

## WEIGHTED INEQUALITIES FOR THE MULTILINEAR HILBERT AND CALDERÓN OPERATORS AND APPLICATIONS

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*Abstract.* We characterize the weighted weak and strong type inequalities for the Hilbert and Calderón multilinear operators. As applications, we characterize a weighted multilinear Hilbert's inequality and extend to the multilinear setting some results on singular integrals due to F. Soria and G. Weiss.

### 1. Introduction and results.

The Hilbert operator, also known as Stieltjes transform, is defined for non negative functions  $f$  on  $(0, \infty)$  by

$$\mathcal{H}f(x) = \int_0^\infty \frac{f(t)}{x+t} dt, \quad x \in (0, \infty).$$

Another classical operator, closely related to  $\mathcal{H}$ , is the Calderón operator  $\mathcal{C}$ , defined also for non negative functions  $f$  on  $(0, \infty)$  by the sum of the Hardy averaging operator  $P$  and its adjoint  $Q$ , i. e.,

$$\mathcal{C}f(x) = Pf(x) + Qf(x) = \frac{1}{x} \int_0^x f(t) dt + \int_x^\infty \frac{f(t)}{t} dt.$$

K. Andersen proved in [1] that if  $p > 1$ , then the weighted inequality

$$\int_0^\infty \mathcal{H}f(x)^p w(x) dx \leq K \int_0^\infty f^p w$$

holds for all non negative  $f$  with a constant  $K$  independent of  $f$  if and only if the positive function  $w$  verifies the following condition: there exists a constant  $K > 0$  such that for all  $b > 0$ , the inequality

$$\left( \int_0^b w \right)^{\frac{1}{p}} \left( \int_0^b \sigma \right)^{\frac{1}{p'}} \leq Kb \tag{1.1}$$

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holds, where  $p'$  is the conjugate exponent of  $p$  and  $\sigma = w^{1-p'}$ .

The same result holds for  $\mathcal{C}$ , since  $\frac{1}{2}\mathcal{C}f(x) \leq \mathcal{H}f(x) \leq \mathcal{C}f(x)$  for all  $f$  and  $x \in (0, \infty)$ .

More recently, J. Duoandikoetxea, F. J. Martín Reyes and S. Ombrosi have studied in [8] the same problem with a different perspective. Specifically, they have defined the maximal operator

$$\mathcal{N}f(x) = \sup_{b>x} \frac{1}{b} \int_0^b |f|, \quad x \in (0, \infty)$$

and proved that if  $p > 1$ , then  $\mathcal{N}$  is bounded in  $L^p(u)$  if and only if  $u$  verifies condition (1.1), which they call  $A_{p,0}$ . Then, they note that

$$Pf(x) \leq \mathcal{N}f(x) \leq \mathcal{C}f(x)$$

and an argument of duality shows that  $\mathcal{C}$  is bounded in  $L^p(u)$  if and only if  $u \in A_{p,0}$ .

In this paper we will deal with the  $m$ -linear Hilbert and Calderón operators. The first one is defined in [5] for  $m$ -tuples  $(f_1, f_2, \dots, f_m)$  of non negative functions on  $(0, \infty)$  by

$$\mathcal{H}(f_1, f_2, \dots, f_m)(x) = \int_{(0, \infty)^m} \frac{f_1(y_1)f_2(y_2) \cdots f_m(y_m)}{(x+y_1+y_2+\cdots+y_m)^m} dy_1 dy_2 \cdots dy_m.$$

We also define the  $m$ -linear Calderón operator as

$$\mathcal{C}(f_1, f_2, \dots, f_m)(x) = \prod_{i=1}^m Pf_i(x) + \sum_{i=1}^m Q(f_i \prod_{\substack{j=1 \\ j \neq i}}^m Pf_j)(x),$$

i. e., as the sum of the  $m$ -linear Hardy averaging operator  $\prod_{i=1}^m Pf_i$  and its  $m$  adjoints  $Q(f_i \prod_{j=1, j \neq i}^m Pf_j)$ ,  $i \in \{1, 2, \dots, m\}$ .

These operators are related as follows: there are two positive constants  $K_1$  and  $K_2$  independent of  $f_1, f_2, \dots, f_m$  and  $x$  such that

$$\mathcal{C}(f_1, f_2, \dots, f_m)(x) \leq K_1 \mathcal{H}(f_1, f_2, \dots, f_m)(x) \leq K_2 \mathcal{C}(f_1, f_2, \dots, f_m)(x). \quad (1.2)$$

Inspired by [8] and [13], we can define a new  $m$ -(sub)linear maximal operator  $\mathcal{N}$  as follows:

$$\mathcal{N}(f_1, f_2, \dots, f_m)(x) = \sup_{b>x} \prod_{j=1}^m \left( \frac{1}{b} \int_0^b f_j \right),$$

which will help us to characterize the weighted weak and strong type inequalities for the operators  $\mathcal{H}$  and  $\mathcal{C}$ .

Our main results characterize the good weights for the operators  $\mathcal{N}$ ,  $\mathcal{H}$  and  $\mathcal{C}$ . The first theorem deals with the strong type inequality for the operator  $\mathcal{N}$ .

**THEOREM 1.** *Let  $p > 0$  and let  $p_1, p_2, \dots, p_m > 1$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$ . Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $(0, \infty)$ ,  $\vec{v} = (v_1, v_2, \dots, v_m)$  and  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}}$ . The next statements are equivalent:*

- (i) The operator  $\mathcal{N}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^p(w)$ .
- (ii) The operator  $\mathcal{N}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .
- (iii)  $\vec{v} \in A_{\vec{p},0}$ , which means that there is  $K > 0$  such that for each  $b > 0$ ,

$$\left(\frac{1}{b} \int_0^b w\right)^{\frac{1}{p}} \prod_{j=1}^m \left(\frac{1}{b} \int_0^b \sigma_j\right)^{\frac{1}{p'_j}} \leq K, \tag{1.3}$$

where  $\sigma_j = v_j^{1-p'_j}$ .

In the following Theorem, we deal with the weak type inequality. We admit  $p_i = 1$  for some  $i$  and we do not require the weights to verify  $w = \prod_{j=1}^m v_j^{\frac{p}{p'_j}}$ .

**THEOREM 2.** Let  $p > 0$  and let  $p_1, p_2, \dots, p_m \geq 1$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$ . Let  $w, v_1, v_2, \dots, v_m$  be positive measurable functions on  $(0, \infty)$  and  $\vec{v} = (v_1, v_2, \dots, v_m)$ . The next statements are equivalent:

- (i) The operator  $\mathcal{N}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .
- (ii)  $(w, \vec{v}) \in A_{\vec{p},0}$ , which means that (1.3) holds, where  $\left(\frac{1}{b} \int_0^b \sigma_j\right)^{\frac{1}{p'_j}}$  is understood as  $(\text{ess inf}_{(0,b)} v_j)^{-1}$  for  $p_j = 1$ .

The next result characterizes the good weights for the strong and weak type inequalities of  $\mathcal{H}$  and  $\mathcal{C}$  in the case  $p_i > 1$  for each  $i$  and  $p > 1$ .

**THEOREM 3.** Let  $p > 1$  and let  $p_1, p_2, \dots, p_m > 1$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$ . Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $(0, \infty)$ ,  $\vec{v} = (v_1, v_2, \dots, v_m)$  and  $w = \prod_{j=1}^m v_j^{\frac{p}{p'_j}}$ . The next statements are equivalent:

- (i) The Hilbert operator  $\mathcal{H}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^p(w)$ .
- (ii) The Hilbert operator  $\mathcal{H}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .
- (iii) The Calderón operator  $\mathcal{C}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^p(w)$ .
- (iv) The Calderón operator  $\mathcal{C}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .
- (v)  $\vec{v} \in A_{\vec{p},0}$ .

Now, we state the weak type theorem in the case  $p_i \geq 1$  and  $p > 0$ . It reads as follows:

**THEOREM 4.** *Let  $p > 0$  and let  $p_1, p_2, \dots, p_m \geq 1$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$  and some  $p_i > 1$ . Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $(0, \infty)$ ,  $\vec{v} = (v_1, v_2, \dots, v_m)$  and  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}}$ . The next statements are equivalent:*

- (i) *The Hilbert operator  $\mathcal{H}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .*
- (ii) *The Calderón operator  $\mathcal{C}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .*
- (iii)  $\vec{v} \in A_{\vec{p},0}$ .

The previous Theorem does not include the extreme case  $p_i = 1$  for all  $i \in \{1, 2, \dots, m\}$ . We deal with it in Theorem 5, where the relationship  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}}$  is not required. The result is the next one:

**THEOREM 5.** *Let  $w, v_1, v_2, \dots, v_m$  be positive measurable functions on  $(0, \infty)$  and  $\vec{v} = (v_1, v_2, \dots, v_m)$ . The next statements are equivalent:*

- (i) *The Hilbert operator  $\mathcal{H}$  is bounded from  $L^1(v_1) \times L^1(v_2) \times \dots \times L^1(v_m)$  to  $L^{\frac{1}{m},\infty}(w)$ .*
- (ii) *The Calderón operator  $\mathcal{C}$  is bounded from  $L^1(v_1) \times L^1(v_2) \times \dots \times L^1(v_m)$  to  $L^{\frac{1}{m},\infty}(w)$ .*
- (iii)  $(w, \vec{v}) \in A_{(1,1,\dots,1),0}$ , which means that there is a positive constant  $K$  such that

$$\left(\frac{1}{b} \int_0^b w\right)^m \leq K \prod_{j=1}^m \operatorname{ess\,inf}_{(0,b)} v_j$$

for all  $b > 0$ .

These results can be extended to higher dimensions. Specifically, let us consider the operator  $\mathcal{N}_n$  defined for  $m$ -tuples  $(f_1, f_2, \dots, f_m)$  of measurable functions on  $\mathbb{R}^n$  and  $x \in \mathbb{R}^n$  by

$$\mathcal{N}_n(f_1, f_2, \dots, f_m)(x) = \sup_{b>|x|} \prod_{j=1}^m \left(\frac{1}{b^n} \int_{|y|<b} |f_j|\right)$$

and the  $n$ -dimensional  $m$ -linear Hilbert operator  $\mathcal{H}_n$  and Calderón operator  $\mathcal{C}_n$  defined by

$$\mathcal{H}_n(f_1, f_2, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} \frac{f_1(y_1) \cdots f_m(y_m)}{(|x| + |y_1| + \dots + |y_m|)^{nm}} dy_1 \dots dy_m$$

and

$$\mathcal{C}_n(f_1, f_2, \dots, f_m)(x) = \prod_{j=1}^m P_n f_j(x) + \sum_{j=1}^m Q_n(f_j \prod_{\substack{i=1 \\ i \neq j}}^m P_n f_i)(x),$$

respectively, where  $P_n$  stands for the  $n$ -dimensional Hardy averaging operator  $P_n$  defined by

$$P_n f(x) = \frac{1}{|x|^n} \int_{|y| < |x|} f(y) dy$$

and  $Q_n$  is its adjoint,

$$Q_n f(x) = \int_{|y| > |x|} \frac{f(y)}{|y|^n} dy.$$

Since the operators  $\mathcal{N}_n$ ,  $\mathcal{H}_n$  and  $\mathcal{C}_n$  are radial, the characterizations of their weighted inequalities are immediate consequences of Theorems 1, 2, 3, 4 and 5. The results are the following ones.

**THEOREM 6.** *Let  $p > 1$  and let  $p_1, p_2, \dots, p_m > 1$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$ . Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $\mathbb{R}^n$ ,  $\vec{v} = (v_1, v_2, \dots, v_m)$  and  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}}$ . The next statements are equivalent:*

- (i) *The operator  $\mathcal{N}_n$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^p(w)$ .*
- (ii) *The operator  $\mathcal{N}_n$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .*
- (iii) *The operator  $\mathcal{C}_n$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^p(w)$ .*
- (iv) *The operator  $\mathcal{C}_n$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .*
- (v) *The operator  $\mathcal{H}_n$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^p(w)$ .*
- (vi) *The operator  $\mathcal{H}_n$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .*
- (vii)  *$\vec{v} \in A_{\vec{p},0}$ , which means that there is  $K > 0$  such that for each  $b > 0$ ,*

$$\left( \frac{1}{b^n} \int_{|x| < b} w \right)^{\frac{1}{p}} \prod_{j=1}^m \left( \frac{1}{b^n} \int_{|x| < b} \sigma_j \right)^{\frac{1}{p_j}} \leq K.$$

**THEOREM 7.** *Let  $p > 0$  and let  $p_1, p_2, \dots, p_m \geq 1$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$  and some  $p_j > 1$ . Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $\mathbb{R}^n$ ,  $\vec{v} = (v_1, v_2, \dots, v_m)$  and  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}}$ . The next statements are equivalent:*

- (i) *The operator  $\mathcal{N}_n$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .*
- (ii) *The operator  $\mathcal{C}_n$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .*
- (iii) *The operator  $\mathcal{H}_n$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ .*
- (iv)  *$\vec{v} \in A_{\vec{p},0}$ .*

**THEOREM 8.** *Let  $w, v_1, v_2, \dots, v_m$  be positive measurable functions on  $\mathbb{R}^n$  and  $\vec{v} = (v_1, v_2, \dots, v_m)$ . The next statements are equivalent:*

- (i) The Hilbert operator  $\mathcal{H}_n$  is bounded from  $L^1(v_1) \times L^1(v_2) \times \dots \times L^1(v_m)$  to  $L^{\frac{1}{m}, \infty}(w)$ .
- (ii) The Calderón operator  $\mathcal{C}_n$  is bounded from  $L^1(v_1) \times L^1(v_2) \times \dots \times L^1(v_m)$  to  $L^{\frac{1}{m}, \infty}(w)$ .
- (iii)  $(w, \vec{v}) \in A_{(1,1,\dots,1),0}$ , which means that there is a positive constant  $K$  such that

$$\left( \frac{1}{b^n} \int_{|x|<b} w \right)^m \leq K \prod_{j=1}^m \operatorname{ess\,inf}_{|x|<b} v_j$$

for all  $b > 0$ .

The first application of the above results deals with Hilbert’s inequality. The boundedness of the Hilbert operator  $\mathcal{H}$  is closely related to the celebrated Hilbert’s inequality [12], which asserts that if  $p > 1$ , then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin \frac{\pi}{p}} \left( \int_0^\infty f^p \right)^{\frac{1}{p}} \left( \int_0^\infty g^{p'} \right)^{\frac{1}{p'}}.$$

It is clear that this inequality holds if and only if

$$\int_0^\infty (\mathcal{H}f)^p \leq \left( \frac{\pi}{\sin \frac{\pi}{p}} \right)^p \int_0^\infty f^p.$$

This relationship remains valid in the weighted case, even in the multilinear setting. As a simple consequence of our previous theorems, we have the following result:

**THEOREM 9.** *Let  $p > 1$  and  $p_1, p_2, \dots, p_m > 1$  with  $\frac{1}{p} = \sum_i \frac{1}{p_i}$ . Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $(0, \infty)$ ,  $\vec{v} = (v_1, v_2, \dots, v_m)$ ,  $w = \prod_j v_j^{\frac{p}{p_j}}$  and  $\sigma = w^{1-p'}$ . Then the weighted multilinear Hilbert’s inequality*

$$\int_{(0,\infty)^{m+1}} \frac{f(y)f_1(y_1)\dots f_m(y_m)}{(y+y_1+\dots+y_m)^m} dy dy_1 \dots dy_m \leq K \|f\|_{p',\sigma} \|f_1\|_{p_1,v_1} \dots \|f_m\|_{p_m,v_m} \quad (1.4)$$

holds if and only if  $\vec{v} \in A_{\vec{p},0}$ .

We only have to observe that, by duality, (1.4) holds if and only if

$$\left( \int_0^\infty \left( \int_{(0,\infty)^m} \frac{f_1(y_1)\dots f_m(y_m)}{(y+y_1+\dots+y_m)^m} dy_1 \dots dy_m \right)^p w(y) dy \right)^{\frac{1}{p}} \leq K \|f_1\|_{p_1,v_1} \dots \|f_m\|_{p_m,v_m}$$

and apply Theorem 3.

The above result extends the weighted multilinear Hilbert’s inequality obtained in [9], where the authors only worked with power weights.

As a consequence of Theorem 6, it is also immediate to characterize a weighted  $n$ -dimensional multilinear Hilbert’s inequality. It is included in the next result.

**THEOREM 10.** *Let  $p > 1$  and  $p_1, p_2, \dots, p_m > 1$  with  $\frac{1}{p} = \sum_i \frac{1}{p_i}$ . Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $\mathbb{R}^n$ ,  $\vec{v} = (v_1, v_2, \dots, v_m)$ ,  $w = \prod_j v_j^{\frac{p}{p_j}}$  and  $\sigma = w^{1-p'}$ . Then the weighted  $n$ -dimensional multilinear Hilbert's inequality*

$$\int_{(\mathbb{R}^n)^{m+1}} \frac{f(y)f_1(y_1)\dots f_m(y_m)}{(|y| + |y_1| + \dots + |y_m|)^{nm}} dy dy_1 \dots dy_m \leq K \|f\|_{p', \sigma} \|f_1\|_{p_1, v_1} \dots \|f_m\|_{p_m, v_m}$$

holds if and only if  $\vec{v} \in A_{\vec{p}, 0}$ .

As a second application, we obtain some weighted inequalities for multilinear singular integrals. If  $T$  is a linear operator bounded on  $L^p(\mathbb{R}^n)$ ,  $p > 1$ , for which there is a constant  $K > 0$  such that

$$|Tf(x)| \leq K \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^n} dy$$

for every  $f \in L^1(\mathbb{R}^n)$  with compact support and every  $x \notin \text{supp}(f)$ , it is well known that  $T$  can be dominated by the sum of a local operator and the  $n$ -dimensional Calderón operator  $P_n + Q_n$ . This kind of estimates allows to get weighted  $L^p$  inequalities for  $T$ , whenever the weights are essentially constant in dyadic crowns and verify the conditions for the Calderón operator to be bounded in the weighted  $L^p$  spaces. This result was proved in [15]. See also [2] and [14] for related results.

We are going to extend this result to the multilinear setting. Specifically, assume that  $T$  is a multilinear operator for which there is a positive constant  $K$  such that

$$|T(f_1, f_2, \dots, f_m)(x)| \leq K \int_{\mathbb{R}^{mm}} \frac{|f_1(y_1)||f_2(y_2)| \dots |f_m(y_m)|}{(|x - y_1| + |x - y_2| + \dots + |x - y_m|)^{mm}} dy_1 dy_2 \dots dy_m \tag{1.5}$$

for each  $m$ -tuple  $(f_1, f_2, \dots, f_m) \in L^1(\mathbb{R}^n) \times \dots \times L^1(\mathbb{R}^n)$  of compactly supported functions and every  $x \notin \bigcap_{i=1}^m \text{supp}(f_i)$ . Such an operator  $T$  will be called a multilinear singular integral. It is clear that Calderón-Zygmund multilinear operators, defined in [11], are multilinear singular integrals, but there are more examples, as, for instance, the rough bilinear singular integrals defined in [6]. Very recently, Grafakos, He and Honzik have proved in [10] that these operators map boundedly  $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$  for  $1 < p_1, p_2 < \infty$  and  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ .

As in the linear case, we can see that a multilinear singular integral  $T$  can be dominated by the sum of a local multilinear operator and the multilinear Calderón operator  $\mathcal{C}_n$ . In fact, for fixed  $x \in \mathbb{R}^n$  we have

$$|T(f_1, f_2, \dots, f_m)(x)| \leq |L(f_1, f_2, \dots, f_m)(x)| + \mathcal{C}_n(|f_1|, |f_2|, \dots, |f_m|)(x), \tag{1.6}$$

where  $L$  is the local part, which will be defined later.

The results are the next ones, where Theorem 11 is the strong type result and Theorems 12 and 13 are the weak type ones.

**THEOREM 11.** *Let  $1 < p, p_1, p_2, \dots, p_m < \infty$  with  $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p}$ ,  $\vec{p} = (p_1, p_2, \dots, p_m)$ . Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $\mathbb{R}^n$  and  $\sigma_i = v_i^{1-p'_i}$ ,  $i = 1, 2, \dots, m$ .*

Let  $w = \prod_{i=1}^m v_i^{\frac{p}{p_i}}$ . Let  $T$  be a multilinear singular integral bounded from  $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$  to  $L^p(\mathbb{R}^n)$ . Assume that there is a positive constant  $K$  such that the inequality

$$\sup_{2^{k-2} < |y| \leq 2^{k+1}} w(y) \leq K \prod_{i=1}^m \left( \inf_{2^{k-2} < |y| \leq 2^{k+1}} v_i(y) \right)^{\frac{p}{p_i}} \tag{1.7}$$

holds for all integer  $k$ . Assume also that  $(v_1, v_2, \dots, v_m) \in A_{\vec{p}, 0}$ . Then,  $T$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^p(w)$ .

**THEOREM 12.** Let  $p > 0$ ,  $1 \leq p_1, p_2, \dots, p_m < \infty$  with  $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p}$  and  $p_j > 1$  for some  $j$ . Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $\mathbb{R}^n$ ,  $\vec{v} = (v_1, v_2, \dots, v_m)$  and  $w = \prod_{i=1}^m v_i^{\frac{p}{p_i}}$ . Let  $T$  be a multilinear singular integral bounded from  $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$  to  $L^{p, \infty}(\mathbb{R}^n)$ . Assume that (1.7) holds and that  $\vec{v} \in A_{\vec{p}, 0}$ . Then,  $T$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p, \infty}(w)$ .

**THEOREM 13.** Let  $v_1, v_2, \dots, v_m$  be positive measurable functions on  $\mathbb{R}^n$  and  $\vec{v} = (v_1, v_2, \dots, v_m)$ . Let  $T$  be a multilinear singular integral bounded from  $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \times \dots \times L^1(\mathbb{R}^n)$  to  $L^{\frac{1}{m}, \infty}(\mathbb{R}^n)$ . Assume that (1.7) holds and that  $(w, \vec{v}) \in A_{(1, 1, \dots, 1), 0}$ . Then,  $T$  is bounded from  $L^1(v_1) \times L^1(v_2) \times \dots \times L^1(v_m)$  to  $L^{\frac{1}{m}, \infty}(w)$ .

It is clear that power weights  $v_i(x) = |x|^{\alpha_i}$  verify condition (1.7), but there are more weights satisfying it. In particular, the weights  $v_i(x) = |x|^{\alpha_i} (\log(1 + |x|))^{\beta_i}$  for suitable  $\alpha_i$  and  $\beta_i$ .

It is worth noting that using Theorems 11, 12 and 13 we obtain weighted weak or strong type inequalities for multilinear singular integrals verifying (1.5) with weights satisfying (1.7) whenever we previously know that the operator is bounded without weights. For the particular case of rough bilinear singular integrals, the only works we know about weighted inequalities are the papers [4] and [7]. Both show results for  $1 < p_1, p_2 < \infty$  and  $p > \frac{1}{2}$  with  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ , but working with weights  $v_1, v_2$  which are separately in the Muckenhoupt classes  $A_{p_1}$  and  $A_{p_2}$ , respectively. The interest of our results is that our weights verify a joint  $A_{\vec{p}, 0}$  condition.

The next sections consist of the proofs of the results. All along the paper, the letter  $K$  stands for a positive constant, not necessarily the same at each occurrence. Moreover, we always understand  $(\frac{1}{t} \int_0^t \sigma_i)^{\frac{1}{p_i}}$  as  $(\text{ess inf}_{(0,t)} v_i)^{-1}$  for  $p_i = 1$ . This does not cause any problem when applying Hölder’s inequality.

## 2. Proof of Theorem 1

*Proof.* The implication (i)  $\Rightarrow$  (ii) is clear. Let us prove the other two implications. (iii)  $\Rightarrow$  (i)

We may assume, without loss of generality, that  $f_1, f_2, \dots, f_m$  are compactly supported. First of all, let us note that  $\mathcal{N}(f_1, f_2, \dots, f_m)$  is decreasing. In fact, if  $x < y$ , each



$b > y$  verifies also  $b > x$ , which implies  $\mathcal{N}(f_1, f_2, \dots, f_m)(x) \geq \mathcal{N}(f_1, f_2, \dots, f_m)(y)$ . Since  $\mathcal{N}(f_1, f_2, \dots, f_m)$  decreases and  $f_i, i \in \{1, 2, \dots, m\}$ , are compactly supported, for every  $k \in \mathbb{Z}$  there is  $b_k > 0$  such that  $O_k = \{x \in (0, \infty) : \mathcal{N}(f_1, f_2, \dots, f_m)(x) > 2^k\} = (0, b_k)$ . Then, we have

$$\begin{aligned}
 \int_0^\infty \mathcal{N}(f_1, f_2, \dots, f_m)^p w &= \sum_{k \in \mathbb{Z}} \int_{\{x/2^k < \mathcal{N}(f_1, f_2, \dots, f_m)(x) \leq 2^{k+1}\}} \mathcal{N}(f_1, f_2, \dots, f_m)^p w \\
 &= \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} \mathcal{N}(f_1, f_2, \dots, f_m)^p w.
 \end{aligned} \tag{2.1}$$

We will need the following Lemma:

LEMMA 1. For each  $k \in \mathbb{Z}$ ,

$$\prod_{i=1}^m \left( \frac{1}{b_k} \int_0^{b_k} f_i \right) = 2^k.$$

*Proof.* Since  $b_k \notin O_k$ , we have that  $\mathcal{N}(f_1, f_2, \dots, f_m)(b_k) \leq 2^k$ . Then for each  $c > b_k$ ,

$$\frac{1}{c^m} \prod_{i=1}^m \left( \int_0^c f_i \right) \leq 2^k,$$

which implies,

$$\frac{1}{b_k^m} \prod_{i=1}^m \left( \int_0^{b_k} f_i \right) \leq 2^k.$$

If we had a strict inequality, as the function

$$\varphi(t) = \frac{1}{t^m} \prod_{i=1}^m \left( \int_0^t f_i \right)$$

is continuous, there would be  $\delta > 0$  such that  $\varphi(c) < 2^k$  for all  $c \in (b_k - \delta, b_k)$ . Let  $x_0 \in (b_k - \delta, b_k)$ . Then, for each  $c > x_0$ ,  $\varphi(c) \leq 2^k$  (it is clear for each  $c$  with  $x_0 < c < b_k$ , but also for  $c$  verifying  $c \geq b_k$ : as they are greater than  $b_k$ , we have also  $\varphi(c) \leq 2^k$ ). Then,  $\mathcal{N}(f_1, f_2, \dots, f_m)(x_0) \leq 2^k$ , what is a contradiction, because  $x_0 \in (0, b_k) = O_k$ .

Applying Lemma 1, multiplying and dividing by  $\prod_{i=1}^m \sigma_i(0, b_k)^p$ , where  $\sigma_i(0, b_k) = \int_0^{b_k} \sigma_i$ , and applying the  $A_{\vec{p}, 0}$  condition for the weights, we have that the right-hand

side of (2.1) is less or equal than

$$\begin{aligned}
 \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} 2^{(k+1)p} w(x) dx &= 2^p \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} 2^{kp} w(x) dx \\
 &= 2^p \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} \frac{1}{b_k^{mp}} \prod_{i=1}^m \left( \int_0^{b_k} f_i \right)^p w(x) dx \\
 &\leq K \sum_{k \in \mathbb{Z}} \prod_{i=1}^m \left( \frac{1}{\sigma_i(0, b_k)} \int_0^{b_k} f_i \right)^p \left( \int_0^{b_k} \sigma_i \right)^{\frac{p}{p_i}}.
 \end{aligned} \tag{2.2}$$

We will need to prove two more Lemmas:

LEMMA 2. For each  $k \in \mathbb{Z}$ ,

$$\frac{b_{k+1}}{b_k} \leq \frac{1}{\sqrt[m]{2}}.$$

*Proof.* Since we have

$$2^k = \frac{1}{b_k^m} \prod_{i=1}^m \left( \int_0^{b_k} f_i \right), \quad 2^{k+1} = \frac{1}{b_{k+1}^m} \prod_{i=1}^m \left( \int_0^{b_{k+1}} f_i \right)$$

and  $b_{k+1} < b_k$ , by dividing we obtain  $\frac{1}{2} \geq \frac{b_{k+1}^m}{b_k^m}$ , which implies  $\frac{b_{k+1}}{b_k} \leq \frac{1}{\sqrt[m]{2}}$ .

LEMMA 3. There is  $K > 0$  such that the inequality

$$\int_0^{b_k} \sigma_i \leq K \int_{b_{k+1}}^{b_k} \sigma_i$$

holds for all  $k \in \mathbb{Z}$  and all  $i \in \{1, 2, \dots, m\}$ .

*Proof.* As a consequence of Lemma 2, we have  $b_k - b_{k+1} \geq b_k - \frac{b_k}{\sqrt[m]{2}} = \left(1 - \frac{1}{\sqrt[m]{2}}\right) b_k$ .

Applying this, the  $A_{\vec{p}, 0}$  condition, Hölder's inequality and also that  $w = \prod_{i=1}^m v_i^{\frac{p}{p_i}}$ , we obtain

$$\begin{aligned}
 \left( \int_0^{b_k} w \right)^{\frac{1}{p}} \prod_{i=1}^m \left( \int_0^{b_k} \sigma_i \right)^{\frac{1}{p_i}} &\leq K b_k^m \leq K (b_k - b_{k+1})^m = K \left( \int_{b_{k+1}}^{b_k} 1 \right)^m \\
 &\leq K \left( \int_{b_{k+1}}^{b_k} w \right)^{\frac{1}{p}} \prod_{i=1}^m \left( \int_{b_{k+1}}^{b_k} \sigma_i \right)^{\frac{1}{p_i}} \\
 &\leq K \left( \int_0^{b_k} w \right)^{\frac{1}{p}} \left( \int_{b_{k+1}}^{b_k} \sigma_1 \right)^{\frac{1}{p_1}} \prod_{i=2}^m \left( \int_0^{b_k} \sigma_i \right)^{\frac{1}{p_i}}.
 \end{aligned}$$

Simplifying,

$$\int_0^{b_k} \sigma_1 \leq K \int_{b_{k+1}}^{b_k} \sigma_1.$$

The argument for  $\sigma_2, \dots, \sigma_m$  is the same as above.

Applying Hölder’s inequality and Lemma 3, we have that the last term in (2.2) is less or equal than

$$\begin{aligned} & K \prod_{i=1}^m \left( \sum_{k \in \mathbb{Z}} \left( \frac{1}{\sigma_i(0, b_k)} \int_0^{b_k} f_i \right)^{p_i} \left( \int_0^{b_k} \sigma_i \right) \right)^{\frac{p}{p_i}} \\ & \leq K \prod_{i=1}^m \left( \sum_{k \in \mathbb{Z}} \left( \frac{1}{\sigma_i(0, b_k)} \int_0^{b_k} f_i \right)^{p_i} \left( \int_{b_{k+1}}^{b_k} \sigma_i \right) \right)^{\frac{p}{p_i}} \\ & = K \prod_{i=1}^m \left( \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} \left( \frac{1}{\sigma_i(0, b_k)} \int_0^{b_k} f_i \right)^{p_i} \sigma_i(x) dx \right)^{\frac{p}{p_i}}. \end{aligned} \tag{2.3}$$

Now, let us consider the maximal operators

$$N_{\sigma_i}(h)(x) = \sup_{b > x} \frac{1}{\sigma_i(0, b)} \int_0^b |h(t)| \sigma_i(t) dt,$$

for  $i \in \{1, 2, \dots, m\}$ . For each  $x \in (b_{k+1}, b_k)$ ,

$$\frac{1}{\sigma_i(0, b_k)} \int_0^{b_k} \frac{f(t)}{\sigma_i(t)} \sigma_i(t) dt \leq N_{\sigma_i} \left( \frac{f}{\sigma_i} \right) (x),$$

$i \in \{1, 2, \dots, m\}$ . Then, applying that the operators  $N_{\sigma_i}$  are bounded in  $L^{p_i}(\sigma_i)$ , we have that the last term in (2.3) is less or equal than

$$\begin{aligned} & K \prod_{i=1}^m \left( \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} \left( N_{\sigma_i} \left( \frac{f}{\sigma_i} \right) (x) \right)^{p_i} \sigma_i(x) dx \right)^{\frac{p}{p_i}} \\ & = K \prod_{i=1}^m \left( \int_0^\infty \left( N_{\sigma_i} \left( \frac{f}{\sigma_i} \right) (x) \right)^{p_i} \sigma_i(x) dx \right)^{\frac{p}{p_i}} \\ & \leq K \prod_{i=1}^m \left( \int_0^\infty \left( \frac{f}{\sigma_i} \right)^{p_i} (x) \sigma_i(x) dx \right)^{\frac{p}{p_i}} \\ & = K \prod_{i=1}^m \|f_i\|_{p_i, \nu_i}^p. \end{aligned}$$

(ii)  $\Rightarrow$  (iii)

Let  $b \in (0, \infty)$ ,  $f_i = \chi_{(0, b)} \sigma_i$ ,  $i \in \{1, 2, \dots, m\}$ ,  $\lambda_0 = \prod_{i=1}^m \left( \frac{1}{b} \int_0^b \sigma_i \right)$  and  $0 < \alpha <$

1. Then  $(0, b) \subset \{x \in (0, \infty) : \mathcal{N}(f_1, f_2, \dots, f_m)(x) > \alpha \lambda_0\}$  and, by (ii), we have

$$\left( \int_0^b w \right)^{\frac{1}{p}} \leq \frac{K}{\alpha \lambda_0} \prod_{i=1}^m \left( \int_0^b \sigma_i \right)^{\frac{1}{p_i}},$$

which means that  $(v_1, v_2, \dots, v_m) \in A_{\vec{p}, 0}$ , letting  $\alpha$  tend to 1.

### 3. Proof of Theorem 2.

(ii)  $\Rightarrow$  (i)

Assume that  $(w, \vec{v}) \in A_{\vec{p}, 0}$ . This means that there is  $K > 0$  such that

$$\left(\frac{1}{b} \int_0^b w\right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{b} \int_0^b \sigma_i\right)^{\frac{1}{p_i}} \leq K \tag{3.1}$$

for all  $b > 0$ . Let  $\lambda > 0$ . Since  $\mathcal{N}(f_1, f_2, \dots, f_m)$  decreases, there is  $b > 0$  such that  $O_\lambda = \{x \in (0, \infty) : \mathcal{N}(f_1, f_2, \dots, f_m)(x) > \lambda\} = (0, b)$ , where  $b$  verifies

$$\prod_{i=1}^m \left(\frac{1}{b} \int_0^b f_i\right) = \lambda.$$

Then, by Hölder’s inequality and condition (3.1),

$$\begin{aligned} \int_{O_\lambda} w &= \int_0^b w = \frac{1}{\lambda^p} \left(\int_0^b w\right) \prod_{i=1}^m \left(\frac{1}{b} \int_0^b f_i\right)^p \\ &\leq \frac{1}{\lambda^p} \left(\int_0^b w\right) \prod_{i=1}^m \left(\frac{1}{b} \int_0^b \sigma_i\right)^{\frac{p}{p_i}} \prod_{i=1}^m \left(\frac{1}{b} \int_0^b f_i^{p_i} v_i\right)^{\frac{p}{p_i}} \\ &\leq \frac{K}{\lambda^p} b^p \prod_{i=1}^m \left(\frac{1}{b} \int_0^b f_i^{p_i} v_i\right)^{\frac{p}{p_i}} = \frac{K}{\lambda^p} \prod_{i=1}^m \|f_i\|_{p_i, v_i}^p. \end{aligned}$$

This proves the weighted weak type inequality.

(i)  $\Rightarrow$  (ii)

Assume now that  $\mathcal{N}$  is bounded from  $L^{p_1}(v_1) \times \dots \times L^{p_m}(v_m)$  to  $L^{p, \infty}(w)$ . Let  $b > 0$ . For each  $i \in \{1, \dots, m\}$  such that  $p_i > 1$ , let  $f_i = \sigma_i \chi_{(0, b)}$  and for each  $i$  such that  $p_i = 1$ , let  $f_i = \chi_{E_i}$ , where  $E_i$  is a measurable subset of  $(0, b)$ . If  $x \in (0, b)$ , we have

$$\mathcal{N}(f_1, \dots, f_m)(x) \geq \prod_{\substack{i=1 \\ p_i \neq 1}}^m \left(\frac{1}{b} \int_0^b \sigma_i\right) \prod_{\substack{i=1 \\ p_i = 1}}^m \left(\frac{1}{b} |E_i|\right).$$

This means that  $(0, b) \subset \{x \in (0, \infty) : \mathcal{N}(f_1, \dots, f_m)(x) > \lambda_0\}$ , where

$$\lambda_0 = \prod_{\substack{i=1 \\ p_i \neq 1}}^m \left(\frac{1}{b} \int_0^b \sigma_i\right) \prod_{\substack{i=1 \\ p_i = 1}}^m \left(\frac{1}{b} |E_i|\right).$$

Then, by the weak type inequality,

$$\int_0^b w \leq \frac{K}{\lambda_0^p} \prod_{\substack{i=1 \\ p_i \neq 1}}^m \left(\int_0^b \sigma_i\right)^{\frac{p}{p_i}} \prod_{\substack{i=1 \\ p_i = 1}}^m \left(\int_{E_i} v_i\right)^p,$$

i.e.,

$$\prod_{i=1}^m \left(\frac{1}{b} |E_i|\right)^p \left(\int_0^b w\right) \prod_{\substack{i=1 \\ p_i \neq 1}}^m \left(\frac{1}{b} \int_0^b \sigma_i\right)^{\frac{p}{p_i}} \prod_{\substack{i=1 \\ p_i \neq 1}}^m b^{-\frac{p}{p_i}} \leq K \prod_{i=1}^m v_i(E_i)^p,$$

which implies

$$\prod_{i=1}^m |E_i|^p \left(\frac{1}{b} \int_0^b w\right) \prod_{\substack{i=1 \\ p_i \neq 1}}^m \left(\frac{1}{b} \int_0^b \sigma_i\right)^{\frac{p}{p_i}} \leq K \prod_{i=1}^m v_i(E_i)^p. \tag{3.2}$$

Let  $\varepsilon > 0$ , let  $y_i \in (0, b)$  for each  $i$  with  $p_i = 1$  and let  $E_i = (y_i - \varepsilon, y_i)$ . Then, by (3.2), we have

$$\left(\frac{1}{b} \int_0^b w\right) \prod_{\substack{i=1 \\ p_i \neq 1}}^m \left(\frac{1}{b} \int_0^b \sigma_i\right)^{\frac{p}{p_i}} \leq K \prod_{i=1}^m \left(\frac{1}{\varepsilon} \int_{y_i-\varepsilon}^{y_i} v_i\right)^p.$$

Letting  $\varepsilon \rightarrow 0^+$ , and applying Lebesgue’s differentiation Theorem, we obtain

$$\left(\frac{1}{b} \int_0^b w\right) \prod_{\substack{i=1 \\ p_i \neq 1}}^m \left(\frac{1}{b} \int_0^b \sigma_i\right)^{\frac{p}{p_i}} \leq K \prod_{i=1}^m v_i(y_i)^p$$

for almost every  $y_i \in (0, b)$ , which is equivalent to  $A_{\vec{p},0}$ .

#### 4. Proof of Theorem 3.

Firstly, by (1.2), it suffices to prove the equivalence of (iii), (iv) and (v). It is clear that (iii) implies (iv). In order to prove the remainder implications, we will need the following lemma.

LEMMA 4. *There exists a positive constant  $K$  such that the inequality*

$$\prod_{i=1}^m P f_i(x) \leq \mathcal{N}(f_1, f_2, \dots, f_m)(x) \leq K \mathcal{C}(f_1, f_2, \dots, f_m)(x) \tag{4.1}$$

holds for all non negative functions  $f_1, f_2, \dots, f_m$  and all  $x \in (0, \infty)$ .

*Proof.* The left hand side inequality in (4.1) is clear. In order to prove the other inequality we will work in the case  $m = 2$ . The general case follows by induction on  $m$ . Let  $f, g$  be positive measurable functions on  $(0, \infty)$ ,  $x \in (0, \infty)$  and  $b > x$ . Then,

$$\begin{aligned} \left(\frac{1}{b} \int_0^b f\right) \left(\frac{1}{b} \int_0^b g\right) &= \frac{1}{b^2} \left(\int_0^x f(t) \left(\int_0^t g(s) ds\right) dt\right) + \frac{1}{b^2} \left(\int_x^b f(t) \left(\int_0^t g(s) ds\right) dt\right) \\ &+ \frac{1}{b^2} \left(\int_0^x f(t) \left(\int_t^b g(s) ds\right) dt\right) + \frac{1}{b^2} \left(\int_x^b f(t) \left(\int_t^b g(s) ds\right) dt\right) \\ &= I + II + III + IV. \end{aligned}$$

It is clear that  $I \leq Pf(x)Pg(x)$ . In order to estimate  $II$ , we work as follows:

$$II \leq \int_x^b \frac{f(t)}{t} \left( \frac{1}{t} \int_0^t g(s) ds \right) dt \leq \int_x^\infty \frac{f(t)}{t} \left( \frac{1}{t} \int_0^t g \right) dt = Q(f \cdot Pg)(x).$$

The estimation of  $III$  is the next one:

$$\begin{aligned} III &= \frac{1}{b^2} \int_0^x f(t) \left( \int_t^x g(s) ds \right) dt + \frac{1}{b^2} \int_0^x f(t) \left( \int_x^b g(s) ds \right) dt \\ &\leq Pf(x)Pg(x) + \frac{1}{b^2} \left( \int_x^b g(s) ds \right) \left( \int_0^x f(t) dt \right) \\ &\leq Pf(x)Pg(x) + \frac{1}{b^2} \int_x^b g(s) \left( \int_0^s f(t) dt \right) ds \\ &\leq Pf(x)Pg(x) + \int_x^b \frac{g(s)}{s} \left( \frac{1}{s} \int_0^s f(t) dt \right) ds \leq Pf(x)Pg(x) + Q(g \cdot Pf)(x). \end{aligned}$$

Finally, by Fubini's theorem, we have

$$\begin{aligned} IV &= \frac{1}{b^2} \int_x^b \left( \int_x^s f(t) dt \right) g(s) ds \leq \frac{1}{b^2} \int_x^b \left( \int_0^s f(t) dt \right) g(s) ds \\ &\leq \int_x^b \frac{g(s)}{s} \left( \frac{1}{s} \int_0^s f \right) ds \leq Q(g \cdot Pf)(x). \end{aligned}$$

Thus,

$$\mathcal{N}(f, g)(x) \leq KPf(x)Pg(x) + Q(g \cdot Pf)(x) + Q(f \cdot Pg)(x) \leq K\mathcal{C}(f, g)(x),$$

what finishes the proof of the Lemma.

(iv)  $\Rightarrow$  (v)

Assume that (iv) holds. Then, by (4.1), the maximal multilinear operator  $\mathcal{N}$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^{p, \infty}(w)$  and applying Theorem 1, this implies  $\vec{v} \in A_{\vec{p}, 0}$ .

(v)  $\Rightarrow$  (iii)

Assume that  $\vec{v} \in A_{\vec{p}, 0}$ . On one hand, by Theorem 1 and (4.1), the multilinear Hardy averaging operator  $\prod_{i=1}^m Pf_i$  is bounded from  $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$  to  $L^p(w)$ . On the other hand, the structure of the condition, the fact that  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}}$  and

$$\frac{1}{p_i} = \frac{1}{p'} + \sum_{\substack{j=1 \\ j \neq i}}^m \frac{1}{p_j}$$

for all  $i \in \{1, 2, \dots, m\}$  imply that

$$(v_1, \dots, v_{i-1}, \sigma, v_{i+1}, \dots, v_m) \in A_{(p_1, \dots, p_{i-1}, p', p_{i+1}, \dots, p_m), 0}$$

and  $\sigma_i = v_1^{\frac{p'_i}{p_1}} \cdots v_{i-1}^{\frac{p'_i}{p_{i-1}}} \sigma^{\frac{p'_i}{p'}} v_{i+1}^{\frac{p'_i}{p_{i+1}}} \cdots v_m^{\frac{p'_i}{p_m}}$  for all  $i \in \{1, 2, \dots, m\}$ , where  $\sigma = w^{1-p'}$ . Therefore, by Theorem 1 and (4.1),  $\mathcal{N}$  and  $\prod_{j=1}^m P f_j$  are bounded from

$$L^{p_1}(v_1) \times \cdots \times L^{p_{i-1}}(v_{i-1}) \times L^{p'}(\sigma) \times L^{p_{i+1}}(v_{i+1}) \times \cdots \times L^{p_m}(v_m) \quad \text{to} \quad L^{p'_i}(\sigma_i)$$

for all  $i \in \{1, 2, \dots, m\}$ . Then, the operators  $Q(f_i \prod_{j=1, j \neq i}^m P f_j)$ ,  $i \in \{1, 2, \dots, m\}$ , which are the  $m$ -linear adjoints of  $\prod_{j=1}^m P f_j$ , are bounded from

$$L^{p_1}(v_1) \times L^{p_2}(v_2) \times \cdots \times L^{p_m}(v_m) \quad \text{to} \quad L^p(w)$$

(see [3]). Thus, the Calderón operator  $\mathcal{C}$  is bounded from

$$L^{p_1}(v_1) \times L^{p_2}(v_2) \times \cdots \times L^{p_m}(v_m) \quad \text{to} \quad L^p(w).$$

### 5. Proofs of Theorems 4 and 5.

We will prove Theorems 4 and 5 simultaneously. It is clear that (i) and (ii) are equivalent and also that (ii)  $\Rightarrow$  (iii) in both results, because the Calderón operator dominates  $\mathcal{N}$  and the weak type boundedness of  $\mathcal{N}$  is equivalent to (iii) (see Theorem 2). Let us see that (iii) implies (i).

Assume that  $\vec{v} \in A_{\vec{p},0}$  (for Theorem 4) or  $(w, \vec{v}) \in A_{\vec{p},0}$  (for Theorem 5). Then, by Theorem 2,  $\mathcal{N}$  is bounded from  $L^{p_1}(v_1) \times \cdots \times L^{p_m}(v_m)$  to  $L^{p,\infty}(w)$ . This implies that the multilinear Hardy operator  $\prod_{i=1}^m P f_i$  is also bounded, since  $\prod_{i=1}^m P f_i \leq \mathcal{N}$ . It only remains to prove that the adjoints of the multilinear Hardy operator are bounded.

For fixed  $i \in \{1, 2, \dots, m\}$ , let us see that  $Q(f_i \prod_{j=1, j \neq i}^m P f_j)$  is bounded. Assume first that  $p_i = 1$ . We will use the next condition, which is a straightforward consequence of  $A_{\vec{p},0}$ : there exists  $K > 0$  such that

$$\left(\frac{1}{b} \int_0^b w\right)^{\frac{1}{p}} \prod_{\substack{j=1 \\ j \neq i}}^m \left(\frac{1}{b} \int_0^b \sigma_j\right)^{\frac{1}{p'_j}} \leq K v_i(t) \tag{5.1}$$

for almost every  $t \in (0, b)$ . Let  $\lambda > 0$ . As  $Q(f_i \prod_{j=1, j \neq i}^m P f_j)$  decreases, the set

$$O_\lambda = \left\{x \in (0, \infty) : Q\left(f_i \prod_{\substack{j=1 \\ j \neq i}}^m P f_j\right)(x) > \lambda\right\}$$

is an interval  $(0, b)$ , where

$$\int_b^\infty \frac{f_i(t)}{t} \prod_{\substack{j=1 \\ j \neq i}}^m \left(\frac{1}{t} \int_0^t f_j\right) dt = \lambda.$$

Then, applying this fact, Hölder’s inequality and (5.1), we have

$$\begin{aligned}
 \int_{O_\lambda} w &= \int_0^b w = \left( \int_0^b w \right) \frac{1}{\lambda^p} \left( \int_b^\infty \frac{f_i(t)}{t} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \int_0^t f_j \right) dt \right)^p \\
 &\leq \frac{1}{\lambda^p} \left( \int_b^\infty \frac{f_i(t)}{t} \left( \int_0^t w \right)^{\frac{1}{p}} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \int_0^t f_j \right) dt \right)^p \\
 &\leq \frac{1}{\lambda^p} \left( \int_b^\infty \frac{f_i(t)}{t} \left( \int_0^t w \right)^{\frac{1}{p}} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \int_0^t f_j^{p_j} v_j \right)^{\frac{1}{p_j}} \left( \frac{1}{t} \int_0^t \sigma_j \right)^{\frac{1}{p_j}} dt \right)^p \\
 &\leq \frac{K}{\lambda^p} \left( \int_b^\infty t^{\frac{1}{p}} \frac{f_i(t)}{t} v_i(t) \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \int_0^t f_j^{p_j} v_j \right)^{\frac{1}{p_j}} dt \right)^p \\
 &\leq \frac{K}{\lambda^p} \left( \int_b^\infty t^{\frac{1}{p}} \frac{f_i(t)}{t} v_i(t) \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \right)^{\frac{1}{p_j}} \|f_j\|_{p_j, v_j} dt \right)^p \\
 &\leq \frac{K}{\lambda^p} \left( \int_b^\infty t^{\frac{1}{p}-1-\sum_{j \neq i} \frac{1}{p_j}} f_i(t) v_i(t) dt \right)^p \prod_{\substack{j=1 \\ j \neq i}}^m \|f_j\|_{p_j, v_j}^p \leq \frac{K}{\lambda^p} \prod_{\substack{j=1 \\ j \neq i}}^m \|f_j\|_{p_j, v_j}^p.
 \end{aligned}$$

This finishes the proof of Theorem 5, since in that result  $p_i = 1$  for all  $i$ . In order to finish the proof of Theorem 4, we have to consider the case  $p_i \neq 1$ . Recall that, in this case, it will be necessary  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}}$ . We will need three Lemmas:

LEMMA 1. *Let  $p > 0$  and let  $p_1, p_2, \dots, p_m \geq 1$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$  and some  $p_j > 1$ . If  $\vec{v} = (v_1, v_2, \dots, v_m) \in A_{\vec{p}, 0}$ , then  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}} \in A_{mp, 0}$ .*

We do not prove Lemma 1 since its proof is essentially included in the one of Theorem 3.6 in [13].

LEMMA 2. *Let  $p > 0$  and  $p_1, p_2, \dots, p_m \geq 1$  with  $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$  and some  $p_j > 1$ . Let  $\vec{v} \in A_{\vec{p}, 0}$  and  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}}$ . Then, there is  $A < 1$  such that*

$$\int_0^b w \leq A \int_0^{2b} w$$

for all  $b > 0$ .



*Proof.* Applying Lemma 1, we have that  $w \in A_{mp,0}$ , i. e., there is  $K > 0$  such that for all  $b > 0$  the inequality

$$\left(\frac{1}{b} \int_0^b w\right)^{\frac{1}{mp}} \left(\frac{1}{b} \int_0^b \sigma\right)^{\frac{1}{(mp)'}} \leq K \tag{5.2}$$

holds, where  $\sigma = w^{1-(mp)'}$ . Let  $b > 0$ . By (5.2) and Hölder’s inequality, we get

$$\begin{aligned} \left(\int_0^{2b} w\right) \left(\int_0^{2b} \sigma\right)^{mp-1} &\leq K 2^{mp} b^{mp} = K 2^{mp} \left(\int_b^{2b} 1\right)^{mp} \\ &= K 2^{mp} \left(\int_b^{2b} w^{\frac{1}{mp}} w^{-\frac{1}{mp}}\right)^{mp} \leq K \left(\int_b^{2b} w\right) \left(\int_b^{2b} \sigma\right)^{mp-1}. \end{aligned}$$

Simplifying, we obtain

$$\int_0^b w \leq K \int_b^{2b} w.$$

This is equivalent to

$$\int_0^b w \leq K \int_0^{2b} w - K \int_0^b w,$$

i. e.,

$$\int_0^b w \leq \frac{K}{K+1} \int_0^{2b} w = A \int_0^{2b} w,$$

where  $A = \frac{K}{K+1} < 1$ .

LEMMA 3. Let  $p > 0$  and  $p_1, p_2, \dots, p_m \geq 1$  with  $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$  and some  $p_j >$

1. Let  $\vec{v} \in A_{\vec{p},0}$  and  $w = \prod_{j=1}^m v_j^{\frac{p}{p_j}}$ . Then, there is  $K > 0$  such that for each  $i \in \{1, 2, \dots, m\}$  with  $p_i > 1$  and each  $b > 0$ ,

$$\left(\int_0^b w\right)^{\frac{1}{p}} \left(\int_b^\infty \frac{\sigma_i(t)}{t^{p_i}} \prod_{\substack{j=1 \\ j \neq i}}^m \left(\left(\frac{1}{t} \int_0^t \sigma_j\right)^{\frac{p_j'}{p_j}} \frac{1}{t^{\frac{p_j'}{p_j}}}\right) dt\right)^{\frac{1}{p_i}} \leq K.$$

*Proof.* We have that

$$\begin{aligned}
 & \left( \int_0^b w \right)^{\frac{1}{p}} \left( \int_b^\infty \frac{\sigma_i(t)}{t^{p'_i}} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \left( \frac{1}{t} \int_0^t \sigma_j \right)^{\frac{p'_j}{p_j}} \frac{1}{t^{\frac{p'_j}{p_j}}} \right) dt \right)^{\frac{1}{p'_i}} \\
 &= \left( \int_0^b w \right)^{\frac{1}{p}} \left( \sum_{k=0}^\infty \int_{2^k b}^{2^{k+1} b} \frac{\sigma_i(t)}{t^{p'_i}} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \left( \frac{1}{t} \int_0^t \sigma_j \right)^{\frac{p'_j}{p_j}} \frac{1}{t^{\frac{p'_j}{p_j}}} \right) dt \right)^{\frac{1}{p'_i}} \\
 &\leq \left( \int_0^b w \right)^{\frac{1}{p}} \left( \sum_{k=0}^\infty \frac{1}{(2^k b)^{mp'_i}} \left( \int_{2^k b}^{2^{k+1} b} \sigma_i \right) \prod_{\substack{j=1 \\ j \neq i}}^m \left( \int_0^{2^{k+1} b} \sigma_j \right)^{\frac{p'_j}{p_j}} \right)^{\frac{1}{p'_i}} \\
 &\leq \left( \sum_{k=0}^\infty \frac{1}{(2^k b)^{mp'_i}} \left( \int_0^b w \right)^{\frac{p'_i}{p}} \left( \int_0^{2^{k+1} b} \sigma_i \right) \prod_{\substack{j=1 \\ j \neq i}}^m \left( \int_0^{2^{k+1} b} \sigma_j \right)^{\frac{p'_j}{p_j}} \right)^{\frac{1}{p'_i}}. \tag{5.3}
 \end{aligned}$$

Now, applying Lemma 2 and condition  $A_{\vec{p},0}$ , the last term of (5.3) is less than

$$\begin{aligned}
 & \left( \sum_{k=0}^\infty \frac{1}{(2^k b)^{mp'_i}} A_{\vec{p}}^{\frac{p'_i}{p}(k+1)} \left( \int_0^{2^{k+1} b} w \right)^{\frac{p'_i}{p}} \left( \int_0^{2^{k+1} b} \sigma_i \right) \prod_{\substack{j=1 \\ j \neq i}}^m \left( \int_0^{2^{k+1} b} \sigma_j \right)^{\frac{p'_j}{p_j}} \right)^{\frac{1}{p'_i}} \\
 &\leq \left( \sum_{k=0}^\infty \frac{1}{(2^k b)^{mp'_i}} (2^{k+1} b)^{mp'_i} A_{\vec{p}}^{\frac{p'_i}{p}(k+1)} \right)^{\frac{1}{p'_i}} = K.
 \end{aligned}$$

Finally, we can complete the proof of the implication (iii)  $\Rightarrow$  (i) of Theorem 4. As in the previous case, we have

$$\int_{\{x \in (0, \infty) / Q(f_i \prod_{\substack{j=1 \\ j \neq i}}^m P f_j)(x) > \lambda\}} w = \int_0^b w = \left( \int_0^b w \right) \frac{1}{\lambda^p} \left( \int_b^\infty \frac{f_i(t)}{t} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \int_0^t f_j \right) dt \right)^p$$

for some  $b > 0$ . Applying Hölder’s inequality and Lemma 3, we obtain

$$\begin{aligned}
 & \left( \int_0^b w \right) \left( \int_b^\infty \frac{f_i(t)}{t} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \int_0^t f_j \right) dt \right)^p \\
 &\leq \left( \int_0^b w \right) \left( \int_b^\infty f_i(t)^{p_i} v_i(t) dt \right)^{\frac{p}{p_i}} \left( \int_b^\infty \frac{\sigma_i(t)}{t^{p'_i}} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \int_0^t f_j \right)^{p'_j} dt \right)^{\frac{p}{p'_i}}
 \end{aligned}$$

$$\begin{aligned} &\leq \|f_i\|_{v_i, p_i}^p \left( \int_0^b w \right) \left( \int_b^\infty \frac{\sigma_i(t)}{t^{p_i}} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \int_0^t \sigma_j \right)^{\frac{p_j'}{p_j}} \left( \frac{1}{t} \int_0^t f_j^{p_j} v_j \right)^{\frac{p_j'}{p_j}} dt \right)^{\frac{p}{p_i}} \\ &\leq \prod_{j=1}^m \|f_j\|_{v_j, p_j}^p \left( \int_0^b w \right) \left( \int_b^\infty \frac{\sigma_i(t)}{t^{p_i}} \prod_{\substack{j=1 \\ j \neq i}}^m \left( \frac{1}{t} \int_0^t \sigma_j \right)^{\frac{p_j'}{p_j}} \frac{1}{t^{\frac{p_i'}{p_i}}} dt \right)^{\frac{p}{p_i}} \\ &\leq K \prod_{j=1}^m \|f_j\|_{v_j, p_j}^p. \end{aligned}$$

**6. Proofs of Theorems 11, 12 and 13.**

Firstly, we will see that if  $T$  is a  $m$ -linear singular integral, then there are a constant  $K > 0$  and a local  $m$ -linear operator  $L$  such that the inequality

$$|T(f_1, f_2, \dots, f_m)(x)| \leq |L(f_1, f_2, \dots, f_m)(x)| + K\mathcal{C}_n(|f_1|, |f_2|, \dots, |f_m|)(x) \tag{6.1}$$

holds for all  $m$ -tuples  $(f_1, f_2, \dots, f_m)$  of measurable functions and all  $x \in \mathbb{R}^n$ . Without loss of generality, we will work in the bilinear case, i. e., assuming  $m = 2$ .

Let  $f, g$  be measurable functions on  $\mathbb{R}^n$  and, following the notation in [15], let, for every integer  $k$ ,  $I_k = \{x : 2^{k-1} \leq |x| < 2^k\}$ ,  $I_k^* = \{x : 2^{k-2} \leq |x| < 2^{k+1}\}$ ,  $f_{k,0} = f\chi_{I_k^*}$ ,  $g_{k,0} = g\chi_{I_k^*}$ ,  $f_{k,1} = f - f_{k,0}$  and  $g_{k,1} = g - g_{k,0}$ . Then

$$T(f, g)(x) = \sum_{k \in \mathbb{Z}} T(f, g)(x)\chi_{I_k}(x) = L(f, g)(x) + G(f, g)(x),$$

where

$$L(f, g) = \sum_{k \in \mathbb{Z}} T(f_{k,0}, g_{k,0})\chi_{I_k} \tag{6.2}$$

is the local part and

$$G(f, g) = \sum_{k \in \mathbb{Z}} T(f_{k,1}, g_{k,0})\chi_{I_k} + \sum_{k \in \mathbb{Z}} T(f_{k,0}, g_{k,1})\chi_{I_k} + \sum_{k \in \mathbb{Z}} T(f_{k,1}, g_{k,1})\chi_{I_k} = I + II + III$$

is the global one.

In order to prove the boundedness of the local part, we will apply condition (1.7) and the fact that  $T$  is bounded without weights. We only show the estimation for the

strong type boundedness, since the weak type one is similar:

$$\begin{aligned}
 \int_{\mathbb{R}^n} |L(f, g)(x)|^p w(x) dx &= \sum_{k \in \mathbb{Z}} \int_{I_k} |T(f_{k,0}, g_{k,0})(x)|^p w(x) dx \\
 &\leq \sum_{k \in \mathbb{Z}} \left( \sup_{x \in I_k^*} w(x) \right) \int_{I_k} |T(f_{k,0}, g_{k,0})(x)|^p dx \\
 &\leq K \sum_{k \in \mathbb{Z}} \left( \sup_{x \in I_k^*} w(x) \right) \left( \int_{I_k^*} |f|^{p_1} \right)^{\frac{p}{p_1}} \left( \int_{I_k^*} |g|^{p_2} \right)^{\frac{p}{p_2}} \\
 &\leq K \sum_{k \in \mathbb{Z}} \left( \int_{I_k^*} |f|^{p_1} v_1 \right)^{\frac{p}{p_1}} \left( \int_{I_k^*} |g|^{p_2} v_2 \right)^{\frac{p}{p_2}} \leq K \|f\|_{p_1, v_1}^p \|g\|_{p_2, v_2}^p,
 \end{aligned}$$

where the last inequality holds by Hölder’s inequality.

For the global part  $G$ , we will see that the operator  $G$  is dominated by the bilinear Calderón operator  $\mathcal{C}_n$  and this immediately gives that  $G$  is bounded by applying Theorems 6, 7 or 8, depending on the case. Let  $x \in I_k$ . Then, by (1.5) and the definitions of  $f_{k,1}$  and  $g_{k,0}$ , we have

$$\begin{aligned}
 |I| &= |T(f_{k,1}, g_{k,0})(x)| \leq \int_{\mathbb{R}^{2n}} \frac{|f_{k,1}(y_1)| |g_{k,0}(y_2)|}{(|x - y_1| + |x - y_2|)^{2n}} dy_1 dy_2 \\
 &= \int_{|y_1| > 2^{k+1}} \int_{2^{k-2} \leq |y_2| < 2^{k+1}} \frac{|f(y_1)| |g(y_2)|}{(|x - y_1| + |x - y_2|)^{2n}} dy_1 dy_2 \\
 &\quad + \int_{|y_1| < 2^{k-2}} \int_{2^{k-2} \leq |y_2| < 2^{k+1}} \frac{|f(y_1)| |g(y_2)|}{(|x - y_1| + |x - y_2|)^{2n}} dy_1 dy_2 \\
 &= A_1 + A_2.
 \end{aligned} \tag{6.3}$$

For the estimation of  $A_1$  we have to take into account that, since  $x \in I_k$  and  $y_1 \notin I_k^*$ , then  $|x - y_1|^{2n} \sim |x|^{2n} + |y_1|^{2n}$ . Therefore

$$\begin{aligned}
 A_1 &\leq \int_{|y_1| > 2^{k+1}} \frac{|f(y_1)|}{|x - y_1|^{2n}} \left( \int_{|y_2| < |y_1|} |g(y_2)| dy_2 \right) dy_1 \\
 &\leq K \int_{|y_1| > 2^{k+1}} \frac{|f(y_1)|}{|x|^{2n} + |y_1|^{2n}} \left( \int_{|y_2| < |y_1|} |g(y_2)| dy_2 \right) dy_1 \\
 &\leq K \int_{|y_1| > |x|} \frac{|f(y_1)|}{|y_1|^n} \left( \frac{1}{|y_1|^n} \int_{|y_2| < |y_1|} |g(y_2)| dy_2 \right) dy_1 \\
 &= K Q_n(|f| P_n |g|)(x).
 \end{aligned}$$

The estimation of  $A_2$  requires to split the integral in  $y_2$  and to observe that  $4|x| > |y_2|$ :

$$\begin{aligned} A_2 &\leq \int_{2^{k-2} \leq |y_2| < |x|} \int_{|y_1| < 2^{k-2}} \frac{|f(y_1)||g(y_2)|}{|x|^{2n}} dy_1 dy_2 \\ &\quad + \int_{|x| \leq |y_2| < 2^{k+1}} \int_{|y_1| < 2^{k-2}} \frac{|f(y_1)||g(y_2)|}{|x|^{2n}} dy_1 dy_2 \\ &\leq P_n |f|(x) P_n |g|(x) + K \int_{|x| \leq |y_2|} \frac{|g(y_2)|}{|y_2|^n} \left( \frac{1}{|y_2|^n} \int_{|y_1| < |y_2|} |f(y_1)| dy_1 \right) dy_2 \\ &= P_n |f|(x) P_n |g|(x) + K Q_n(|g| P_n |f|)(x). \end{aligned}$$

The estimation of  $II$  is similar. Finally, for the estimation of  $III$ , we observe that, since  $x \in I_k$  and  $y_i \notin I_k^*$  for  $i = 1, 2$ , then  $|x - y_i| \sim |x| + |y_i|$  for  $i = 1, 2$  and, therefore,

$$|III| \leq K \int_{\mathbb{R}^{2n}} \frac{|f_{k,1}(y_1)||g_{k,1}(y_2)|}{(|x| + |y_1| + |y_2|)^{2n}} dy_1 dy_2 \leq K \mathcal{H}_n(f, g)(x).$$

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