[\varphi_1\varphi_2](t)}R_{\omega_2}[1](t)+\frac{1}{8}\frac{\{R_{\omega_2}[(\varphi_1+\varphi_2)f](t)]\}^2}{R_{\omega_2}[\varphi_1\varphi_2](t)}R_{\omega_1}[1](t)\\ -R_{\omega_1}[f](t)R_{\omega_2}[f](t)=G_{\omega_1,\omega_2}(f,\varphi_1,\varphi_2)(t)\end{aligned}\tag{3.6}$$

Analogously, it is clear

$$\begin{aligned}1/2R_{\omega_1}[g^2](t)R_{\omega_2}[1](t)+1/2R_{\omega_2}[g^2](t)R_{\omega_1}[1](t)-R_{\omega_1}[g](t)R_{\omega_2}[g](t)\\ \leqslant\frac{1}{8}\frac{\{R_{\omega_1}[(\varphi_1+\varphi_2)g](t)\}^2}{R_{\omega_1}[\varphi_1\varphi_2](t)}R_{\omega_2}[1](t)+\frac{1}{8}\frac{\{R_{\omega_2}[(\varphi_1+\varphi_2)g](t)]\}^2}{R_{\omega_2}[\varphi_1\varphi_2](t)}R_{\omega_1}[1](t)\\ -R_{\omega_1}[g](t)R_{\omega_2}[g](t)=G_{\omega_1,\omega_2}(g,\varphi_1,\varphi_2)(t)\end{aligned}\tag{3.7}$$

Thus, by (3.3), (3.5), (3.6) and (3.7), we come to inequality (3.1).

Corollary 3.1. Let f and g be two positive integrable functions on $[a,\infty)$. Assume that there exist four integrable functions φ_1, φ_2, ψ_1 and ψ_2 on $[a,\infty)$ satisfying $(H1)$. Then, for $t>a$, $k>0$, $a\geqslant 0$, $\alpha>0$, $r\in\mathbb{R}\setminus\{-1\}$ and $\omega\in\Omega$, the following inequality holds:

$$\begin{aligned}|R_\omega[fg](t)R_\omega[1](t)-R_\omega[g](t)R_\omega[f](t)|\\ \leqslant[G_{\omega,\omega}(f,\varphi_1,\varphi_2)(t)G_{\omega,\omega}(g,\varphi_1,\varphi_2)(t)]^{1/2}\end{aligned}\tag{3.8}$$

where

$$G_{\omega,\omega}(u,v,w)(t)=\frac{1}{4}\frac{[R_\omega[(v+w)u](t)]^2}{R_\omega[vw](t)}R_\omega[1]-(R_\omega[u](t))^2$$

Besides, if f and g satisfy $(H2)$, then

$$G_{\omega,\omega}(f,m,M)(t)=\frac{(M-m)^2}{4Mm}(R_\omega[f](t))^2,\tag{3.9}$$

and

$$G_{\omega,\omega}(g,n,N)(t)=\frac{(N-n)^2}{4Nn}(R_\omega[g](t))^2.\tag{3.10}$$

REMARK 3.1. *The above results can be obtained analogously for the operator R'_ω defined in formula (1.14).*

REMARK 3.2. *One can see easily that when $\omega=\omega_1=\omega_2$ is defined as remark 1.1, then the results in [1] are the same up to some constants.*

4. Examples

The following applications show the multipurpose of the operator R_ω, R'_ω and the inequalities proved.

PROPOSITION 4.1. *If m, n, p, and q are positive real numbers satisfying the condition $(p-m)(q-n) \leqslant 0$ and $k>0$, then*

$$\frac{\beta_k^{[a,t]}(p+k,n+k)\beta_k^{[a,t]}(m+k,q+k)}{\left[\beta_k^{[a,t]}\left(\frac{p+m}{2}+k,\frac{q+n}{2}+k\right)\right]^2} \leqslant \frac{1}{4}\left(\sqrt{\left(\frac{a}{b}\right)^{\frac{p-m}{2k}}\left(\frac{1-b}{1-a}\right)^{\frac{q-n}{2k}}}+\sqrt{\left(\frac{b}{a}\right)^{\frac{p-m}{2k}}\left(\frac{1-a}{1-b}\right)^{\frac{q-n}{2k}}}\right)^2. \tag{4.1}$$

Proof. Setting, $f(\tau)=\tau^{\frac{p-m}{2k}}$ and $g(\tau)=\tau^{\frac{q-n}{2k}}$, one can convinced that they are synchronous on $[0,1]$. Furthermore,

$$\omega(t,\tau)=\frac{t\tau^{m/k}(1-\tau)^{n/k}}{k}, \quad 0<a\leqslant t\leqslant b<1, \quad \tau\leqslant t,$$

we get

$$R'_\omega[f^2](t)=\frac{1}{k}\int_a^t \tau^{p/k}(1-\tau)^{n/k}d\tau=\beta_k^{[a,t]}(p+k,n+k), \quad t\in[a,b],$$

where the notation $\beta_k^{[a,t]}$ is the same defined in remark 1.2. And,

$$R'_\omega[g^2](t)=\frac{1}{k}\int_a^t \tau^{m/k}(1-\tau)^{q/k}d\tau=\beta_k^{[a,t]}(m+k,q+k), \quad t\in[a,b].$$

Now, note that for any $\tau\in[a,b]$

$$\varphi_1=a^{\frac{p-m}{2k}}\leqslant\tau^{\frac{p-m}{2k}}\leqslant b^{\frac{p-m}{2k}}=\varphi_2, \quad \psi_1=(1-b)^{\frac{q-n}{2k}}\leqslant(1-\tau)^{\frac{q-n}{2k}}\leqslant(1-a)^{\frac{q-n}{2k}}=\psi_2,$$

Thus,

$$\begin{aligned}[R'_\omega[(\varphi_1\psi_1+\varphi_2\psi_2)fg](t)]^2 &= [C_{p,m}^{q,n}(k,a,b)]^2[R'_\omega[fg](t)]^2\\ &= [C_{p,m}^{q,n}(k,a,b)]^2\left(\frac{1}{k}\int_a^t \tau^{\frac{p+m}{2k}}(1-\tau)^{\frac{q+n}{2k}}d\tau\right)^2\\ &= [C_{p,m}^{q,n}(k,a,b)]^2\left[\beta_k^{[a,t]}\left(\frac{p+m}{2}+k,\frac{q+n}{2}+k\right)\right]^2.\end{aligned}$$

where $C_{p,m}^{q,n}(k,a,b)=a^{\frac{p-m}{2k}}(1-b)^{\frac{q-n}{2k}}+b^{\frac{p-m}{2k}}(1-a)^{\frac{q-n}{2k}}$. Finally, by Lemma 2.1, remark 3.1 and some straightforward calculus, we come to the desired statement (4.1).

PROPOSITION 4.2. *If f and g be two positive integrable functions satisfying $(H2)$. Then, for $t > a$, $k > 0$, $a \geqslant 0$, $\alpha > 0$ and $s \in \mathbb{R} \setminus \{-1\}$, the following inequality holds:*

$$\left| \frac{(t^{s+1} - a^{s+1})^{\alpha/k}}{(1+s)^{\alpha/k}\Gamma_k(\alpha+k)} {}_k^sJ_a^\alpha fg(t) - {}_k^sJ_a^\alpha f(t) \right| \leqslant \frac{(M-m)(N-n)}{4\sqrt{MmNn}} {}_k^sJ_a^\alpha g(t) {}_k^sJ_a^\alpha f(t). \tag{4.2}$$

Proof. By Corollary 3.1, formula (3.9) and (3.10), we get that inequality (3.8) is true. Besides, if $\omega(t,\tau) = (t^{s+1} - \tau^{s+1})^{\frac{\alpha}{k}} \tau^s$ for $t \in [a,b]$, $t \geqslant \tau \geqslant a$, then by remark 1.3, $R_\omega[f](t) = {}_k^sJ_a^\alpha f(t)$. Hence, inequality (3.8) becomes to

$$|{}_k^sJ_a^\alpha fg(t) {}_k^sJ_a^\alpha 1(t) - {}_k^sJ_a^\alpha g(t) {}_k^sJ_a^\alpha f(t)| \leqslant \left[\frac{(M-m)^2}{4Mm} ({}_k^sJ_a^\alpha f(t))^2 \frac{(N-n)^2}{4Nn} ({}_k^sJ_a^\alpha f(t))^2 \right]^{1/2} \tag{4.3}$$

where it is easy to prove that

$${}_k^sJ_a^\alpha 1(t) = \frac{(t^{s+1} - a^{s+1})^{\alpha/k}}{(1+s)^{\alpha/k}\Gamma_k(\alpha+k)}$$

Hence, we arrive to inequality (4.2).

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