

THE DIRICHLET PROBLEM FOR A SUB-ELLIPTIC EQUATION WITH SINGULAR NONLINEARITY ON THE HEISENBERG GROUP

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Abstract. This paper studies the following singular sub-elliptic equation:

$$\begin{cases} -\Delta_H u = \frac{h(\xi)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{H}^n$ is a smooth bounded domain, $\gamma > 0$ and $h \geq 0$. We first use the Schauder's fixed point theorem and approximating method to prove the existence of solutions to the above equation. We then obtain the uniqueness result by proving a weak comparison principle and further deduce that the solution is cylindrically symmetric under some necessary structural conditions on Ω and h .

1. Introduction and main results

The Heisenberg group \mathbb{H}^n is the space \mathbb{R}^{2n+1} endowed with the group action \circ defined by

$$\xi_0 \circ \xi = (x_0 + x, y_0 + y, t_0 + t + 2(x \cdot y_0 - y \cdot x_0)),$$

where \cdot denotes the inner product in \mathbb{R}^n and

$$\xi_0 = (x_0, y_0, t_0) = (x_{01}, \dots, x_{0n}, y_{01}, \dots, y_{0n}, t_0), \quad \xi = (x, y, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t).$$

There is a natural group of dilations on \mathbb{H}^n given by $\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$ for any $\lambda > 0$. Hence, $\delta_\lambda(\xi_0 \circ \xi) = \delta_\lambda(\xi_0) \circ \delta_\lambda(\xi)$ and $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ, \delta_\lambda)$ is a homogeneous Lie group and the number $Q = 2n + 2$ is the homogeneous dimension of \mathbb{H}^n . For any $\xi \in \mathbb{H}^n$, let us define the norm

$$|\xi|_H = ((x^2 + y^2)^2 + t^2)^{1/4},$$

which is homogeneous of degree one with respect to the dilation δ_λ . Therefore the associated distance between two points ξ, η of \mathbb{H}^n is defined by $d(\xi, \eta) = |\eta^{-1} \circ \xi|_H$,

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where η^{-1} denotes the inverse of η with respect to the group action \circ . Similarly, the set $B_r(\xi) = \{\eta \in \mathbb{H}^n : d(\xi, \eta) < r\}$ denotes the Heisenberg ball of radius r centered at ξ . Meanwhile, the natural volume in \mathbb{H}^n is the Haar measure, which coincides with Lebesgue measure L^{2n+1} in \mathbb{R}^{2n+1} . Therefore $|B_r(\xi)| = \alpha_Q r^Q$, where $\alpha_Q = |B_1(0)|$. In addition, the Heisenberg group \mathbb{H}^n corresponding Lie Algebra of left-invariant vector fields is generated by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

for $j = 1, \dots, n$. The second order self-adjoint operator: $\Delta_H = \sum_{j=1}^n (X_j^2 + Y_j^2)$, that is,

$$\Delta_H = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} + 4(x_j^2 + y_j^2) \frac{\partial^2}{\partial t^2}$$

is usually called the Kohn Laplacian or Heisenberg Laplacian. This type of operator has been successfully applied to the Brownian motion, to the kinetic theory of gases and to the mathematical models in theoretical physics and diffusion processes, see [1, 2, 3]. Let

$$\nabla_H = (X_1, \dots, X_n, Y_1, \dots, Y_n),$$

and A denotes the following $(2n+1) \times (2n+1)$ matrix

$$\begin{pmatrix} I_n & 0 & 2y^T \\ 0 & I_n & -2x^T \\ 2y & -2x & 4(x^2 + y^2) \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix and $x^2 + y^2 = \sum_{j=1}^n (x_j^2 + y_j^2)$. Then

$$\Delta_H u = \operatorname{div}(A(\xi) \nabla u),$$

which implies that the Kohn Laplacian $\Delta_{\mathbb{H}}$ is very degenerate, since $\det(A(\xi)) = 0$ for any $\xi \in \mathbb{H}^n$. However, it is easy to check that X_j and Y_j satisfy

$$[X_i, Y_j] = -4T \delta_{ij}, \quad [X_i, X_j] = [Y_i, Y_j] = 0$$

for any $i, j \in \{1, \dots, n\}$. This means that the vector fields X_j, Y_j and their first order commutator span the whole Lie algebra. Therefore the operator Δ_H satisfies the Hörmander rank condition, see [4, 3], which implies that Δ_H is hypoelliptic and Bony's maximum principle is satisfied, see [5]. On the other hand, a basic role in the functional analysis on \mathbb{H}^n is played by the following Sobolev type inequality:

$$\|\phi\|_{Q^*}^2 \leq B_Q \|\nabla_H \phi\|_2^2, \quad \forall \phi \in C_0^\infty(\mathbb{H}^n), \quad (1)$$

where $Q^* = \frac{2Q}{Q-2}$ and B_Q is the best Sobolev constant, see Jerison and Lee [6], and here and below $\|\cdot\|_p$ denotes the usual L^p -norm. Let Ω be a smooth bounded domain

of \mathbb{H}^n , we define the associated Sobolev space as following

$$S^1(\Omega) = \{u \in L^2(\Omega) : \nabla_H u \in L^2(\Omega)\}.$$

The space $S^1(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{S^1(\Omega)} = \|u\|_2 + \|\nabla_H u\|_2. \tag{2}$$

Let $S_0^1(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm (2). It follows from (1) that the norm $\|\cdot\|_{S^1(\Omega)}$ is equivalent in $S_0^1(\Omega)$ to that generated by the inner product

$$\langle u, v \rangle_{S_0^1} = \int_{\Omega} \nabla_H u \nabla_H v d\xi.$$

Therefore $(S_0^1(\Omega), \|\cdot\|_{S_0^1})$ is a Hilbert space. In particular, when $1 \leq p < Q^*$, $S_0^1(\Omega)$ is compactly embedded in $L^p(\Omega)$, and when $p = Q^*$, the embedding is continuous. This means that there exists $B_p > 0$ such that for any $1 \leq p \leq Q^*$,

$$\|\phi\|_p \leq B_p \|\phi\|_{S_0^1}, \quad \forall \phi \in L^p(\Omega). \tag{3}$$

Besides, we also need to use the local Sobolev space $S_{loc}^2(\Omega)$ which is defined by saying that $u \in S_{loc}^1(\Omega)$ if and only if $u \in S^1(\omega)$ for any $\omega \subset\subset \Omega$.

In this paper we study the following singular sub-elliptic equation:

$$\begin{cases} -\Delta_H u = \frac{h(\xi)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where Ω is a smooth bounded domain of the Heisenberg group \mathbb{H}^n , and γ is a positive parameter, and h is a nonnegative function and it is not identically zero. The aim of this paper is to prove existence and uniqueness of the solution to (4), and then deduce that the solution is cylindrically symmetric under some necessary structural conditions on h and Ω . Here, we say that a weak solution of the equation in (4) is a function $u \in S_{loc}^1(\Omega)$ such that, for any $\omega \subset\subset \Omega$, there exists $c_\omega > 0$ such that $u \geq c_\omega$ a.e. in ω and

$$\int_{\Omega} \nabla_H u \nabla_H \varphi d\xi = \int_{\Omega} \frac{h\varphi}{u^\gamma} d\xi, \quad \forall \varphi \in C_c^\infty(\Omega). \tag{5}$$

Since the solution u may do not belong to $S_0^1(\Omega)$, $u|_{\partial\Omega} = 0$ has to be understood in a generalized meaning, that is, u is nonnegative and $(u - \varepsilon)^+ \in S_0^1(\Omega)$ for any $\varepsilon > 0$.

The Dirichlet problems of singular elliptic equations appeared in many applied fields, such as the theory of heat conduction in electrically conducting materials and pseudoplastic fluids, the binary communications by signals and etc, see [7, 8, 9]. Meanwhile, since the work of Stuart [10] and Grandall and Rabinowitz et al. [11], singular elliptic problems in \mathbb{R}^n have been widely studied by many scholars and many meaningful results are obtained by using the fixed point theory, variational methods and some

analysis techniques. Here, with respect to the case of bounded domain Ω , we refer the reader to Canino [12], Canino and Degiovanni [13], Boccardo and Orsina [14], Canino and Sciunzi [15], Canino and Grandinetti et al. [16, 17], Canino and Montoro et al. [18], Canino and Esposito et al. [19] while for the whole space \mathbb{R}^n , please see Kusano and Swanson [20], Lair and Shaker [21], Alves and Goncalves et al. [22]. However, to the best of our knowledge, there are few results about the sub-elliptic equations with singular nonlinearities $\frac{h(\xi)}{u^\gamma}$ on the Heisenberg group \mathbb{H}^n .

We first have the following existence theorem of weak solutions.

THEOREM 1.1. (A) If $\gamma \geq 1$ and $h \in L^1(\Omega)$, then (4) has a weak solution u in $S_{loc}^1(\Omega)$ and $u^{\frac{\gamma+1}{2}} \in S_0^1(\Omega)$. (B) If $0 < \gamma < 1$ and $h \in L^m(\Omega)$ with $m = \frac{2Q}{Q+2+\gamma(Q-2)}$, then (4) has a weak solution u in $S_0^1(\Omega)$.

Next, we will give the uniqueness and symmetry results of the solution. Firstly, we say that a function $u(z, t)$ is cylindrically symmetric if there exists $\xi_0 \in \mathbb{H}^n$ such that $v(\xi) = u(\xi_0 \circ \xi)$ is a two variables function, i.e., $v(z, t) = v(r, t)$ with $r = |z|$. Without loss of generality, we always suppose that ξ_0 occurring in the definition is 0. In addition, we say that a domain $\Omega \subset \mathbb{H}^n$ is a cylinder if there exists a cylindrical function Φ such that $\xi \in \Omega$ if and only if $\Phi(\xi) < 0$. As an example, the Heisenberg ball $B_r(\xi_0)$ is a cylinder, see [27].

THEOREM 1.2. Under the same assumptions as in Theorem 1.1, the solution of problem (4) is unique.

THEOREM 1.3. Under the same assumptions as in Theorem 1.1, let Ω be a bounded cylinder defined by a cylindrical function Φ . Suppose that h is cylindrically symmetric and u is a weak solution of problem (4). Then u is cylindrically symmetric. Further, if we assume that $h(r, t) = h(r, -t)$ and $\Phi(r, t) = \Phi(r, -t)$. Then, $u(r, t) = u(r, -t)$ for any $(z, t) \in \Omega$.

The symmetry of the solutions is widely studied in recent decades, many classical results were obtained by using the method of moving planes, which goes back to Alexandrov [23] and was first used by Serrin in [24]. Also, the method was developed and improved by Berestycki and Nirenberg et al. in [25] and [26]. Moreover, in [28], the method was successfully applied in the Heisenberg group. Especially, by a generalization of the moving plane method, Birindelli and Prajapat [27] proved a partial symmetry result of positive solutions of the sub-elliptic equation $-\Delta_H u = f(u)$ in a bounded cylinder domain of \mathbb{H}^n , with the Dirichlet zero boundary conditions, where f and u were assumed to be a Lipschitz function and a cylindrical solution, respectively. But many scholars believe that the assumption on u may be extra. Indeed, this is a difficult open problem for sub-elliptic equations, see [27, 29, 28, 30]. Here, we want to emphasize that the nonlinearity $\frac{h(\xi)}{u^\gamma}$ of problem (4) is not Lipschitz continuous in the u -variable at $u = 0$. Meanwhile, there is not any condition on u in Theorem 1.3, since the uniqueness of the solution is given by Theorem 1.2.

2. Existence of solutions

In this section, we use the Schauder's fixed point theorem and approximating method to prove Theorem 1.1. We first study the approximation of (4) by truncating the singular term $\frac{h(\xi)}{u^\gamma}$. That is, we first consider the following sub-elliptic equation:

$$\begin{cases} -\Delta_H u_n = \frac{h_n(\xi)}{(u_n + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \quad (6)$$

where $h_n(\xi) = \min\{h(\xi), n\}$ for fixed $n \in \mathbb{N}$. For our ease, sometimes we will make use of the notation $h_n = h_n(\xi)$.

LEMMA 2.1. *Problem (6) has a nonnegative weak solution u_n in $S_0^1(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let $v \in L^2(\Omega)$, for fixed $n \in \mathbb{N}$, we consider:

$$\begin{cases} -\Delta_H w = \frac{h_n(\xi)}{(|v| + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (7)$$

Let us define $J : S_0^1(\Omega) \rightarrow \mathbb{R}$, $\forall w \in S_0^1(\Omega)$,

$$J(w) = \frac{1}{2} \int_{\Omega} |\nabla_H w|^2 d\xi - \int_{\Omega} \frac{h_n(\xi)w}{(|v| + \frac{1}{n})^\gamma} d\xi,$$

then critical points of the functional J are weak solutions of problem (7), and vice-versa. Moreover, it is easy to show that the functional J is coercive and weakly lower semi-continuous in $S_0^1(\Omega)$. Hence, it follows from [34, Theorem 1.2] that (7) has at least one weak solution $w \in S_0^1(\Omega)$. Now we prove the weak solution of (7) is unique. Assume that w_1 and w_2 are two weak solutions of (7) in $S_0^1(\Omega)$, then we have

$$\int_{\Omega} \nabla_H w_1 \nabla \varphi d\xi = \int_{\Omega} \nabla_H w_2 \nabla \varphi d\xi = \int_{\Omega} \frac{h_n(\xi)\varphi}{(|v| + \frac{1}{n})^\gamma} d\xi \quad (8)$$

for any $\varphi \in S_0^1(\Omega)$. Let $\varphi = w_1 - w_2$ in (8), it follows that

$$\int_{\Omega} |\nabla_H (w_1 - w_2)|^2 d\xi = 0.$$

Therefore we get $w_1 = w_2$. This proves that the weak solution of (7) is unique.

Next, we use the Schauder's fixed point theorem to prove the existence of solutions to problem (6). Let w be the unique weak solution of (7) and define $T : v \rightarrow w$ for any $v \in L^2(\Omega)$. From Hölder's inequality, one has

$$\begin{aligned} \int_{\Omega} |\nabla_H w|^2 d\xi &= \int_{\Omega} \frac{h_n(\xi)w}{(|v| + \frac{1}{n})^\gamma} d\xi \\ &\leq n^{\gamma+1} \sqrt{|\Omega|} \left(\int_{\Omega} |w|^2 d\xi \right)^{1/2}. \end{aligned} \quad (9)$$

It follows from (3) and (9) that

$$\|w\|_{S_0^1} \leq B_2 \sqrt{|\Omega|} n^{\gamma+1}. \quad (10)$$

Let

$$K = \{v \in S_0^1(\Omega) : \|v\|_{S_0^1} \leq B_2 \sqrt{|\Omega|} n^{\gamma+1}\},$$

then K is invariant under the action of T , that is, $T(K) \subset K$.

In order to apply the Schauder's fixed point theorem. We first show the continuity of T as an operator from $S_0^1(\Omega)$ to $S_0^1(\Omega)$. Let $\{v_k\} \subset S_0^1(\Omega)$ and $v_k \rightarrow v$ in $S_0^1(\Omega)$ as $k \rightarrow \infty$. Thus one has

$$v_k \rightarrow v \text{ in } L^{Q^*}(\Omega) \text{ and } v_k(\xi) \rightarrow v(\xi) \text{ a.e. in } \Omega \text{ as } k \rightarrow \infty.$$

For convenience, let $w_k = T(v_k)$, $w = T(v)$ and $\bar{w}_k(\xi) = w_k(\xi) - w(\xi)$. It follows from (7) and Hölder's inequality that

$$\begin{aligned} \int_{\Omega} |\nabla_H \bar{w}_k|^2 d\xi &= \int_{\Omega} \left(\frac{h_n(\xi)}{(|v_k| + \frac{1}{n})^\gamma} - \frac{h_n(\xi)}{(|v| + \frac{1}{n})^\gamma} \right) \bar{w}_k d\xi \\ &\leq \left(\int_{\Omega} \left| \frac{h_n(\xi)}{(|v_k| + \frac{1}{n})^\gamma} - \frac{h_n(\xi)}{(|v| + \frac{1}{n})^\gamma} \right|^{\frac{2Q}{Q+2}} d\xi \right)^{\frac{Q+2}{2Q}} \|\bar{w}_k\|_{L^{Q^*}(\Omega)} \\ &\leq B_{Q^*} \left(\int_{\Omega} \left| \frac{h_n(\xi)}{(|v_k| + \frac{1}{n})^\gamma} - \frac{h_n(\xi)}{(|v| + \frac{1}{n})^\gamma} \right|^{\frac{2Q}{Q+2}} d\xi \right)^{\frac{Q+2}{2Q}} \|\bar{w}_k\|_{S_0^1}. \end{aligned} \quad (11)$$

Notice that

$$\left| \frac{h_n(\xi)}{(|v_k| + \frac{1}{n})^\gamma} - \frac{h_n(\xi)}{(|v| + \frac{1}{n})^\gamma} \right| \leq \frac{h_n(\xi)}{(|v_k| + \frac{1}{n})^\gamma} + \frac{h_n(\xi)}{(|v| + \frac{1}{n})^\gamma} \leq 2n^{\gamma+1}.$$

Therefore, from (11) and Lebesgue dominated convergence theorem, we have

$$\|\bar{w}_k\|_{S_0^1} = \|w_k - w\|_{S_0^1} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (12)$$

This proves the continuity of T as an operator from $S_0^1(\Omega)$ to $S_0^1(\Omega)$.

Secondly, we prove that T is a compact operator. Indeed, let $\{v_k\} \subset S_0^1(\Omega)$ be a bounded sequence. Without loss of generality, we can assume that $v_k \rightarrow v$ in $S_0^1(\Omega)$ and $v_k(\xi) \rightarrow v(\xi)$ a.e. in Ω . Let $w_k = T(v_k)$, it follows from (10) that there exists $C > 0$ such that $\|w_k\|_{S_0^1} < C$. Therefore, up to a subsequence, there exists $w \in S_0^1(\Omega)$ such that $w_k \rightharpoonup w$ in $S_0^1(\Omega)$ and $w_k \rightarrow w$ a.e. in Ω . By Lebesgue dominated convergence theorem, one has

$$\int_{\Omega} \nabla_H w \nabla_H \varphi d\xi = \int_{\Omega} \frac{h_n(\xi) \varphi}{(|v| + \frac{1}{n})^\gamma} d\xi, \quad \forall \varphi \in S_0^1(\Omega).$$

This means that $w = T(v)$. Therefore, Proceeding as in the proof of (12), we have

$$\lim_{k \rightarrow \infty} \|w_k - w\|_{S_0^1} = \lim_{k \rightarrow \infty} \|T(v_k) - T(v)\|_{S_0^1} = 0.$$

This proves that T is a compact operator.

Therefore, for fixed $n \in \mathbb{N}$, it follows from Schauder’s fixed point theorem that there exists $u_n \in S_0^1(\Omega)$ such that $u_n = Tu_n$. That is, u_n satisfies the following equation

$$\begin{cases} -\Delta_H u_n = \frac{h_n(\xi)}{(|u_n| + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

By the definition of $h_n(\xi)$, we have

$$\frac{h_n(\xi)}{(|u_n| + \frac{1}{n})^\gamma} \leq n^{\gamma+1}.$$

This means that $\frac{h_n(\xi)}{(|u_n| + \frac{1}{n})^\gamma} \in L^\infty(\Omega)$ for $n \in \mathbb{N}$. By the subelliptic estimates and embedding of the non-isotropic Sobolev spaces, one gets $u_n \in L^\infty(\Omega)$, although here $\|u_n\|_\infty$ may depend on n , see [35, 36, 37]. In addition, we also have $-\Delta_H u_n \geq 0$ in Ω , since $\frac{h_n(\xi)}{(|u_n| + \frac{1}{n})^\gamma} \geq 0$. Hence, $u_n \geq 0$ in Ω . That is to say that $u_n \in S_0^1(\Omega) \cap L^\infty(\Omega)$ is a non-negative weak solution of problem (6). \square

LEMMA 2.2. *For all $n \in \mathbb{N}$, the solution sequence $\{u_n\}$ of (6) satisfies $u_{n+1} \geq u_n$ and $u_n > 0$ in Ω . Further, for any $\omega \subset\subset \Omega$, there exists positive constant c_ω such that*

$$u_n(\xi) \geq c_\omega \text{ a.e. in } \omega. \tag{14}$$

Proof. For any $\gamma > 0$, by the definition of $h_n(\xi)$ and (6), we have in the distributional meaning

$$\begin{aligned} -\Delta_H(u_n - u_{n+1}) &= \frac{h_n(\xi)}{(u_n + \frac{1}{n})^\gamma} - \frac{h_{n+1}(\xi)}{(u_{n+1} + \frac{1}{n+1})^\gamma} \\ &\leq h_{n+1}(\xi) \left[\frac{(u_{n+1} + \frac{1}{n+1})^\gamma - (u_n + \frac{1}{n+1})^\gamma}{(u_{n+1} + \frac{1}{n+1})^\gamma (u_n + \frac{1}{n+1})^\gamma} \right]. \end{aligned} \tag{15}$$

Let us choose $(u_n - u_{n+1})^+$ as a test function in (15) and note that

$$\left[\left(u_{n+1} + \frac{1}{n+1}\right)^\gamma - \left(u_n + \frac{1}{n+1}\right)^\gamma \right] (u_n - u_{n+1})^+ \leq 0. \tag{16}$$

Then we obtain

$$\int_\Omega |\nabla_H(u_n - u_{n+1})^+|^2 d\xi \leq 0, \tag{17}$$

which implies that $u_n \leq u_{n+1}$ in Ω . Besides, it follows from Lemma 2.1 that $u_1 \in S_0^1(\Omega)$ and

$$\begin{cases} -\Delta_H u_1 = \frac{h_1(\xi)}{(u_1+1)^\gamma} & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \quad (18)$$

Since $0 \leq h_1(\xi)(u_1+1)^{-\gamma} \leq 1$ and $u_1 \geq 0$, we can use the Strong Maximum Principle [38, Lemma 2.7] to get $u_1 \equiv 0$ or $u_1 > 0$ in Ω . Furthermore, it follows from $h_1(\xi) \not\equiv 0$ and (18) that $u_1 > 0$ in Ω . This completes the proof of Lemma 2.2. \square

REMARK 2.3. It is easy to see that the solution of (6) is unique. Indeed, assume that u_n and v_n are two solutions of (6) in $S_0^1(\Omega) \cap L^\infty(\Omega)$. Then, we have in the distributional meaning

$$-\Delta_H(u_n - v_n) = h_n(\xi) \left[\frac{(v_n + \frac{1}{n})^\gamma - (u_n + \frac{1}{n})^\gamma}{(u_n + \frac{1}{n})^\gamma (v_n + \frac{1}{n})^\gamma} \right]. \quad (19)$$

Now let us choose $(u_n - v_n)^+$ and $(u_n - v_n)^-$ as test functions in (19), respectively. Then, proceeding as in the proof of Lemma 2.2, one obtain $u_n = v_n$.

LEMMA 2.4. *Let u_n be the unique nonnegative solution of problem (6). Then,*

(A) *If $\gamma \geq 1$ and $h \in L^1(\Omega)$, then $u_n^{\frac{\gamma+1}{2}}$ is uniformly bounded in $S_0^1(\Omega)$ and u_n is uniformly bounded in $L^s(\Omega)$ with $s = \frac{Q(\gamma+1)}{Q-2}$. Further, when $\gamma > 1$, u_n is uniformly bounded in $S_{loc}^1(\Omega)$.*

(B) *If $0 < \gamma < 1$ and $h \in L^m(\Omega)$ with $m = \frac{2Q}{Q+2+\gamma(Q-2)}$, then u_n is uniformly bounded in $S_0^1(\Omega)$.*

Proof. (A) Let us choose u_n^γ as a test function in (6), then we have

$$\int_{\Omega} \nabla_H u_n \nabla_H u_n^\gamma d\xi = \int_{\Omega} \frac{h_n(\xi) u_n^\gamma}{(u_n + \frac{1}{n})^\gamma} d\xi \leq \|h\|_1.$$

Noting that

$$\frac{4\gamma}{(\gamma+1)^2} \int_{\Omega} \left| \nabla_H (u_n^{\frac{\gamma+1}{2}}) \right|^2 d\xi = \gamma \int_{\Omega} |\nabla_H u_n|^2 u_n^{\gamma-1} d\xi = \int_{\Omega} \nabla_H u_n \nabla_H u_n^\gamma d\xi.$$

Then, we have

$$\left\| u_n^{\frac{\gamma+1}{2}} \right\|_{S_0^1} = \int_{\Omega} \left| \nabla_H (u_n^{\frac{\gamma+1}{2}}) \right|^2 d\xi \leq \frac{(\gamma+1)^2}{4\gamma} \|h\|_1, \quad (20)$$

which means $u_n^{\frac{\gamma+1}{2}}$ is uniformly bounded in $S_0^1(\Omega)$. In addition, by (3), one has

$$\|u_n\|_s = \|u_n^{\frac{\gamma+1}{2}}\|_{L^{Q^*}(\Omega)}^{\frac{2}{\gamma+1}} \leq B_{Q^*}^{\frac{2}{\gamma+1}} \|u_n^{\frac{\gamma+1}{2}}\|_{S_0^1}^{\frac{2}{\gamma+1}}.$$

From this and (20), we know that $u_n^{\frac{\gamma+1}{2}}$ is uniformly bounded in $S_0^1(\Omega)$ with $s = \frac{Q(\gamma+1)}{Q-2}$.

Next, we prove that u_n is uniformly bounded in $S_{loc}^1(\Omega)$ if $\gamma > 1$. In fact, let $\varphi \in C_0^\infty(\Omega)$ and $\omega = \{\xi \in \Omega : \varphi \neq 0\}$. Then $u_n \varphi^2$ can be used as a test function in (6). From (14) and the weighted Young's inequality, one obtains

$$\begin{aligned} \int_{\Omega} |\nabla_H u_n|^2 \varphi^2 d\xi &= \int_{\Omega} \frac{h_n(\xi) u_n \varphi^2}{(u_n + \frac{1}{n})^\gamma} d\xi - 2 \int_{\Omega} \varphi u_n \nabla_H u_n \nabla_H \varphi d\xi \\ &\leq c_\omega^{1-\gamma} \int_{\Omega} h_n(\xi) \varphi^2 d\xi + \frac{1}{2} \int_{\Omega} |\nabla_H u_n|^2 \varphi^2 d\xi \\ &\quad + 2 \int_{\Omega} |\nabla_H \varphi|^2 u_n^2 d\xi. \end{aligned} \quad (21)$$

Therefore, (21) becomes

$$\begin{aligned} \int_{\Omega} |\nabla_H u_n|^2 \varphi^2 d\xi &\leq 2c_\omega^{1-\gamma} \int_{\Omega} h_n(\xi) \varphi^2 d\xi + 4 \int_{\Omega} |\nabla_H \varphi|^2 u_n^2 d\xi \\ &\leq 2c_\omega^{1-\gamma} \|\varphi\|_\infty^2 \|h\|_1 + 4 \|\nabla_H \varphi\|_\infty^2 \int_{\Omega} u_n^2 d\xi \\ &\leq 2c_\omega^{1-\gamma} \|\varphi\|_\infty^2 \|h\|_1 + 4 \|\nabla_H \varphi\|_\infty^2 |\Omega|^{\frac{s-2}{2}} \|u_n\|_s^2. \end{aligned} \quad (22)$$

From this and the uniform boundedness of $\|u_n\|_s$, we conclude that u_n is uniformly bounded in $S_{loc}^1(\Omega)$.

(B) When $0 < \gamma < 1$, we choose u_n as a test function in (6), it follows from (3) that

$$\begin{aligned} \int_{\Omega} |\nabla_H u_n|^2 d\xi &= \int_{\Omega} \frac{h_n(\xi) u_n}{(u_n + \frac{1}{n})^\gamma} d\xi \leq \int_{\Omega} h u_n^{1-\gamma} d\xi \leq \|h\|_m \left(\int_{\Omega} u_n^{Q^*} \right)^{1/m'} \\ &\leq C \|h\|_m \left(\int_{\Omega} |\nabla_H u_n|^2 d\xi \right)^{Q^*/(2m')}, \end{aligned} \quad (23)$$

where m' is the conjugate number of m . Note that $\frac{Q^*}{2m'} = \frac{1-\gamma}{2} \in (0, 1)$, it follows from (23) that u_n is bounded in $S_0^1(\Omega)$. \square

Proof of Theorem 1.1. Since we have a priori estimates on u_n from Lemma 2.4, we can easily prove the existence of the solutions of (4). Indeed, note that $m = 1$ if $\gamma = 1$. Then, for $0 < \gamma \leq 1$, by Lemma 2.4, let $u_n \in S_0^1(\Omega)$ be the nonnegative weak solution of (6). Meanwhile, for all $n \in \mathbb{N}$, the sequence $\{u_n\}$ is uniformly bounded in $S_0^1(\Omega)$. Then, there exists $u \in S_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } S_0^1(\Omega), \\ u_n(\xi) \rightarrow u(\xi) & \text{a.e. in } \Omega. \end{cases} \quad (24)$$

On the one hand, it follows from Lemma 2.2 and (24) that $u > 0$ in Ω and $u|_{\Omega} = 0$. On the other hand, since $u_n \in S_0^1(\Omega)$ is the weak solution of problem (6), one has

$$\int_{\Omega} \nabla_H u_n \nabla_H \varphi d\xi = \int_{\Omega} \frac{h_n(\xi) \varphi}{(u_n + \frac{1}{n})^\gamma} d\xi, \quad \forall \varphi \in C_0^\infty(\Omega). \quad (25)$$

Now let $\omega = \{\xi \in \Omega : \varphi \neq 0\}$, it follows from Lemma 2.2 that

$$0 \leq \left| \frac{h_n(\xi)\varphi}{(u_n + \frac{1}{n})^\gamma} \right| \leq \frac{\|\varphi\|_\infty}{c_\omega^\gamma} h(\xi),$$

Therefore from (25) and Lebesgue dominated convergence theorem, we get

$$\int_\Omega \nabla_H u \nabla_H \varphi d\xi = \int_\Omega \frac{h\varphi}{u^\gamma} d\xi, \quad \forall \varphi \in C_0^\infty(\Omega).$$

That is, when $0 < \gamma \leq 1$ and $h \in L^1(\Omega)$, problem (4) has a solution u in $S_0^1(\Omega)$.

For the case of $\gamma > 1$, since u_n is uniformly bounded in $S_{loc}^1(\Omega)$, the proof is almost identical with the case $0 < \gamma \leq 1$, we omit it. \square

3. Uniqueness and symmetry

In this section, we will prove Theorems 1.2 and 1.3. Or, more specifically, we first prove a weak comparison principle and then conclude the uniqueness of solution. Subsequently, we prove that the solution to (4) is cylindrically symmetric.

For fixed $k \in \mathbb{N}$, let

$$f_k(s) = \begin{cases} -\min\{\frac{1}{s^\gamma}, k\} & \text{if } s > 0, \\ -k & \text{if } s \leq 0. \end{cases}$$

Moreover, let F_k denote the primitive of f_k (i.e. $F'(s) = f_k(s)$) such that $F_k(1) = 0$. Now, we define the functional $I_k : S_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$,

$$I_k(\varphi) = \frac{1}{2} \int_\Omega |\nabla_H \varphi|^2 d\xi + \int_\Omega h(\xi) F_k(\varphi) d\xi, \quad \forall \varphi \in S_0^1(\Omega).$$

Let $\psi \in S_{loc}^1(\Omega)$ be a fixed weak supersolution to the equation $\Delta_H z = h(\xi) F_k'(z)$, i.e.

$$\int_\Omega \nabla_H \psi \nabla_H \phi d\xi \geq - \int_\Omega h F_k'(\psi) \phi d\xi \quad (26)$$

for any $\phi \in S_0^1(\Omega) \cap L_c^\infty(\Omega)$ with $\phi \geq 0$ almost everywhere in Ω . Let

$$K_\psi = \{\varphi \in S_0^1(\Omega) : 0 \leq \varphi \leq \psi \text{ a.e. in } \Omega\}.$$

Obviously, K_ψ is a closed and convex subset of $S_0^1(\Omega)$. And then there exists $w \in K_\psi$ such that

$$I_k(w) = \min_{\varphi \in K_\psi} I_k(\varphi).$$

Therefore, for any $\varphi \in w + (S_0^1(\Omega) \cap L_c^\infty(\Omega))$ with $0 \leq \varphi \leq \psi$, we have the following variational inequality

$$\int_\Omega \nabla_H w \nabla_H (\varphi - w) d\xi \geq - \int_\Omega h F_k'(w) (\varphi - w) d\xi. \quad (27)$$

In fact, for $0 \leq \lambda \leq 1$, it follows from the convexity of K_ψ that $w + \lambda(\varphi - w) \in K_\psi$. Therefore we get

$$I_k(w + \lambda(\varphi - w)) \geq I_k(w).$$

Therefore,

$$\lim_{\lambda \rightarrow 0^+} \frac{I_k(w + \lambda(\varphi - w)) - I_k(w)}{\lambda} \geq 0.$$

From this and a direct computation, we know that the inequality (27) is true.

In what follows, we prove a weak comparison principle, which is the key of the proof of Theorem 1.2.

LEMMA 3.1. *Let $\gamma > 1$ and $u \in S_{loc}^1(\Omega)$ be such that*

$$\int_{\Omega} \nabla_H u \nabla_H \phi d\xi \leq \int_{\Omega} hu^{-\gamma} \phi d\xi, \quad \forall \phi \in S_0^1(\Omega) \cap L_c^\infty(\Omega) \quad (28)$$

with $\phi > 0$ a.e. in Ω . If $(u - \varepsilon)^+ \in S_0^1(\Omega)$ for any $\varepsilon > 0$. Then $u \leq \psi$ a.e. in Ω , where ψ is used as in (26).

Proof. For any $\phi \in C_c^\infty(\Omega)$ with $\phi \geq 0$ a.e. in Ω , we first claim that

$$\int_{\Omega} \nabla_H w \nabla_H \phi d\xi \geq - \int_{\Omega} hF'_k(w) \phi d\xi, \quad (29)$$

where w is as above. Indeed, let us define a cut-off function $0 \leq \eta \leq 1$ such that

$$\eta(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| \geq 2. \end{cases}$$

For $k \geq 1$ and $t > 0$, let

$$\phi_k = \eta\left(\frac{w}{k}\right)\phi, \quad \phi_{k,t} = \min\{w + t\phi_k, \psi\}.$$

Obviously, we have $w \leq \phi_{k,t} \leq \psi$ and $\phi_{k,t} \in w + (S_0^1(\Omega) \cap L_c^\infty(\Omega))$ by construction. Now, for the sake of simplicity, let $\tilde{\phi}_{k,t} = \phi_{k,t} - w - t\phi_k$, then $\tilde{\phi}_{k,t} \leq 0$. It follows from (26) that

$$\int_{\Omega} (\nabla_H \psi \nabla_H \tilde{\phi}_{k,t} + hF'_k(\psi) \tilde{\phi}_{k,t}) d\xi \leq 0. \quad (30)$$

Meanwhile, it follows from (27) that

$$\int_{\Omega} \nabla_H w \nabla_H (\phi_{k,t} - w) d\xi \geq \int_{\Omega} hF'_k(w) (\phi_{k,t} - w) d\xi. \quad (31)$$

Note that $\tilde{\phi}_{k,t} = 0$ if $w + t\phi_k \leq \psi$. Hence, from (30) and (31), one has

$$\begin{aligned}
& \int_{\Omega} (|\nabla_H(\phi_{k,t} - w)|^2 + h(F'_k(\phi_{k,t}) - F'_k(w))(\phi_{k,t} - w)) d\xi \\
& \leq \int_{\Omega} (\nabla_H \phi_{k,t} \nabla_H(\phi_{k,t} - w) + hF'_k(\phi_{k,t})(\phi_{k,t} - w)) d\xi \\
& = \int_{\Omega} (\nabla_H \phi_{k,t} \nabla_H \tilde{\phi}_{k,t} + hF'_k(\phi_{k,t})\tilde{\phi}_{k,t}) d\xi + t \int_{\Omega} (\nabla_H \phi_{k,t} \nabla_H \phi_k + hF'_k(\phi_{k,t})\phi_k) d\xi \\
& = \int_{\Omega} (\nabla_H \psi \nabla_H \tilde{\phi}_{k,t} + hF'_k(\psi)\tilde{\phi}_{k,t}) d\xi + t \int_{\Omega} (\nabla_H \phi_{k,t} \nabla_H \phi_k + hF'_k(\phi_{k,t})\phi_k) d\xi \\
& \leq t \int_{\Omega} (\nabla_H \phi_{k,t} \nabla_H \phi_k + hF'_k(\phi_{k,t})\phi_k) d\xi. \tag{32}
\end{aligned}$$

From this, we get

$$\int_{\Omega} h(F'_k(\phi_{k,t}) - F'_k(w))(\phi_{k,t} - w) d\xi \leq t \int_{\Omega} (\nabla_H \phi_{k,t} \nabla_H \phi_k + hF'_k(\phi_{k,t})\phi_k) d\xi. \tag{33}$$

By applying $\phi_{k,t} - w \leq t\phi_k$ and (33), one has

$$\int_{\Omega} h|F'_k(\phi_{k,t}) - F'_k(w)|\phi_k d\xi \geq - \int_{\Omega} (\nabla_H \phi_{k,t} \nabla_H \phi_k + hF'_k(\phi_{k,t})\phi_k) d\xi. \tag{34}$$

Since $\phi_{k,t}(\xi) \rightarrow w(\xi)$ a.e. in Ω as $t \rightarrow 0^+$, it follows from (34) and Lebesgue dominated convergence theorem that

$$\int_{\Omega} (\nabla_H w \nabla_H \phi_k + hF'_k(w)\phi_k) d\xi \geq 0. \tag{35}$$

Further, since $\phi_k(\xi) \rightarrow \phi(\xi)$ a.e. in Ω as $k \rightarrow \infty$, we get the inequality (29) from (35) and Lebesgue dominated convergence theorem, that is, we complete the proof of the claim.

Now, we use (29) to prove (28). In fact, for $\varepsilon > 0$, we have $(u - w - \varepsilon)^+ \in S_0^1(\Omega)$. Then there exists $\{\varphi_n\} \subset C_c^\infty(\Omega)$ such that

$$\varphi_n \rightarrow (u - w - \varepsilon)^+ \text{ in } S_0^1(\Omega).$$

Now, let us define the truncation function for fixed $l \in \mathbb{N}^+$,

$$T_l(s) = \max\{-l, \min\{l, s\}\}.$$

Therefore, it follows from (29) that

$$\int_{\Omega} \nabla_H w \nabla_H T_l((u - w - \varepsilon)^+) d\xi \geq - \int_{\Omega} hF'_k(w)T_l((u - w - \varepsilon)^+) d\xi. \tag{36}$$

Let

$$\tilde{\varphi}_{l,n} = T_l(\min\{(u - w - \varepsilon)^+, \varphi_n^+\}),$$

then $\tilde{\varphi}_{l,n} \in S_0^1(\Omega) \cap L_c^\infty(\Omega)$. Therefore, it follows from (28) that

$$\int_{\Omega} \nabla_H u \nabla_H \tilde{\varphi}_{l,n} d\xi \leq \int_{\Omega} \frac{h}{u^\gamma} \tilde{\varphi}_{l,n} d\xi. \tag{37}$$

Let $n \rightarrow \infty$ in (37), one gets

$$\int_{\Omega} \nabla_H u \nabla_H T_l((u-w-\varepsilon)^+) d\xi \leq \int_{\Omega} \frac{h}{u^\gamma} T_l((u-w-\varepsilon)^+) d\xi. \tag{38}$$

Now let us choose ε such that $\varepsilon^{-\gamma} < k$. Then from (36), (38) and $F'_k(s) = f_k(s)$, one has

$$\begin{aligned} \int_{\Omega} |\nabla_H T_l((u-w-\varepsilon)^+)|^2 d\xi &\leq \int_{\Omega} \nabla_H(u-w) \nabla_H T_l((u-w-\varepsilon)^+) d\xi \\ &\leq \int_{\Omega} h(u^{-\gamma} + F'_k(w)) T_l((u-w-\varepsilon)^+) d\xi \\ &\leq \int_{\Omega} h(-F'_k(u) + F'_k(w)) T_l((u-w-\varepsilon)^+) d\xi \\ &\leq 0. \end{aligned} \tag{39}$$

It follows that $T_l((u-w-\varepsilon)^+) = 0$ and hence one has

$$u \leq w + \varepsilon \leq \psi + \varepsilon \text{ a.e. in } \Omega,$$

which means that $u \leq \psi$ a.e. in Ω from the arbitrariness of ε . The proof is completed. \square

Proof of Theorem 1.2. Let u and ψ be two solutions of problem (4). When $0 < \gamma \leq 1$. It follows from Theorem 1.1 that u and ψ belong to $S_0^1(\Omega)$ and hence $(u-\psi)^+ \in S_0^1(\Omega)$. Then there exists $\{\varphi_n\} \subset C_c^\infty(\Omega)$ such that

$$\varphi_n \rightarrow (u-\psi)^+ \text{ in } S_0^1(\Omega).$$

Now let us define

$$\widehat{\varphi}_n = \min\{(u-\psi)^+, \varphi_n^+\}.$$

It follows that $\widehat{\varphi}_n \in S_0^1(\Omega)$ and $\text{supp}(\widehat{\varphi}_n) \subset \Omega$. This is to say that $\widehat{\varphi}_n$ can be used as a test function. Hence, by the definition of $\widehat{\varphi}_n$, one gets

$$\int_{\Omega} \nabla_H(u-\psi) \nabla_H \widehat{\varphi}_n d\xi = \int_{\Omega} h \widehat{\varphi}_n (u^{-\gamma} - \psi^{-\gamma}) d\xi \leq 0.$$

By the Lebesgue dominated convergence theorem, one has

$$\int_{\Omega} |\nabla_H(u-\psi)^+|^2 \xi \leq 0.$$

This means that $u \leq \psi$ in Ω . Similarly, we can also conclude that $\psi \leq u$ in Ω . Then, $u = \psi$ and thus the solution of (4) with $0 < \gamma \leq 1$ is unique.

For the case of $\gamma > 1$, by Theorem 1.1, we know $u^{\frac{\gamma+1}{2}} \in S_0^1(\Omega)$. In order to exploit Lemma 3.1, we will prove that $(u - \varepsilon)^+$ and $(\psi - \varepsilon)^+$ belong to $S_0^1(\Omega)$ for any $\varepsilon > 0$. In fact, let $\varphi_n \in C_c^\infty(\Omega)$ and $\varphi_n \rightarrow u^{\frac{\gamma+1}{2}}$ in $S_0^1(\Omega)$. We define

$$\widehat{\varphi}_n = (\varphi_n^{\frac{2}{\gamma+1}} - \varepsilon)^+$$

Then $\widehat{\varphi}_n$ is uniformly bounded in $S_0^1(\Omega)$ and

$$\widehat{\varphi}_n \rightarrow (u - \varepsilon)^+ \text{ in } S_0^1(\Omega),$$

which means that $(u - \varepsilon)^+ \in S_0^1(\Omega)$. Note that both u and ψ satisfy (26) and (28). It follows from Lemma 3.1 that $u \leq \psi$ in Ω . In the same way, we can also prove that $(\psi - \varepsilon)^+ \in S_0^1(\Omega)$ and thus $\psi \leq u$ in Ω . So, $u = \psi$ for $\gamma > 1$. The proof of Theorem 1.2 is completed. \square

Proof of Theorem 1.3. Let u be a solution of (4) and \mathcal{S} denote a unitary rotation in \mathbb{C}^n . We define $u_{\mathcal{S}}(z, t) = u(\mathcal{S}z, t)$ for any $(z, t) \in \mathbb{C}^n \times \mathbb{R}$. Since Δ_H is invariant w.r.t. \mathcal{S} and h is cylindrically symmetric, then we have

$$-\Delta_H u_{\mathcal{S}}(z, t) = -\Delta_H u(\mathcal{S}z, t) = \frac{h(\mathcal{S}z, t)}{u(\mathcal{S}z, t)^\gamma} = \frac{h(z, t)}{u_{\mathcal{S}}(z, t)^\gamma}.$$

This means that $u_{\mathcal{S}}$ is also a solution of (4). By the uniqueness (Theorem 1.2) of the solution of (4), we obtain $u = u_{\mathcal{S}}$ for any unitary rotation \mathcal{S} in \mathbb{C}^n . This is to say that the solution u is a function of $(|z_1|, |z_2|, \dots, |z_n|, t)$. Therefore (4) becomes

$$\begin{cases} -(\Delta_z u + 4|z|^2 \partial_{tt} u) = \frac{h(\xi)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (40)$$

Now let \mathcal{R} denote a real rotation around t -axis in \mathbb{R}^{2n} and $u_{\mathcal{R}}(z, t) = u(\mathcal{R}z, t)$. Because of the fact that the operator $-(\Delta_z + 4|z|^2 \partial_{tt})$ is also invariant w.r.t. \mathcal{R} . Moreover, proceeding as in the proof of Theorem 1.2, we can also conclude that the solution of problem (40) is unique. Therefore one gets $u(z, t) = u(\mathcal{R}z, t)$ for any rotation \mathcal{R} in \mathbb{R}^{2n} . This means that u is cylindrically symmetric. Further, if $h(r, t) = h(r, -t)$ and $\Phi(r, t) = \Phi(r, -t)$, then we have $u(r, -t) = u(r, t)$ by the same reason. This completes the proof of Theorem 1.3. \square

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