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PROOF OF ONE OPEN INEQUALITY OF LAUB–ILANI TYPE

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Dedicated to my son Ladislav

(*Communicated by J. Pečarić*)

Abstract. In this paper, we prove one new algebraic trigonometric inequality of Laub-Ilani type. This inequality was posted as the Conjecture 2.4 in the paper A. Y. Özban, "New Algebraic-Trigonometric Inequalities of Laub-Ilani type", Bull. Aust. Math. Soc., 2017, doi 10.1017/S0004972717000156.

1. Introduction

Inequalities with power functions have many important applications. Luab-Ilani type inequalities appeared in the "Problems and Solutions" section of the American Mathematical Monthly as Problem E3116 [4]. Inequalities with power functions can be found in mathematical analysis and in other theories like ordinary differential equations, probability theory and statistics, chemistry, economics, mathematical physics, mathematical biology. In the paper [1] the following conjecture was published.

CONJECTURE 1. If $0 < x < y \leqslant \pi/2$, then

$$\cos\left(x^x\right)+\cos\left(y^y\right)<\cos\left(x^y\right)+\cos\left(y^x\right). \tag{1}$$

The aim of this paper is to prove the Conjecture 1.

2. Methods

In this paper, methods of mathematical and numerical analysis are used. We use also the software MATLAB for some computing.

3. Results and discussion

In this section we prove the Conjecture 1.

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3.1. Lemmas and theorem

LEMMA 1. *Let* $p(u) = u\ln(u)\cos(u) + \sin(u)\ln(u) + \sin(u)$ *for* $0 < u < 1$. *Then*

1. *there is only one* u_0 *such that* $0 < u_0 < 1$, $p(u) < 0$ *for* $0 < u < u_0$ *and* $p(u) > 0$ *for* $u_0 < u < 1$ *(*$u_0 \doteq 0.58782$*);*

2. $p(u) > -0.440495$ *for* $0 < u < 1$*;*

3. *if* $p(u^*) = \min\limits_{0<u<1} p(u)$ *then* $0.213 < u^* < 0.214$

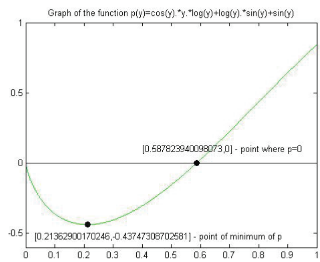

Figure 1: Graph of the function p(y)

Proof. We show that $p'(u) > 0$ for $u > 1/e$ and $p''(u) > 0$ for $0 < u < 1/2$. From $p(0) = 0$, $p(1/2) = -0.157033...$, $p(1) = \sin(1) = 0.84$ we obtain the assertion 1. From $p(0.58782) = -7 * 10^{-6}$, $p(0.58783) = 10^{-5}$ we get $0.58782 < u_0 < 0.58783$. We get

$$p'(u) = 2\cos(u)(1+\ln(u)) + \frac{\sin(u)}{u} - u\ln(u)\sin(u),$$
$$p''(u) = -3\sin(u)(1+\ln(u)) + \frac{3u\cos(u) - \sin(u)}{u^2} - u\ln(u)\cos(u).$$

Using $(u-1)/u < \ln(u) < u-1$ for $0 < u < 1$ we obtain

$$u^2 p''(u) \geqslant q(u) = -(3u^3+1)\sin(u) + (u^3 - u^4 + 3u)\cos(u).$$

Using $u^3 - u^4 > 0$ and $u^3 < u$ we get

$$q(u) \geqslant r(u) = -(3u+1)\sin(u) + 3u\cos(u).$$

To prove $r(u) > 0$ it suffices to show that

$$s(u) = \tan(u) - 1 + \frac{1}{3u+1} < 0.$$

But it follows from $s(0)=0$, $s(1/2)=-0.05369751$ and

$$s''(u)=\frac{2\sin(u)}{\cos^3(u)}+\frac{18}{(3u+1)^3}>0.$$

So p is a convex function on $(0,1/2)$.

Next it is evident that $p'(u)>0$ for $u>1/e$.

From $p'(0)=-\infty$, $p'(u)$ is an increasing function for $0<u<1/2$, $p'(0.5)=1.663584$, $p'(0.213)=-0.006139538\ldots$, $p'(0.214)=0.0036126766$ we obtain $0.213<u^*<0.214$.

Now we prove the assertion 2. Because of $0.213<u^*<0.214$ we have

$$\begin{aligned}p(u)&=\sin(u)+\sin(u)\ln(u)+u\ln(u)\cos(u)\geqslant\\&\sin(0.213)+\sin(0.214)\ln(0.213)+0.214\ln(0.213)\cos(0.213)\geqslant-0.440495.\end{aligned}$$

(We used the monotonicity of $\sin(u),\ln(u),\cos(u)$ on $(0.213,0.214)$.) The proof is complete.

LEMMA 2. *Let* $0<a<x<y<1$. *Then*

$$\frac{xy^x}{yx^y}\geqslant C_a=f(a) \text{ where} \tag{2}$$
$$f(u)=e^{\ln(u)+(1-u)\ln\left(\frac{\ln(u)}{u-1}\right)+1-u}.$$

The constant C_a *is the best possible.*

Proof. (2) is equivalent to

$$f_1(x,y)=\ln(x)+x\ln(y)-\ln(y)-y\ln(x)-\ln(C_a)\geqslant 0.$$

$$\text{If}\quad \frac{\partial f_1(x,y)}{\partial y}=\frac{x-1}{y}-\ln(x)=0\quad\text{then}\quad y_0=\frac{x-1}{\ln(x)}.$$

Next

$$\frac{\partial^2 f_1(x,y)}{\partial y^2}=\frac{1-x}{y^2}>0.$$

So for each x such that $0<a<x<y<1$, $f_1(y)$ is a convex function. If we show $f_1(y_0)\geqslant 0$ then $f_1(y)\geqslant 0$ for $0<a<x<y<1$. Put

$$f_2(x)=f_1(x,y_0)=\ln(x)+(1-x)\ln\left(\frac{-\ln(x)}{1-x}\right)+1-x-\ln(C_a).$$

To show $f_2(x)>0$ it suffices to prove $f_2''(x)<0$ because of $f_2(x=1)=-\ln(C_a)\geqslant 0$ and $f_2(x=a)=0$. Some calculation gives

$$\frac{df_2(x)}{dx}=\frac{1}{x}+\ln(1-x)+\frac{1-x}{x\ln(x)}-\ln(-\ln(x)),$$

$$\frac{d^2 f_2(x)}{dx^2} = \frac{1}{(1-x)x^2 \ln^2(x)} \alpha(x),$$

where

$$\alpha(x) = (x-1-x^2)\ln^2(x) - x(1-x)\ln(x) + (x-1-\ln(x))(1-x).$$

$\alpha(x) < 0$ is equivalent to

$$l(x) = \ln^2(x) - \frac{(1-x^2)}{-x^2+x-1}\ln(x) - \frac{(1-x)^2}{-x^2+x-1} > 0,$$

which is evident because of

$$l(x) = \left(\ln(x) - \frac{(1-x^2)}{2(-x^2+x-1)}\right)^2 + \frac{(1-x)^2(3x^2-6x+3)}{4(-x^2+x-1)^2}.$$

Because of $f_2(a) = 0$ we get C_a is the best constant. It completes the proof.

LEMMA 3. *Let* $0 < a < 1$, $0 < x_2 < 1$ *such that* $f'(x_2) > 0$, *where* $f(x) = (\ln(a)/\ln(x))^x$ *for* $0 < x < 1$. *Then*

$$f(x) = \left(\frac{\ln(a)}{\ln(x)}\right)^x \geqslant K_{a,x_2} = e^{\frac{x_2}{\ln(x_2)}} \quad \textit{for} \quad 0 < x < 1.$$

Proof. Some calculations give

$$f'(x) = \left(\frac{\ln(a)}{\ln(x)}\right)^x \left[\ln\left(\frac{\ln(a)}{\ln(x)}\right) - \frac{1}{\ln(x)}\right],$$

$$f''(x) = \left(\tfrac{\ln(a)}{\ln(x)}\right)^x \left\{\left[\ln\left(\tfrac{\ln(a)}{\ln(x)}\right) - \tfrac{1}{\ln(x)}\right]^2 - \tfrac{1}{x\ln(x)} + \tfrac{1}{x\ln^2(x)}\right\} \geqslant 0.$$

So $f(x)$ is a convex function on $(0,1)$, $f'(x)$ is an increasing function on $(0,1)$. It is easy to see that $\lim_{x\to 0^+} f'(x) = -\infty$, $\lim_{x\to 0^+} f(x) = 1$. Assumptions give there is only one x_0 such that $0 < x_0 < x_2$, $f'(x_0) = 0$ and x_0 is the point of minimum f on $(0,x_2)$. Because of $f'(x_2) > 0$ we obtain x_0 is the minimum of f on $(0,1)$. We also have $\ln\left(\frac{\ln(a)}{\ln(x_0)}\right) = \frac{1}{\ln(x_0)}$.

Put $s(x) = e^{x/\ln(x)}$ for $x \in (0,1)$. Then

$$s'(x) = s(x)\frac{\ln(x)-1}{\ln^2(x)} < 0.$$

So $s(x)$ is a decreasing function on $(0,1)$. It implies $e^{x/\ln(x)} \geqslant e^{x_2/\ln(x_2)}$ for $0 < x \leqslant x_2$. Because of $f(x_0) = e^{x_0/log(x_0)}$ the proof is complete.

Table 1: Values of the coefficient K_{a,x_2}

a	x_2	$f'(x_2)$	$K_{a,x_2} \geqslant$
0.588	0.29762704	$3.4 * 10^{-8}$	0.782247
0.404	0.19068045	$5 * 10^{-9}$	0.891308

NOTE 1. Some applications of the previous lemma.

THEOREM 1. *Let* $0 < x, y \leqslant \pi/2$. *Then*

$$\cos(x^x) + \cos(y^y) < \cos(x^y) + \cos(y^x). \tag{3}$$

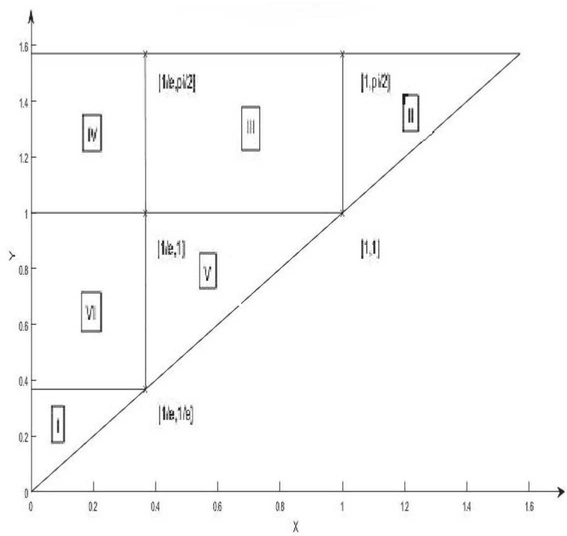

Figure 2: Cases of proof

Proof. We can suppose $0 < x < y \leqslant \pi/2$. There are six cases:

1. $0 < x < y \leqslant 1/e$;
2. $1 \leqslant x < y \leqslant \pi/2$;
3. $1/e \leqslant x \leqslant 1 \leqslant y \leqslant \pi/2$;
4. $0 < x \leqslant 1/e,\ 1 \leqslant y \leqslant \pi/2$;
5. $1/e \leqslant x < y \leqslant 1$;
6. $0 < x \leqslant 1/e \leqslant y \leqslant 1$.

Put

$$g(x,y) = \cos(x^x) + \cos(y^y) - \cos(x^y) - \cos(y^x).$$

Proof of the case 1. Let $0 < x < y \leqslant 1/e$.

Put $g_1(t) = \cos(t^x) - \cos(t^y)$ for $x \leqslant t \leqslant y$. It is evident that if $g_1(x) < g_1(y)$ then $g(x,y) < 0$. So it suffices to prove $g_1'(t) > 0$. We have

$$g_1'(t) = -\sin(t^x)xt^{x-1} + \sin(t^y)yt^{y-1}.$$

Denote $\chi(x,t,y) = g_1'(t)$. Then $\chi_x'(x,t,y) = -t^x p(t^x)$ and $\chi_y'(x,t,y) = t^y p(t^y)$, where p is defined in Lemma 1. Next we have $(1/e)^{1/e} \leqslant x^x \leqslant t^x \leqslant y^x$ because of $x \leqslant t \leqslant y \leqslant 1/e$. Lemma 1 implies $\chi_x'(x,t,y) \neq 0$ ($p(u) > 0$ for $u > (1/e)^{1/e} = 0.6922$). So χ does not have local extremes on $W = \{(x,t,y); 0 < x < t < y < 1/e\}$. It implies if we show $\chi \geqslant 0$ on the boundary of W then the proof of the case 1 will be done.

We need to show the following five inequalities:

(i) $\chi \geqslant 0$ for $t = x$ which is

$$z_1 = -x\sin(x^x)x^x + y\sin(x^y)x^y > 0;$$

(ii) $\chi \geqslant 0$ for $t = y$ which is

$$z_1 = x\sin(y^x)y^x - y\sin(y^y)y^y < 0;$$

(iii) $\chi \geqslant 0$ for $x = 0$ which is

$$z_1 = t\sin(t^y)t^y > 0;$$

(iv) $\chi \geqslant 0$ for $y = 1/e$ which is

$$z_1 = x\sin(t^x)t^x - \frac{1}{e}\sin\left(t^{1/e}\right)t^{1/e} < 0;$$

(v) $\chi \geqslant 0$ for $t = y = x$ which is

$$z_1 = 0.$$

The cases (v),(iii) are evident. We show (i). We have $z_{1y}' = x^y p(x^y)$. From Lemma 1 we obtain $z_{1y}'(y = x) = x^x p(x^x) \geqslant 0$. Let x be fixed. Then $u = x^y$ is a decreasing function in y. Because of $z_1(y = x) = 0$, $z_{1y}'(y = x) > 0$, $u = x^y$ is a decreasing function to prove the case (i) it suffices to show $z_1(y = 1/e) > 0$. This follows from Lemma 1.

We have

$$z_1(y = 1/e) = F(x) = -x\sin(x^x)x^x + \frac{1}{e}\sin\left(x^{1/e}\right)x^{1/e}.$$

Let $0 < a < x < b \leqslant 1/e$. Put $t = x^{1/e}$. Then $0 < a^{1/e} < t < b^{1/e} \leqslant (1/e)^{1/e}$ and $(x^x)\sin(x^x) < (a^a)\sin(a^a)$. ($(x^x)\sin(x^x)$ is a decreasing function on $(0,1/e)$.) So $F \geqslant 0$ for $0 < a < x < b \leqslant 1/e$ is equivalent to

$$G(t) = \sin(t)\frac{1}{e} - t^{e-1}(a^a)\sin(a^a) > 0$$

for $0 < u = a^{1/e} < t < v = b^{1/e} \leqslant (1/e)^{1/e}$. We have

$$G''(t) = -\sin(t)\frac{1}{e} - (e-1)(e-2)t^{e-3}(a^a)\sin(a^a) < 0.$$

So G is a concave function on (u,v). If we show $G(u) \geqslant 0$, $G(v) \geqslant 0$ then $G(t) \geqslant 0$ for $t \in (u,v)$.

Table 2: Values of the function G

a	b	u	v	G(u)	G(v)
0	0.04	0	0.306003	0	0.002253
0.04	0.08	0.306003	0.394883	0.0606333	0.011782
0.08	0.138	0.394883	0.482590	0.0566311	0.001007
0.138	0.21	0.482590	0.5631945	0.0562648	0.0021249
0.21	0.29	0.5631945	0.6342015	0.05201544	0.0015938

The proof of $F(x) > 0$ for $x \in (0, 0.29>$ follows from the table 2.

Now we show $F(x) > 0$ for $x \in (0.29, 1/e>$.

Put

$$\alpha(t) = \frac{1}{e}x^{1/e}\sin(x^{1/e}) - t\sin(x^t)x^t$$

for $0.29 < x < t < 1/e$. We show $\alpha'(t) < 0$. From $\alpha(1/e) = 0$ we obtain $F(x) > 0$ for $x \in (0.29, 1/e>$.

$$\alpha'(t) = -x^t p(x^t).$$

Next we have

$$x^t \geqslant \min\left\{0.29^{1/e}, 0.29^{0.29}, \left(\frac{1}{e}\right)^{1/e}, \left(\frac{1}{e}\right)^{0.29}\right\} = 0.6342015.$$

It implies $p(x^t) \geqslant p(0.6342015) = 0.0900053109 > 0$.

The proof of the case 1 is complete.

Proof of the case 2. Let $1 \leqslant x < y \leqslant \pi/2$.

We show $g'_x(x,y) > 0$. Because of $g(x=y,y) = 0$ we obtain the proof of the case 2. Put

$$q(y) = g'_x(x,y) = -\sin(x^x)x^x(1+\ln(x)) + \sin(x^y)yx^{y-1} + \sin(y^x)y^x\ln(y).$$

From $q(y=x)=0$ it suffices to show

$$q'_y(x,y) = \cos(x^y)x^y\ln(x^y)x^{y-1} + \sin(x^y)x^{y-1} + \sin(x^y)yx^{y-1}\ln(x) +$$
$$\cos(y^x)y^{x-1}y^x\ln(y^x) + \sin(y^x)y^{x-1}\ln(y^x) + \sin(y^x)y^{x-1} \geqslant 0$$

which is evident.

Proof of the case 3. Let $1/e \leqslant x \leqslant 1 \leqslant y \leqslant \pi/2$.

We show again $g'_x(x,y) > 0$ and $g(x=1,y) < 0$ for $1 \leqslant y \leqslant \pi/2$. We have $g(x=1,y) = \cos(y^y) - \cos(y) < 0$ because of $y \leqslant y^y \leqslant \pi$. Next we show $g'_x(x,y) > 0$. We have

$$g'_x(x,y) = -\sin(x^x)x^x(1+\ln(x)) + \sin(x^y)yx^{y-1} + \sin(y^x)y^x\ln(y).$$

There are two cases.

$$3\text{A:}\quad \frac{1}{e} \leqslant x \leqslant 0.72, \qquad 3\text{B:}\quad 0.72 \leqslant x \leqslant 1.$$

Proof of the case 3A. From $1 \leqslant y^x < \pi/2$ it suffices to show that

$$h_1(x,y) = -x\sin(x^x)x^x(1+\ln(x)) + \sin(x^y)x^y + \sin(1)\ln(y^x) > 0.$$

Because of $x^y \geqslant x^{\pi/2}$ we show

$$h_2(x,y) = -x\sin(x^x)x^x(1+\ln(x)) + \sin\left(x^{\pi/2}\right)x^{\pi/2} + \sin(1)\ln(y^x) > 0.$$

From $h'_{2y}(x,y) = x\sin(1)/y > 0$ it suffices to prove

$$d(x) = -x\sin(x^x)x^x(1+\ln(x)) + \sin\left(x^{\pi/2}\right)x^{\pi/2} > 0.$$

It is easy to see that $(\sin(x^x)x^x(1+\ln(x)))'_x \geqslant 0$. Let $1/e \leqslant a \leqslant x \leqslant b \leqslant 0.72$. Put

$$d_{a,b}(x) = -\sin\left(b^b\right)b^b(1+\ln(b)) + \sin\left(x^{\pi/2}\right)x^{\pi/2-1}.$$

If we prove $d_{a,b}(x) \geqslant 0$ on (a,b) then we obtain $d(x) \geqslant 0$ on (a,b). To prove $d_{a,b}(x) \geqslant 0$ on (a,b) it suffices to show $d_{a,b}(a) \geqslant 0$ because of

$$d'_{a,b}(x) = \pi\cos\left(x^{\pi/2}\right)x^{\pi-2}/2 + \left(\frac{\pi}{2}-1\right)\sin\left(x^{\pi/2}\right)x^{\pi/2-2} \geqslant 0.$$

So from the following table 3 we get $d \geqslant 0$ on $(1/e, 0.76)$.

Proof of the case 3B. Let $0.72 \leqslant x \leqslant 1 \leqslant y \leqslant \pi/2$. We show again $g'_x(x,y) \geqslant 0$. We have

$$xg'_x(x,y) = -x\sin(x^x)x^x(1+\ln(x)) + \sin(x^y)yx^y + x\sin(y^x)y^x\ln(y).$$

Put

$$z(x,y) = -x\sin(x^x)x^x(1+\ln(x)) + \sin(x^y)yx^y.$$

Table 3: Values of a, b, $d_{a,b}$

a	b	$d_{a,b}$
1/e	0.47	0.00578269
0.47	0.55	0.004607584
0.55	0.62	0.0082388258
0.62	0.69	0.0056917399
0.69	0.76	0.001336697

We prove $z(x,y=1)>0$ and $z'_y(x,y)\geqslant 0$. It completes our proof of the case 3. Really,

$$z(x,y=1)=-x\sin(x^x)x^x(1+\ln(x))+x\sin(x)\geqslant 0$$

can be shown if we prove

$$q=\sin(x)-\sin(1)x^{x+1}\geqslant 0.$$

(We used $x^x\leqslant 1$ and $\ln(x)\leqslant x-1$.)

$q\geqslant 0$ is equivalent to

$$q_1=-\ln(\sin(x))+\ln(\sin(1))+(x+1)\ln(x)\leqslant 0.$$

We see $q_1(x=1)=0$. So it suffices to prove $q'_1\geqslant 0$. We get

$$q'_1=\ln(x)+\frac{1+x}{x}-\cot(x).$$

We use $\ln(x)>(x-1)/x$. So it suffices to show $2-\cot(x)\geqslant 0$. From monotonicity of $\cot(x)$ we obtain $\cot(x)\leqslant\cot(0.72)=1.14016...<2$. Now we show $z'_y(x,y)\geqslant 0$. From $z'_y(x,y)=x^y p(x^y)$ it suffices to prove $p(x^y)\geqslant 0$. We have

$$x^y\geqslant\min\left\{0.72,1,0.72^{\pi/2}\right\}=0.596895643881881.$$

This implies $p(x^y)\geqslant p(0.596895643881881)=0.017291369410603>0$. The proof of the case 3 is complete.

Proof of the case 4. Let $0<x\leqslant 1/e$, $1\leqslant y\leqslant\pi/2$.

We have

$$g'_x(x,y)=-\sin(x^x)x^x(1+\ln(x))+\sin(x^y)yx^{y-1}+\sin(y^x)y^x\ln(y)>0.$$

So the function $h(x)=g(x,y)$ is an increasing on $(0,1/e)$ for fixed y. If we show $h(1/e)=t(y)<0$ on $1\leqslant y\leqslant\pi/2$ then the proof of the case 4 will be done. We have

$$t(y)=\cos\left(\left(\frac{1}{e}\right)^{1/e}\right)+\cos(y^y)-\cos\left(\left(\frac{1}{e}\right)^{y}\right)-\cos\left(y^{1/e}\right).$$

It is evident that $y^y \geqslant 1$ so $\cos(y^y) \leqslant \cos(1)$. To prove the case 4 it suffices to show

$$t_1(y) = \cos\left(\left(\frac{1}{e}\right)^{1/e}\right) + \cos(1) - \cos\left(\left(\frac{1}{e}\right)^{y}\right) - \cos\left(y^{1/e}\right) < 0$$

which can be rewriting as

$$1.3101 < \cos\left(\left(\frac{1}{e}\right)^{y}\right) + \cos\left(y^{1/e}\right).$$

Because of $\left(\frac{1}{e}\right)^y < 1/e$ it suffices to prove $1.3101 - \cos(1/e) < \cos(y^{1/e})$ which can be rewriting as $\cos(y^{1/e}) > 0.3771$. We have $y^{1/e} < (\pi/2)^{1/e} = 1.1807$. It implies $\cos(y^{1/e}) > \cos(1.1807) = 0.3803 > 0.3771$ which completes the proof of the case 4.

Proof of the case 5. Let $1/e \leqslant x < y \leqslant 1$.

We have

$$yg'_y(x,y) = -\sin(y^y)\,y^y(1+\ln(y))y + \sin(x^y)\,x^y\ln(x^y) + \sin(y^x)\,xy^x.$$

Because of $g(x, y = x) = 0$ it suffices to show $yg'_y(x,y) < 0$. Put $h(x) = yg'_y(x,y)$ for fixed $1 \geqslant y > x$. We obtain

$$h'_x(x) = yx^y p(x^y) + xy^x p(y^x).$$

From $h(x = y) = 0$ it suffices to prove $h'_x(x) > 0$. It is evident that $p(y^x) > 0$ ($y^x \geqslant x^x \geqslant (1/e)^{1/e}$ and $p((1/e)^{1/e}) = 0.207402\ldots$. So $h'_x(x) > 0$ is equivalent to

$$h_1(x,y) = p(x^y) + C_{1/e}p(y^x) > 0.$$

Lemma 2 gives $C_{1/e} \geqslant 0.925018$.

Let $x^y > 0.4788$ then (see Lemmas 1,2)

$$h_1(x,y) = p(x^y) + C_{1/e}p(y^x) > p(0.4788) + 0.925018p((1/e)^{1/e}) = 2.882 * 10^{-4} > 0.$$

If $0.422 < x^y \leqslant 0.4788$ then $y \geqslant -\ln(0.4788)/(-\ln(x)) \geqslant -\ln(0.4788) \geqslant 0.7364$ and $y^x \geqslant 0.7364^x \geqslant 0.7364$. So

$$h_1(x,y) = p(x^y) + C_{1/e}p(y^x) > p(0.422) + 0.925018p(0.7364) = 8.15 * 10^{-4} > 0.$$

If $x^y \leqslant 0.422$ then $y \geqslant -\ln(0.422)/(-\ln(x)) \geqslant -\ln(0.422) \geqslant 0.8627$ and $y^x \geqslant 0.8627^x \geqslant 0.8627$. So

$$h_1(x,y) = p(x^y) + C_{1/e}p(y^x) > p(1/e) + 0.925018p(0.8627) = 0.17894 > 0.$$

This completes the proof of the case 5.

Proof of the case 6. Let $0 < x \leqslant 1/e \leqslant y \leqslant 1$.

There are five cases.

6A: $0 < x \leqslant 0.08$,

6B: $0.08 \leqslant x \leqslant 0.13$,

6C: $0.13 \leqslant x \leqslant 0.165$,

6D: $0.165 \leqslant x \leqslant 0.188$,

6E: $0.188 \leqslant x \leqslant 1/e$.

Proof of the case 6A.

We have $0 < x \leqslant 0.08$ and $(-\cos(y^x))'_y = \sin(y^x)xy^{x-1} > 0$. So $-\cos(y^x)) < -\cos(1)$. It implies

$$g(x,y) \leqslant g_1(x,y) = \cos(x^x) + \cos(y^y) - \cos(1) - \cos(x^y).$$

Some calculation gives

$$g'_{1y}(x,y) = -\sin(y^y)y^y(1+\ln(y)) + \sin(x^y)x^y\ln(x) < 0.$$

So it suffices to prove $g_1(x, y = 1/e) < 0$.

Denote

$$\beta(x) = g_1(x, 1/e) = \cos(x^x) + \cos\left(\left(\frac{1}{e}\right)^{1/e}\right) - \cos(1) - \cos\left(x^{1/e}\right).$$

Because of

$$\beta(0.08) = \cos\left(0.08^{0.08}\right) + \cos\left(\left(\frac{1}{e}\right)^{1/e}\right) - \cos(1) - \cos\left(0.08^{1/e}\right) = -0.009123... < 0$$

it suffices to prove $\beta'(x) > 0$. But it follows from

$$\beta'(x) = -\sin(x^x)x^x(1+\ln(x)) + \frac{1}{e}\sin\left(x^{1/e}\right)x^{1/e-1} > 0.$$

Proof of the case 6B.

We have $0.08 \leqslant x \leqslant 0.13$. Next

$$g(x,y) \leqslant g_2(x,y) = \cos\left(0.13^{0.13}\right) + \cos(y^y) - \cos(x^y) - \cos\left(y^{0.08}\right).$$

It is evident that $g'_{2x}(x,y) = y\sin(x^y)x^{y-1} > 0$. So it suffices to prove

$$g_3(y) = \cos\left(0.13^{0.13}\right) + \cos(y^y) - \cos(0.13^y) - \cos\left(y^{0.08}\right) < 0.$$

Because of $g_3(y = 1/e) = -0.0041331...$ it suffices to show $g'_3(y) < 0$.

Some calculation gives

$$g'_3(y) = -\sin(y^y)y^y(1+\ln(y)) + \sin(0.13^y)0.13^y\ln(0.13) + \sin\left(y^{0.08}\right)0.08y^{-0.92}.$$

We use $y^y \geqslant (1/e)^{1/e}$ so $\sin(y^y) \geqslant \sin((1/e)^{1/e})$, next $y^{0.08} \leqslant 1$ and $y^{-0.92} \leqslant e^{0.92}$. It implies

$$g_3'(y) \leqslant g_4(y) = -\sin\left(\left(\frac{1}{e}\right)^{1/e}\right)\left(\frac{1}{e}\right)^{1/e}(1+\ln(y))+ \sin(0.13^y)0.13^y\ln(0.13)+\sin(1)0.08e^{0.92}.$$

We show $g_4(y) \leqslant 0$. There are two cases

$$\text{6Bb:}\quad \frac{1}{e} \leqslant y \leqslant 0.5, \qquad \text{6Ba:}\quad 0.5 \leqslant y \leqslant 1.$$

Let us consider the case 6Ba. We have $0.13^y > 0.13$. So

$$g_4(y) \leqslant g_{4a}(y) = -\sin\left(\left(\frac{1}{e}\right)^{1/e}\right)\left(\frac{1}{e}\right)^{1/e}(1+\ln(y))+ \sin(0.13)0.13\ln(0.13)+\sin(1)0.08e^{0.92}.$$

Because of $g_{4a}(y=0.5) = -0.00102...$ it suffices to show $g_{4a}' < 0$ which is evident.

Let us consider the case 6Bb. We have $0.13^{0.5} \leqslant 0.13^y \leqslant 0.13^{1/e}$. So

$$g_4(y) \leqslant g_{4b}(y) = -\sin\left(\left(\frac{1}{e}\right)^{1/e}\right)\left(\frac{1}{e}\right)^{1/e}(1+\ln(y))+ \sin(0.13^{0.5})0.13^{0.5}\ln(0.13)+\sin(1)0.08e^{0.92}.$$

Similarly from $g_{4b}(y=1/e) = -0.0905...$ we obtain $g_4(y) \leqslant 0$ for $1/e \leqslant y \leqslant 0.5$.

Proof of the case 6C.

We have $0.13 \leqslant x \leqslant 0.165$. Next $y^x \leqslant y^{0.13}$. It implies

$$g(x,y) \leqslant g_5(x,y) = \cos\left(0.165^{0.165}\right)+\cos\left(y^y\right)-\cos(x^y)-\cos\left(y^{0.13}\right).$$

From $(-\cos(x^y))_x' = \sin(x^y)yx^{y-1} \geqslant 0$ we obtain

$$g_5(x,y) \leqslant g_6(y) = \cos\left(0.165^{0.165}\right)+\cos\left(y^y\right)-\cos(0.165^y)-\cos\left(y^{0.13}\right).$$

Because of $g_6(y=1/e) = -0.002316...$ it suffices to prove $g_6'(y) < 0$. We have

$$g_6'(y) = g_7(y) = -\sin\left(y^y\right)y^y(1+\ln(y))+ \sin(0.165^y)0.165^y\ln(0.165)+\sin(y^{0.13})0.13y^{0.13-1}.$$

Next

$$g_7(y) \leqslant g_8(y) = -\sin\left(\left(\frac{1}{e}\right)^{1/e}\right)\left(\frac{1}{e}\right)^{1/e}(1+\ln(y))+ \sin(0.165^y)0.165^y\ln(0.165)+\sin(1)0.13e^{0.87}.$$

Now we prove $g_8(y) < 0$. There are three cases.

g8a: $1/e \leqslant y \leqslant 0.5$, and we have $0.165^y \geqslant 0.165^{0.5}$,

g8b: $0.5 \leqslant y \leqslant 0.6$, and we have $0.165^y \geqslant 0.165^{0.6}$,

g8c: $0.6 \leqslant y \leqslant 1$, and we have $0.165^y \geqslant 0.165$.

Put

$$g_a(y) = -\sin\left(\left(\frac{1}{e}\right)^{1/e}\right)\left(\frac{1}{e}\right)^{1/e}(1+\ln(y)) + \sin(0.165^a)0.165^a\ln(0.165) + \sin(1)0.13e^{0.87}.$$

The proof of the case 6C follows from the following Table 4. ($g'_{ay}(y) < 0$).

Table 4: Values of $a, y,\ g_a(y)$

a	y	$g_a(y)$
0.5	1/e	-0.02808298
0.6	0.5	-0.077845216
1	0.6	-0.003834876

Proof of the case 6D.
Let us considere $0.165 \leqslant x \leqslant 0.188$. We obtain

$$g(x,y) \leqslant g_9(x,y) = \cos\left(0.188^{0.188}\right) + \cos\left(y^y\right) - \cos(y^{0.165}) - \cos\left(x^y\right).$$

$(-\cos(x^y))'_x \geqslant 0$ implies

$$g_9(x,y) \leqslant g_{10}(y) = \cos\left(0.188^{0.188}\right) + \cos\left(y^y\right) - \cos(y^{0.165}) - \cos\left(0.188^y\right).$$

Because of $g_{10}(y = 1/e) = -0.0041266924$ it suffices to prove $g_{11} = g'_{10y} \leqslant 0$. We have

$$g_{11} = -\sin\left(y^y\right)y^y(1+\ln(y)) + \sin(0.188^y)0.188^y\ln(0.188) + \sin(y^{0.165})0.165e^{0.165-1}.$$

There are four cases.

g11a: $1/e \leqslant y \leqslant 0.48$, and we have $0.188^y \geqslant 0.188^{0.48}$,

g11b: $0.48 \leqslant y \leqslant 0.62$, and we have $0.188^y \geqslant 0.188^{0.62}$,

g11c: $0.62 \leqslant y \leqslant 0.8$, and we have $0.188^y \geqslant 0.188^{0.8}$.

g11d: $0.8 \leqslant y \leqslant 1$, and we have $0.188^y \geqslant 0.188$.

Put

$$g_b(y) = -\sin\left(\left(\frac{1}{e}\right)^{1/e}\right)\left(\frac{1}{e}\right)^{1/e}(1+\ln(y))$$
$$+\sin(0.188^b)0.188^b\ln(0.188)+\sin(1)0.165e^{0.835}.$$

The proof of the case 6D follows from the following Table 5. ($g'_{by}(y) < 0$).

Table 5: Values of $b, y,\ g_b(y)$

b	y	$g_b(y)$
0.48	1/e	-0.00478284
0.62	0.48	-0.0035201217
0.8	0.62	-0.024537437
1	0.8	-0.0819205844

Proof of the case 6E. Let us consider $0.188 \leqslant x < y < 1$. There are four cases.

6E1: $x^y \geqslant 0.588$,

6E2: $0.404 \leqslant x^y \leqslant 0.588$,

6E3: $0.218 \leqslant x^y \leqslant 0.404$,

6E4: $x^y \leqslant 0.218$.

It is easy to show that

$$x\left(yg'_y(x,y)\right)'_x = yx^y p(x^y) + xy^x p(y^x),$$

where p is defined in Lemma 1. Put

$$\lambda(x,y) = yx^y p(x^y) + xy^x p(y^x).$$

If we show $\lambda(x,y) > 0$ for $0.188 \leqslant x < y < 1$ then $yg'_y(x,y)$ is an increasing function in x on $(0.188, y)$. Because of $yg'_y(x=y,y) = 0$ we obtain $g'_y(x,y) < 0$ for fixed $0.188 < x$ and $y \in (x,1)$. Because of $g(x, y=x) = 0$ we get $g(x,y) < 0$ for $0.188 \leqslant x < y < 1$.

From the previous parts we have if $p(x^y) + C_{0.188}p(y^x) \geqslant 0$ then $\lambda(x,y) > 0$ for $0.188 \leqslant x < y < 1$.

Proof of the case E1. Let $x^y \geqslant 0.588$. Lemma 1 gives $p(x^y) > 0$. We have always $p(y^x) > 0$, So $p(x^y) + C_{0.188}p(y^x) \geqslant 0$.

Proof of the case E2. Let $0.404 \leqslant x^y \leqslant 0.588$. Then $y^x \geqslant (\ln(0.588)/\ln(x))^x$. From Note 1 we obtain $y^x \geqslant K_{0.588,x_2} > 0.7822473$.

So

$$p(x^y) + C_{0.188}p(y^x) \geqslant p(0.404) + 0.76097p(0.7822473) = 7*10^{-4} > 0.$$

Proof of the case E3. Let $0.218 \leqslant x^y \leqslant 0.404$. Then $y^x \geqslant (\ln(0.404)/\ln(x))^x$. From Note 1 we obtain $y^x \geqslant K_{0.404,x_2} > 0.89130833$.

So

$$p(x^y) + C_{0.188} p(y^x) \geqslant p(0.218) + 0.76097 p(0.89130833) = 0.037405 > 0.$$

Proof of the case E4. Let $x^y \leqslant 0.218$. Then we have $f'(x = 0.188) = 0.49683 > 0$, $f(x = 0.188) = 0.9827128 > 0$, where f is a function defined in Lemma 3. From Lemma 1 we obtain

$$p(x^y) + C_{0.188} p(y^x) \geqslant -0.4405 + 0.76097 p(0.9827128) = 0.3674646 > 0.$$

The proof is complete.

4. Conclusion

In this paper, we proved the following open problem:

If $0 < x < y \leqslant \pi/2$, then

$$\cos(x^x) + \cos(y^y) < \cos(x^y) + \cos(y^x).$$

5. Competing interests

The author declares that he has no competing interests.

6. Authors' contributions

The author completed the paper and approved the final manuscripts.

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