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SOME INEQUALITIES INVOLVING OPERATOR MEANS AND MONOTONE CONVEX FUNCTIONS

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Abstract. Utilizing the Mond-Pečarić method and the properties of operator means, we present some Ando's type inequalities. Some inequalities in the existing literature are also generalized by our results.

1. Introduction

Throughout this paper, $\mathbb{B}(\mathscr{H})$ is the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. For convenience, we denote $A \geqslant 0$ (respectively, $A > 0$) if A is a positive (respectively, positive invertible) operator. The order between operators is that in which $A \leqslant B$ means $B - A$ is positive. We denote by $\mathbf{1}_{\mathscr{H}}$ the identity operator of $\mathscr{H}$. We also denote by $\mathscr{C}([m,M])$ the set of all real valued continuous functions on an interval $[m,M]$. A linear map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is positive if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$. It is said to be normalized if $\Phi(\mathbf{1}_{\mathscr{H}}) = \mathbf{1}_{\mathscr{K}}$. A continuous real valued function f defined on an interval J is called operator monotone if $A \geqslant B$ implies that $f(A) \geqslant f(B)$ for all self-adjoint operators A, B with spectra in J.

The theory for connections and means of pairs of positive operators has been developed by Kubo and Ando in [8]. A binary operation $(A,B) \mapsto A \ \sigma \ B$ on the set of positive invertible operators is a connection if it satisfies the following axiomatic properties:

(M1) monotonicity $A \leqslant C$ and $B \leqslant D$ implies $A \ \sigma \ B \leqslant C \ \sigma \ D$,

(M2) transformer inequality $T^*(A \ \sigma \ B)T \leqslant (T^*AT) \ \sigma \ (T^*BT)$ for every operator T,

(M3) upper continuity $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \ \sigma \ B_n \downarrow A \ \sigma \ B$.

A mean is a connection with the following

(M4) normalization condition $\mathbf{1}_{\mathscr{H}} \ \sigma \ \mathbf{1}_{\mathscr{H}} = \mathbf{1}_{\mathscr{H}}$.

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The isomorphism $\sigma \mapsto f_\sigma$ defined by $f_\sigma(t)\mathbf{1}_{\mathscr{H}} = \mathbf{1}_{\mathscr{H}}\ \sigma\ (t\mathbf{1}_{\mathscr{H}})$, $t > 0$, establishes an isomorphism between the class of connections and the class of nonnegative operator monotone functions on $(0,\infty)$. Moreover, for positive invertible operators A, B

$$A\ \sigma\ B = A^{1/2} f_\sigma\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

An immediate consequence of the above that every connection σ possesses the following properties:

(M1′) subadditivity $(A\ \sigma\ B) + (C\ \sigma\ D) \leqslant (A+C)\ \sigma\ (B+D)$ for invertible C,

(M2′) positively homogeneous $a(A\ \sigma\ B) = (aA)\ \sigma\ (aB)$ for $a > 0$.

The well-known Ando's inequality [2] asserts that if A, B are two positive operators and Φ is a positive linear map, then

$$\Phi(A\sharp_v B) \leqslant \Phi(A)\sharp_v \Phi(B) \tag{1}$$

where $A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}$, $v \in [0,1]$. A fact worth noting here is that (1) is true for any arbitrary operator mean σ (see [1]),

$$\Phi(A\ \sigma\ B) \leqslant \Phi(A)\ \sigma\ \Phi(B). \tag{2}$$

As it is mentioned in [9, Lemma 3.1], inequality (2) is equivalent to Jensen's operator inequality [3,4] (also known as the Choi-Davis-Jensen inequality):

$$\Phi(f_\sigma(A)) \leqslant f_\sigma(\Phi(A)),$$

where f_σ is an operator monotone function. We refer the reader to [11, 12, 14] for recent developments of Ando and Jensen inequality.

This article aims to provide new operator inequalities containing (2). These inequalities are given in Section 2. Actually, we state companion inequalities to (2) involving concave/convex functions. Our main idea and technical tool is the Mond-Pečarić method. Section 3 collects a few application of the results obtained in the previous section.
Our inequalities mainly extend the results appeared in [5,6] and [10].

2. Main results

The following notations will be used in this section:

$$a_f = \frac{f(M)-f(m)}{M-m} \qquad \text{and} \qquad b_f = \frac{Mf(m)-mf(M)}{M-m}.$$

In the following two theorems we will give our main results. These results nicely extend the main theorems of [10].

THEOREM 1. *Let* $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ *be a normalized positive linear mapping,* σ *be an arbitrary mean,* A,B,C,D *be positive operators on* $\mathscr{H}$ *satisfying* $A \geqslant B$, $C \geqslant D$ *and* $M\mathbf{1}_{\mathscr{H}} \geqslant A,C \geqslant m\mathbf{1}_{\mathscr{H}}$, $N\mathbf{1}_{\mathscr{H}} \geqslant B,D \geqslant n\mathbf{1}_{\mathscr{H}}$ *for some scalars* $0 < m < M$, $0 < n \leqslant N$.

If $f \in \mathscr{C}([m,M])$ *is a concave function such that* $a_f, b_f \geqslant 0$ *and* $g \in \mathscr{C}([n,N])$, $f,g > 0$, *then for every* $\alpha \in \mathbb{R}$

$$\Phi(f(A))\ \sigma\ \Phi(f(C)) \geqslant \alpha\Phi(g(B\ \sigma\ D)) + \underline{\beta}\mathbf{1}_{\mathscr{K}}, \tag{3}$$

$$\Phi(f(A))\ \sigma\ \Phi(f(C)) \geqslant \alpha g(\Phi(B\ \sigma\ D)) + \underline{\beta}\mathbf{1}_{\mathscr{K}}, \tag{4}$$

where $\underline{\beta} = \min_{n \leqslant t \leqslant N}\{a_f t + b_f - \alpha g(t)\}$.

Proof. If f is a concave function and $a_f \geqslant 0$, then

$$f(A) \geqslant a_f A + b_f \mathbf{1}_{\mathscr{H}} \geqslant a_f B + b_f \mathbf{1}_{\mathscr{H}} \quad \text{and} \quad f(C) \geqslant a_f C + b_f \mathbf{1}_{\mathscr{H}} \geqslant a_f D + b_f \mathbf{1}_{\mathscr{H}}.$$

Using the monotonicity property (M1) of operator means, we infer

$$f(A)\ \sigma\ f(C) \geqslant \left(a_f B + b_f \mathbf{1}_{\mathscr{H}}\right)\ \sigma\ \left(a_f D + b_f \mathbf{1}_{\mathscr{H}}\right).$$

By the subadditivity (M1$'$), positively homogeneous (M2$'$), and normalization (M4) of σ for $a_f, b_f \geqslant 0$, we get

$$\begin{aligned}\left(a_f B + b_f \mathbf{1}_{\mathscr{H}}\right)\ \sigma\ \left(a_f D + b_f \mathbf{1}_{\mathscr{H}}\right) &\geqslant \left(a_f B\right)\ \sigma\ \left(a_f D\right) + \left(b_f \mathbf{1}_{\mathscr{H}}\right)\ \sigma\ \left(b_f \mathbf{1}_{\mathscr{H}}\right) \\ &\geqslant a_f (B\ \sigma\ D) + b_f \mathbf{1}_{\mathscr{H}}.\end{aligned}$$

Combining the two inequalities above we have

$$f(A)\ \sigma\ f(C) \geqslant a_f (B\ \sigma\ D) + b_f \mathbf{1}_{\mathscr{H}}.$$

Applying a normalized positive linear mapping Φ on the above inequality and using (2) we get

$$\Phi(f(A))\ \sigma\ \Phi(f(B)) \geqslant a_f \Phi(B\ \sigma\ D) + b_f \mathbf{1}_{\mathscr{K}}.$$

Therefore, from the fact that $N\mathbf{1}_{\mathscr{H}} \geqslant B\ \sigma\ D \geqslant n\mathbf{1}_{\mathscr{H}}$ we have

$$\begin{aligned}\Phi(f(A))\ \sigma\ \Phi(f(C)) - \alpha\Phi(g(B\ \sigma\ D)) &\geqslant a_f \Phi(B\ \sigma\ D) + b_f \mathbf{1}_{\mathscr{K}} - \alpha\Phi(g(B\ \sigma\ D)) \\ &= \Phi\left(a_f (B\ \sigma\ D) + b_f \mathbf{1}_{\mathscr{H}} - \alpha g(B\ \sigma\ D)\right) \\ &\geqslant \min_{n \leqslant t \leqslant N}\left\{a_f t + b_f - \alpha g(t)\right\}\mathbf{1}_{\mathscr{K}}\end{aligned}$$

which gives the inequality (3).

Also,

$$\begin{aligned}\Phi(f(A))\ \sigma\ \Phi(f(C)) - \alpha g(\Phi(B\ \sigma\ D)) &\geqslant a_f \Phi(B\ \sigma\ D) + b_f \mathbf{1}_{\mathscr{K}} - \alpha g(\Phi(B\ \sigma\ D)) \\ &\geqslant \min_{n \leqslant t \leqslant N}\left\{a_f t + b_f - \alpha g(t)\right\}\mathbf{1}_{\mathscr{K}}\end{aligned}$$

which gives the inequality (4). $\square$

REMARK 1.

i) The assumption: $f>0$ is a concave function such that $a_f, b_f \geqslant 0$ in Theorem 1 can be replace by a stronger condition: $f>0$ is an increasing concave function and $b_f \geqslant 0$.

ii) Let the assumptions of Theorem 1 be satisfied. Then the following ratio type inequalities hold

$$\Phi(f(A))\ \sigma\ \Phi(f(C)) \geqslant \underline{\alpha}\,\Phi(g(B\ \sigma\ D)),$$
$$\Phi(f(A))\ \sigma\ \Phi(f(C)) \geqslant \underline{\alpha}\,g(\Phi(B\ \sigma\ D)),$$

where $\underline{\alpha} = \min_{n\leqslant t\leqslant N}\{\frac{a_f t+b_f}{g(t)}\}$.

THEOREM 2. *Let $\Phi:\mathbb{B}(\mathscr{H})\to\mathbb{B}(\mathscr{K})$ be a normalized positive linear mapping, σ be an arbitrary mean, A,B,C,D be positive operators on $\mathscr{H}$ satisfying $A\geqslant B$, $C\geqslant D$ and $M\mathbf{1}_{\mathscr{H}}\geqslant B,D\geqslant m\mathbf{1}_{\mathscr{H}}$, $N\mathbf{1}_{\mathscr{H}}\geqslant A,C\geqslant n\mathbf{1}_{\mathscr{H}}$ for some scalars $0<m<M$, $0<n\leqslant N$.*

If $f\in\mathscr{C}([m,M])$ is a convex function such that $a_f\geqslant 0$ and $g\in\mathscr{C}([n,N])$, $f,g>0$, then for every $\alpha\in\mathbb{R}$

$$\Phi(f(B\ \sigma\ D)) \leqslant \alpha\, g(\Phi(A)\ \sigma\ \Phi(C)) + \overline{\beta}\,\mathbf{1}_{\mathscr{K}}, \tag{5}$$
$$f(\Phi(B\ \sigma\ D)) \leqslant \alpha\, g(\Phi(A)\ \sigma\ \Phi(C)) + \overline{\beta}\,\mathbf{1}_{\mathscr{K}}, \tag{6}$$

where $\overline{\beta} = \max_{n\leqslant t\leqslant N}\{a_f t + b_f - \alpha\, g(t)\}$.

But, if $f>0$ is a concave function such that $a_f\leqslant 0$, then the reverse inequalities are valid in (5) and (6) with min *instead* max *in $\overline{\beta}$.*

Proof. Since f is a convex function and $a_f\geqslant 0$, then using (2) and the monotonicity property (M1), we obtain

$$\begin{aligned}\Phi(f(B\ \sigma\ D)) &\leqslant a_f\Phi(B\ \sigma\ D) + b_f\mathbf{1}_{\mathscr{K}}\\ &\leqslant a_f(\Phi(B)\ \sigma\ \Phi(D)) + b_f\mathbf{1}_{\mathscr{K}}\\ &\leqslant a_f(\Phi(A)\ \sigma\ \Phi(C)) + b_f\mathbf{1}_{\mathscr{K}}.\end{aligned}$$

Therefore, the fact $n\mathbf{1}_{\mathscr{K}}\leqslant\Phi(A),\Phi(C)\leqslant N\mathbf{1}_{\mathscr{K}}$ implies $n\mathbf{1}_{\mathscr{K}}\leqslant\Phi(A)\ \sigma\ \Phi(C)\leqslant N\mathbf{1}_{\mathscr{K}}$, and we have

$$\begin{aligned}\Phi(f(B\ \sigma\ D)) - \alpha\, g(\Phi(A)\ \sigma\ \Phi(C)) &\leqslant a_f(\Phi(A)\ \sigma\ \Phi(C)) + b_f\mathbf{1}_{\mathscr{K}} - \alpha\, g(\Phi(A)\ \sigma\ \Phi(C))\\ &\leqslant \max_{n\leqslant t\leqslant N}\left\{a_f t + b_f - \alpha g(t)\right\}\mathbf{1}_{\mathscr{K}}\end{aligned}$$

which gives the inequality (5).

Also,

$$\begin{aligned}&f(\Phi(B\ \sigma\ D)) - \alpha g(\Phi(A)\ \sigma\ \Phi(C))\\ &\leqslant a_f\Phi(B\ \sigma\ D) + b_f\mathbf{1}_{\mathscr{K}} - \alpha g(\Phi(A)\ \sigma\ \Phi(C))\\ &\leqslant a_f(\Phi(A)\ \sigma\ \Phi(C)) + b_f\mathbf{1}_{\mathscr{K}} - \alpha g(\Phi(A)\ \sigma\ \Phi(C))\\ &\leqslant \max_{n\leqslant t\leqslant N}\left\{a_f t + b_f - \alpha g(t)\right\}\mathbf{1}_{\mathscr{K}}\end{aligned}$$

which gives the inequality (6). □

REMARK 2.

i) The assumption: $f>0$ is a convex function such that $a_f \geqslant 0$ in Theorem 2 can be replace by a stronger condition: f is an increasing convex function.

ii) Let the assumptions of Theorem 2 be satisfied. Then the following ratio type inequalities

$$\Phi(f(B\,\sigma\,D)) \leqslant \overline{\alpha}\, g(\Phi(A)\,\sigma\,\Phi(C)),$$
$$f(\Phi(B\,\sigma\,D)) \leqslant \overline{\alpha}\, g(\Phi(A)\,\sigma\,\Phi(C))$$

hold, where $\overline{\alpha} = \max_{n\leqslant t\leqslant N}\{\frac{a_f t+b_f}{g(t)}\}$.

iii) Under the same assumptions as in Theorem 2 and if f is convex such that $a_f \geqslant 0$, then we obtain the following obvious inequalities

$$\Phi(f(B\,\sigma\,D)) \leqslant \alpha\Phi\big(g\big(A\,\sigma\,C\big)\big) + \overline{\beta}\mathbf{1}_{\mathscr{K}},$$
$$f(\Phi(B\,\sigma\,D)) \leqslant \alpha\Phi\big(g\big(A\,\sigma\,C\big)\big) + \overline{\beta}\mathbf{1}_{\mathscr{K}},$$

for the same $\overline{\beta}$ as above.

In the next theorems we observe order between the mean of operators for some functions g. First, by using the idea of [13, Theorem 2.2], we will give the following theorem in which the bounds of operators A and C are not needed in the constant β_1.

THEOREM 3. *Let $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ be a normalized positive linear mapping, σ be an arbitrary mean, A,B,C,D be positive operators on $\mathscr{H}$ satisfying $A \geqslant B$, $C \geqslant D$ and $M\mathbf{1}_{\mathscr{H}} \geqslant B,D \geqslant m\mathbf{1}_{\mathscr{H}}$ for some scalars $0<m<M$. If $f \in \mathscr{C}([m,M])$ is a concave function, $g \in \mathscr{C}(J)$ be a decreasing concave, where J be any interval $J \supseteq [m,M]\cup \mathsf{Sp}(A)\cup \mathsf{Sp}(C)$, $f,g>0$ and $\alpha>0$ such that $\beta_1 = \min_{m\leqslant t\leqslant M}\{a_f t + b_f - \alpha g(t)\} \geqslant 0$, then*

$$\Phi(f(B))\ \sigma\ \Phi(f(D)) \geqslant \alpha\ \Phi(g(A)\ \sigma\ g(C)) + \beta_1 \mathbf{1}_{\mathscr{K}}. \tag{7}$$

Proof. Let $x\in\mathscr{H}$ be any unit vector and $A \geqslant B$. By the concavity decreasing function αg, we get

$$\alpha\langle g(A)x,x\rangle \leqslant \alpha g(\langle Ax,x\rangle) \leqslant \alpha g(\langle Bx,x\rangle).$$

Next, since $m \leqslant \langle Bx,x\rangle \leqslant M$, $\beta_1 := \min_{m\leqslant t\leqslant M}\{a_f t + b_f - \alpha g(t)\}$ and f is concave on $[m,M]$, then

$$\alpha g(\langle Bx,x\rangle) + \beta_1 \leqslant a_f\langle Bx,x\rangle + b_f \leqslant \langle f(B)x,x\rangle.$$

Therefore, combining the above two inequalities we have

$$\alpha\langle g(A)x,x\rangle + \beta_1 \leqslant \alpha g(\langle Bx,x\rangle) + \beta_1 \leqslant \langle f(B)x,x\rangle.$$

So, we obtain the following order

$$f(B) \geqslant \alpha\, g(A) + \beta_1 \mathbf{1}_{\mathscr{H}}.$$

Also, if $C \geqslant D$, then

$$f(D) \geqslant \alpha\, g(C) + \beta_1 \mathbf{1}_{\mathscr{H}}.$$

Using the above two inequality, the monotonicity property (M1), the subadditivity (M1$'$), positively homogeneous (M2$'$) and normalization (M4), we get

$$f(B)\ \sigma\ f(D) \geqslant (\alpha\, g(A) + \beta_1 \mathbf{1}_{\mathscr{H}})\ \sigma\ (\alpha\, g(C) + \beta_1 \mathbf{1}_{\mathscr{H}}) \geqslant \alpha(g(A)\ \sigma\ g(C)) + \beta_1 \mathbf{1}_{\mathscr{H}}.$$

Applying a normalized positive linear mapping and using (2) we get

$$\Phi(f(B))\ \sigma\ \Phi(f(D)) \geqslant \Phi(f(B)\ \sigma\ f(D)) \geqslant \alpha\ \Phi(g(A)\ \sigma\ g(C)) + \beta_1 \mathbf{1}_{\mathscr{K}},$$

which gives (7). □

But, if g is a increasing concave function in Theorem 3, then by using the idea of [10, Theorem 5.1] in which the bounds of operators B and D are not needed in the constant β_1.

THEOREM 4. *Let A,B,C,D be positive operators on $\mathscr{H}$ satisfying $A \geqslant B$, $C \geqslant D$ and $m\mathbf{1}_{\mathscr{H}} \leqslant A, C \leqslant M\mathbf{1}_{\mathscr{H}}$ for some scalars $0 < m < M$ and Φ, σ, f, β_1 be as in Theorem 3. If $g \in \mathscr{C}(J)$ be a increasing concave, where J be any interval $J \supseteq [m,M] \cup \mathrm{Sp}(B) \cup \mathrm{Sp}(D)$, $f,g > 0$ and $\alpha > 0$ such that $\beta_1 \geqslant 0$, then*

$$\Phi(f(A))\ \sigma\ \Phi(f(C)) \geqslant \alpha\ \Phi(g(B)\ \sigma\ g(D)) + \beta_1 \mathbf{1}_{\mathscr{K}}. \tag{8}$$

Proof. We use the same technique as in the proof of Theorem 3 and we will omit details. □

REMARK 3.

i) Let the assumptions of Theorem 3 be satisfied. Then the following ratio type inequality

$$\Phi(f(B))\ \sigma\ \Phi(f(D)) \geqslant \alpha_1\ \Phi(g(A)\ \sigma\ g(C))$$

holds, where $\alpha_1 = \min\limits_{m \leqslant t \leqslant M} \{\frac{a_f t + b_f}{g(t)}\} > 0$. Similarly, using Theorem 4 we can obtain the ratio type of inequality. We omit the details.

ii) Let A,B,C,D, Φ, σ, m,M, J be as in Theorem 3. If $f \in \mathscr{C}([m,M])$ is convex, $g \in \mathscr{C}(J)$ is increasing convex $f,g > 0$ and $\alpha > 0$ such that $\beta_2 = \max\limits_{m \leqslant t \leqslant M} \{a_f t + b_f - \alpha g(t)\} \geqslant 0$, then

$$f(\Phi(B\ \sigma\ D)) \leqslant \alpha\, g\big(\Phi(A)\ \sigma\ \Phi(C)\big) + \beta_2 \mathbf{1}_{\mathscr{K}}. \tag{9}$$

But, if A,B,C,D be as in Theorem 4, f is convex, g is decreasing convex and $\alpha > 0$ such that $\beta_2 \geqslant 0$, then

$$f(\Phi(A\ \sigma\ C)) \leqslant \alpha\, g\big(\Phi(B)\ \sigma\ \Phi(D)\big) + \beta_2 \mathbf{1}_{\mathscr{K}}. \tag{10}$$

We will prove (9). Since g is increasing convex and $\alpha > 0$, then

$$\alpha \langle g(\Phi(A)\ \sigma\ \Phi(C))x,x\rangle \geqslant \alpha g(\langle \Phi(B\ \sigma\ D)x,x\rangle),$$

for every unit $x \in \mathscr{K}$. Since f is convex, then

$$\alpha g(\langle \Phi(B\ \sigma\ D)x,x\rangle) + \beta_2 \geqslant a_f \langle \Phi(B\ \sigma\ D)x,x\rangle + b_f \geqslant \langle f(\Phi(B\ \sigma\ D))x,x\rangle.$$

Combining the above inequalities we obtain the desired inequality (9).

3. Consequences and applications

In this section we give some results by using the inequalities in the section above for special cases of functions or operators or means.

Setting $f \equiv g$, $A = B$ and $C = D$ in Theorem 1, we obtain the following corollary.

COROLLARY 1. *Let $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ be a normalized positive linear mapping, σ be an arbitrary mean and A,B be positive operators satisfying $M\mathbf{1}_{\mathscr{H}} \geqslant A,B \geqslant m\mathbf{1}_{\mathscr{H}}$ for some scalars $0 < m < M$.*

If $f \in \mathscr{C}([m,M])$ is a concave function such that $f > 0$, $a_f, b_f \geqslant 0$, then for every $\alpha \in \mathbb{R}$

$$\begin{aligned} \Phi(f(A))\ \sigma\ \Phi(f(B)) &\geqslant \alpha \Phi(f(A\ \sigma\ B)) + \beta \mathbf{1}_{\mathscr{K}}, \\ \Phi(f(A))\ \sigma\ \Phi(f(B)) &\geqslant \alpha f(\Phi(A\ \sigma\ B)) + \beta \mathbf{1}_{\mathscr{K}}, \end{aligned} \tag{11}$$

where $\beta = \min\limits_{m \leqslant t \leqslant M} \{a_f t + b_f - \alpha f(t)\}$. Also,

$$\begin{aligned} \Phi(f(A))\ \sigma\ \Phi(f(B)) &\geqslant \alpha_1 \Phi(f(A\ \sigma\ B)), \\ \Phi(f(A))\ \sigma\ \Phi(f(B)) &\geqslant \alpha_1 f(\Phi(A\ \sigma\ B)), \end{aligned} \tag{12}$$

hold, where $\alpha_1 = \min\limits_{m \leqslant t \leqslant M} \{\frac{a_f t + b_f}{f(t)}\}$.

Setting $f(t) = t^p$ in Corollary 1, we obtain the following results.

COROLLARY 2. *Let Φ, σ, A,B be as in Corollary 1. If $0 < m < M$, $\alpha > 0$ and $0 \leqslant p \leqslant 1$, then*

$$\Phi(A^p)\ \sigma\ \Phi(B^p) \geqslant \alpha \Phi^p(A\ \sigma\ B) + \beta_p \mathbf{1}_{\mathscr{K}} \geqslant \alpha \Phi((A\ \sigma\ B)^p) + \beta_p \mathbf{1}_{\mathscr{K}}, \tag{13}$$

where $\beta_p = \min\limits_{m \leqslant t \leqslant M} \{\frac{M^p - m^p}{M-m} t + \frac{Mm^p - mM^p}{M-m} - \alpha t^p\}$.

We especially have

$$\Phi(A^p)\ \sigma\ \Phi(B^p) \geqslant \alpha_p \Phi^p(A\ \sigma\ B) \geqslant \alpha_p \Phi((A\ \sigma\ B)^p), \tag{14}$$

where $\alpha_p = \min\limits_{m \leqslant t \leqslant M} \{(\frac{M^p - m^p}{M-m} t + \frac{Mm^p - mM^p}{M-m})/t^p\}$.

Proof. Since $f(t)=t^p$ for $p\in[0,1]$ is positive increasing concave on $[0,\infty)$ then $a_{t^p}=\frac{M^p-m^p}{M-m}>0$, $b_{t^p}=\frac{Mm^p-mM^p}{M-m}>0$ and $\alpha_p>0$. Moreover, since f is operator concave, then Jensen's operator inequality give $\Phi^p(A\,\sigma\,B)\geqslant\Phi((A\,\sigma\,B)^p)$. So, (13) follows from (11) and (14) follows from (12). □

In Corollary 1, f must be a concave function to ensure $a_f,b_f\geqslant 0$. However, for the case $\sigma=\nabla_v$ (where "∇_v" denotes the operator arithmetic mean of two positive operators, i.e., $A\nabla_v B=(1-v)A+vB$, $v\in[0,1]$), the conditions $a_f,b_f\geqslant 0$ can be dropped. As some applications of Corollary 1 and since $a(A\nabla_v B)=(aA)\nabla_v(aB)$ for all $a\in\mathbb{R}$, we get the following corollary.

COROLLARY 3. *Let $A,B\in\mathbb{B}(\mathscr{H})$ such that $m\mathbf{1}_{\mathscr{H}}\leqslant A,B\leqslant M\mathbf{1}_{\mathscr{H}}$ for some scalars $0<m<M$ and $v\in[0,1]$.*

If $f\in\mathscr{C}([m,M])$ is a concave function such that $f>0$, then the difference type inequality

$$f(A\nabla_v B)+\beta\mathbf{1}_{\mathscr{H}}\leqslant f(A)\nabla_v f(B) \tag{15}$$

holds, where $\beta=\min_{t\in[m,M]}\left\{a_f t+b_f-f(t)\right\}$.

Additionally, the ratio type inequality

$$f(A\nabla_v B)\leqslant\frac{1}{\alpha}\left(f(A)\nabla_v f(B)\right) \tag{16}$$

holds, where $\alpha=\min_{t\in[m,M]}\left\{\frac{a_f t+b_f}{f(t)}\right\}$.

If f is a convex function, then the reverse inequalities are valid in (15) and (16) with max *instead* min *in β and α.*

REMARK 4. The inequality (15) was shown previously in [5, Lemma 3.2] by an entirely different method.

COROLLARY 4. *Let $A,B\in\mathbb{B}(\mathscr{H})$ such that $0<m\mathbf{1}_{\mathscr{H}}\leqslant A,B\leqslant M\mathbf{1}_{\mathscr{H}}$.*

(a) *If $f\in\mathscr{C}([m,M])$ is a concave function such that $f>0$, then for any $v\in[0,1]$*

$$f(A\,\sharp_v\,B)\leqslant\frac{1}{\alpha}\left(f(A)\,\sharp_v\,f(B)\right) \tag{17}$$

where $\alpha=\min_{t\in[m,M]}\left\{\frac{a_f t+b_f}{f(t)}\right\}$. Additionally, the following difference type inequality holds:

$$f(A\,\sharp_v\,B)\leqslant f(A)\,\sharp_v\,f(B)-\beta\mathbf{1}_{\mathscr{H}} \tag{18}$$

where $\beta=\min_{t\in[m,M]}\left\{a_f t+b_f-f(t)\right\}$.

If $f>0$ is a convex function, then the reverse inequalities are valid in (17) and (18) with max *instead* min *in β and α.*

(b) *If* $g:[m,M]\to(0,\infty)$ *is a continuous convex function, then for any* $v\in[0,1]$

$$g(A\,\sharp_v\,B)\leqslant\gamma(g(A)\,\sharp_v\,g(B)) \tag{19}$$

where $\gamma=\max\limits_{t\in[m,M]}\left\{g(t)\left(\frac{1/g(M)-1/g(m)}{M-m}t+\frac{M/g(m)-m/g(M)}{M-m}\right)\right\}$.

(c) *If* $g:[m,M]\to(0,\infty)$ *is an operator monotone decreasing and* $\Phi:\mathbb{B}(\mathscr{H})\to\mathbb{B}(\mathscr{K})$ *be a normalized positive linear mapping, then*

$$g(\Phi(A\,\sharp_v\,B))\leqslant\Phi(g(A\,\sharp_v\,B))\leqslant\gamma(\Phi(g(A))\,\sharp_v\,\Phi(g(B))) \tag{20}$$

with min *instead* max *in* γ.

Proof.

(a) It is easy to see that $a(A\sharp_v B)=(aA)\sharp_v(aB)$ for all $a\in\mathbb{R}$. So get (17) and (18) applying Corollary 3.1.

(b) If g is positive concave, then $\frac{1}{g}$ is convex. Applying (17) for $f=1/g$, we have

$$\frac{1}{g(A\,\sharp_v\,B)}\geqslant\frac{1}{\gamma}\left(\frac{1}{g(A)}\,\sharp_v\,\frac{1}{g(B)}\right)=\frac{1}{\gamma}\cdot\frac{1}{g(A)\,\sharp_v\,g(B)}$$

where $\gamma=\max\limits_{t\in[m,M]}\left\{\frac{a_{1/g}t+b_{1/g}}{(1/g)(t)}\right\}$. By taking the inverse of both sides we deduce the desired inequality (19).

(c) Since g is operator monotone decreasing on $(0,\infty)$, so $1/g$ is operator monotone on $(0,\infty)$. It follows that $1/g$ is operator concave, so it is a concave function. Now applying (17) and Jensen's operator inequality we deduce the desired reverse inequality in (19) with min instead max in γ. □

REMARK 5. It should be remarked here that the inequalities (17) and (19) can be regarded as a reverse of [7, Theorem 3] and [7, Theorem 4], respectively.

Setting $f(t)=t^p$ and $g(t)=t^q$ in Remark 3 (ii), we obtain the following corollary.

COROLLARY 5. *Let* $\Phi:\mathbb{B}(\mathscr{H})\to\mathbb{B}(\mathscr{K})$ *be a normalized positive linear mapping,* σ *be an arbitrary mean,* A,B,C,D *be positive operators on* $\mathscr{H}$ *satisfying* $A\geqslant B$, $C\geqslant D$ *and* $\alpha>0$ *such that* $\beta=\max\limits_{m\leqslant t\leqslant M}\{\frac{M^p-m^p}{M-m}t+\frac{Mm^p-mM^p}{M-m}-\alpha t^q\}\geqslant 0$.

- *If* $M\mathbf{1}_{\mathscr{H}}\geqslant B,D\geqslant m\mathbf{1}_{\mathscr{H}}$ *for some scalars* $0<m<M$ *and* $p\in(-\infty,0]\cup[1,\infty)$, $q\in[1,\infty)$, *then*

$$\Phi^p(B\,\sigma\,D)\leqslant\alpha\left(\Phi(A)\,\sigma\,\Phi(C)\right)^q+\beta\mathbf{1}_{\mathscr{K}}$$

and

$$\Phi^p(B\,\sigma\,D)\leqslant\alpha_1\left(\Phi(A)\,\sigma\,\Phi(C)\right)^q,$$

where $\alpha_1=\max\limits_{m\leqslant t\leqslant M}\left\{(\frac{M^p-m^p}{M-m}t+\frac{Mm^p-mM^p}{M-m})/t^q\right\}$.

- *But, if* $M\mathbf{1}_{\mathscr{H}} \geqslant A, C \geqslant m\mathbf{1}_{\mathscr{H}}$ *for some scalars* $0 < m < M$ *and* $p \in (-\infty, 0] \cup [1, \infty)$, $q \in (-\infty, 0]$, *then*

$$\Phi^p (A \ \sigma \ C) \leqslant \alpha \left(\Phi(B) \ \sigma \ \Phi(D)\right)^q + \beta \mathbf{1}_{\mathscr{K}}$$

and

$$\Phi^p (A \ \sigma \ C) \leqslant \alpha_1 \left(\Phi(B) \ \sigma \ \Phi(D)\right)^q.$$

REMARK 6. By putting $A = C$, $B = D$, $p = q$, and $\Phi(T) = T \ (T \geqslant 0)$ in Corollary 5 we get [6, Theorem 2.1].

Setting $f \equiv g$ in Theorem 3 and Theorem 4, and $\sigma = \nabla_v$ or $\sigma = \sharp_v$, the condition $\beta_1 \geqslant 0$ can be dropped. For example, we obtain the following inequalities applying Theorem 3 and Remark 3 (ii) on the geometric mean.

COROLLARY 6. *Let* $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ *be a normalized positive linear mapping,* $v \in [0,1]$, A, B, C, D *be positive operators on* $\mathscr{H}$ *satisfying* $A \geqslant B$, $C \geqslant D$ *and* $M\mathbf{1}_{\mathscr{H}} \geqslant B, D \geqslant m\mathbf{1}_{\mathscr{H}}$ *for some scalars* $0 < m < M$.

If $f \in \mathscr{C}(J)$ *is a decreasing concave, where* J *be any interval* $J \supseteq [m, M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(C)$, $f > 0$, *then*

$$\Phi(f(B)) \ \sharp_v \ \Phi(f(D)) \geqslant \Phi(f(A) \ \sharp_v \ f(C)) + \min_{t \in [m,M]} \left\{a_f t + b_f - f(t)\right\} \mathbf{1}_{\mathscr{K}}$$

and

$$\Phi(f(B)) \ \sharp_v \ \Phi(f(D)) \geqslant \min_{m \leqslant t \leqslant M} \left\{\frac{a_f t + b_f}{f(t)}\right\} \Phi(f(A) \ \sharp_v \ f(C)).$$

But, if $f \in \mathscr{C}(J)$ *is increasing convex, then*

$$f(\Phi(A \ \sharp_v \ C)) \leqslant \alpha f\left(\Phi(B) \ \sharp_v \ \Phi(D)\right) + \max_{t \in [m,M]} \left\{a_f t + b_f - f(t)\right\} \mathbf{1}_{\mathscr{K}} \tag{21}$$

and

$$f(\Phi(A \ \sharp_v \ C)) \leqslant \max_{m \leqslant t \leqslant M} \left\{\frac{a_f t + b_f}{f(t)}\right\} f\left(\Phi(B) \ \sharp_v \ \Phi(D)\right).$$

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