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AN IMPROVED RESULT OF A WEIGHTED TRIGONOMETRIC INEQUALITY IN ACUTE TRIANGLES WITH APPLICATIONS

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Abstract. An improved inequality of a weighted trigonometric inequality in acute triangles is established by using the simplest arithmetic-geometric mean inequality, which also is an improvement of the well known Wolstenholme inequality for non-obtuse triangles. Its two equivalent weighted inequalities for the strengthened versions of the Erdös-Mordell inequality and Barrow's inequality are obtained. Some applications are given by new results and five relevant interesting conjectures are also put forward.

1. Introduction

For any triangle ABC and real numbers x,y,z we have the following well known Wolstenholme inequality

$$x^2+y^2+z^2 \geqslant 2(yz\cos A+zx\cos B+xy\cos C), \tag{1}$$

where A,B,C denote the angles of the triangle ABC. Equality holds if and only if $x:y:z=\sin A:\sin B:\sin C$.

Inequality (1) has already appeared in Wolstenholme's book [17] in 1867. A number of triangle inequalities can be derived from this inequality. In the seventh chapter of my recent monograph [5], the author specially discussed applications of the Wolstenholm inequality (1).

In 1994, the author established the following similar inequality for the acute triangle ABC in a Chinese paper [6]:

$$x^2+y^2+z^2 \geqslant 4(yz\cos B\cos C+zx\cos C\cos A+xy\cos A\cos B), \tag{2}$$

with equality if and only if the acute ABC is equilateral and $x=y=z$.

In fact, there are several equivalent forms of inequality (2). By the substitution $x \to x\cos A$ etc., we see that it is equivalent to

$$\begin{aligned} &x^2\cos^2 A+y^2\cos^2 B+z^2\cos^2 C\\ &\geqslant 4(yz\cos^2 B\cos^2 C+zx\cos^2 C\cos^2 A+xy\cos^2 A\cos^2 B), \end{aligned} \tag{3}$$

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which gives a weighted generalization of the following Oppenheim's non-obtuse triangle inequality (see [14] and [11, pp.31-32]):

$$\cos^2 A+\cos^2 B+\cos^2 C \geqslant 4\left(\cos^2 B\cos^2 C+\cos^2 C\cos^2 A+\cos^2 A\cos^2 B\right). \tag{4}$$

By the transformations $A \to (\pi - A)/2$ etc., we know again that inequality (2) is equivalent to

$$x^2+y^2+z^2 \geqslant 4\left(yz\sin\frac{B}{2}\sin\frac{C}{2}+zx\sin\frac{C}{2}\sin\frac{A}{2}+xy\sin\frac{A}{2}\sin\frac{B}{2}\right), \tag{5}$$

where the triangle ABC is arbitrary. Clearly, inequality (5) also is equivalent to

$$\frac{x^2}{\sin^2\frac{A}{2}}+\frac{y^2}{\sin^2\frac{B}{2}}+\frac{z^2}{\sin^2\frac{C}{2}} \geqslant 4(yz+zx+xy). \tag{6}$$

Furthermore, it is easy to show that inequality (6) is equivalent to the following algebraic inequality (see [5, Corollary 2.19]):

$$x^2\frac{v+w}{u}+y^2\frac{w+u}{v}+z^2\frac{u+v}{w} \geqslant 4\left(yz\frac{u}{v+w}+zx\frac{v}{w+u}+xy\frac{w}{u+v}\right), \tag{7}$$

where u,v,w are arbitrary positive numbers. Equality holds if and only $x=y=z$ and $u=v=w$.

In [5], the author gives some applications of (6) and (7). For example, we used inequality (6) to deduce the following weighted trigonometric inequality (see [5, Corollary 17.15]):

$$\begin{aligned} &x^2\left(\cos\frac{B}{2}\cos\frac{C}{2}\right)^2+y^2\left(\cos\frac{C}{2}\cos\frac{A}{2}\right)^2+z^2\left(\cos\frac{A}{2}\cos\frac{B}{2}\right)^2 \\ &\geqslant yz(\sin B\sin C)^2+zx(\sin C\sin A)^2+xy(\sin A\sin B)^2, \end{aligned} \tag{8}$$

and the weighted geometric inequality (see [5, Corollary 2.18]):

$$x^2R_2R_3+y^2R_3R_1+z^2R_1R_2 \geqslant 4(yzw_2w_3+zxw_3w_1+xyw_1w_2), \tag{9}$$

where R_1,R_2,R_3 denote the distances from a point P inside triangle ABC to the vertices A,B,C of ABC and w_1,w_2,w_3 denote the lengths of the internal bisectors of $\angle BPC,\angle CPA,\angle APB$, respectively.

In addition, the author obtains various generalizations and strengthened versions of inequality (2) in the monograph [5]. Owing to the limitation of space, we do not introduce related results here.

In this paper, we give an improvement of inequality (2) (which also is an improvement of the Wolstenholme inequality (1) for non-obtuse triangles) and prove its two equivalent propositions. Some applications of our new results are also given.

2. Main result and its proof

The acute triangle inequality (2) can be improved and extended to non-obtuse triangles as follows:

THEOREM 1. *For the non-obtuse triangle* ABC *and arbitrary real numbers* x,y,z, *the following inequality holds:*

$$x^2+y^2+z^2 \geqslant (z\cos B+y\cos C)^2+(x\cos C+z\cos A)^2+(y\cos A+x\cos B)^2, \quad (10)$$

If ABC *is an acute triangle, then the equality in* (10) *holds if and only if* $x:y:z=\sin A:\sin B:\sin C$*; If* ABC *is a right triangle with* $A=\pi/2$, *then the equality in* (10) *holds if and only if* $y=z$ *and* $B=C=\pi/4$.

Since $(z\cos B+y\cos C)^2 \geqslant 4yz\cos B\cos C$, thus inequality (10) obviously is an improvement of inequality (2). In fact, inequality (10) also is an improvement of the Wolstenholme inequality (1) for the non-obtuse triangle ABC (see Remark 2 below).

Clearly, inequality (10) is equivalent to

$$\begin{aligned}&\left(1-\cos^2 B-\cos^2 C\right)x^2+\left(1-\cos^2 C-\cos^2 A\right)y^2+\left(1-\cos^2 A-\cos^2 B\right)z^2\\ &\geqslant 2(yz\cos B\cos C+zx\cos C\cos A+xy\cos A\cos B),\end{aligned} \quad (11)$$

and equivalent to

$$\begin{aligned}&x^2\cos^2 A+y^2\cos^2 B+z^2\cos^2 C\\ &\geqslant (y+z)^2\cos^2 B\cos^2 C+(z+x)^2\cos^2 C\cos^2 A+(x+y)^2\cos^2 A\cos^2 B,\end{aligned} \quad (12)$$

which clearly is stronger than (3). Equality in (12) holds if and only if $x\cot A=y\cot B=z\cot C$.

Next, we give a straightforward proof of Theorem 1.

Proof. We divide our arguments into the following two cases.

Case 1. $\triangle ABC$ is a right triangle with $A=\pi/2$.

In this case, inequality (10) becomes

$$x^2+y^2+z^2 \geqslant (z\cos B+y\cos C)^2+x^2(\cos^2 C+\cos^2 B), \quad (13)$$

which is required to prove. By Cauchy inequality, we have

$$(z\cos B+y\cos C)^2 \leqslant (y^2+z^2)(\cos^2 B+\cos^2 C). \quad (14)$$

Since $A=\pi/2$, we have $\cos^2 B+\cos^2 C=1$. Thus, (13) follows from (14). In view of the equality condition of Cauchy inequality and $A=\pi/2$,we easily further know that the equality in (13) holds if and only if $y=z$ and $B=C=\pi/4$.

Case 2. $\triangle ABC$ is an acute triangle.

Firstly,we shall prove that for any triangle ABC and real numbers x,y,z holds:

$$\begin{aligned}&\left(z\sin\frac{B}{2}+y\sin\frac{C}{2}\right)^2+\left(x\sin\frac{C}{2}+z\sin\frac{A}{2}\right)^2+\left(y\sin\frac{A}{2}+x\sin\frac{B}{2}\right)^2\\ &\leqslant x^2+y^2+z^2,\end{aligned} \quad (15)$$

with equality if and only if $x:y:z=\cos\frac{A}{2}:\cos\frac{B}{2}:\cos\frac{C}{2}$.

In the sequel, we let a,b,c denote the sides BC,CA,AB of $\triangle ABC$ respectively, and let $s=(a+b+c)/2$. By the half-angle formula

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}, \tag{16}$$

we have

$$\sin\frac{B}{2}\sin\frac{C}{2}=\frac{s-a}{a}\sqrt{\frac{(s-b)(s-c)}{bc}}. \tag{17}$$

Thus, we use the simplest arithmetic-geometric inequality to obtain that

$$2yz\sin\frac{B}{2}\sin\frac{C}{2}\leqslant\frac{s-a}{a}\left(z^2\frac{s-b}{c}+y^2\frac{s-c}{b}\right) \tag{18}$$

with equality if and only if

$$z^2\frac{s-b}{c}=y^2\frac{s-c}{b}. \tag{19}$$

Let Q_0 be the value of the left hand of (15), by (18) and formula (16), we have

$$\begin{aligned}Q&=x^2\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)+y^2\left(\sin^2\frac{C}{2}+\sin^2\frac{A}{2}\right)+z^2\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}\right)\\&\quad+2\left(yz\sin\frac{B}{2}\sin\frac{C}{2}+zx\sin\frac{C}{2}\sin\frac{A}{2}+xy\sin\frac{A}{2}\sin\frac{B}{2}\right)\\&=\frac{s-a}{a}\left(\frac{s-c}{c}+\frac{s-b}{b}\right)x^2+\frac{s-b}{b}\left(\frac{s-a}{a}+\frac{s-c}{c}\right)y^2\\&\quad+\frac{s-c}{c}\left(\frac{s-b}{b}+\frac{s-a}{a}\right)z^2+\frac{s-a}{a}\left(z^2\frac{s-b}{c}+y^2\frac{s-c}{b}\right)\\&\quad+\frac{s-b}{b}\left(x^2\frac{s-c}{a}+z^2\frac{s-a}{c}\right)+\frac{s-c}{c}\left(y^2\frac{s-a}{b}+x^2\frac{s-b}{a}\right).\end{aligned}$$

Also, it is easy to verify the following identity

$$\frac{s-a}{a}\left(\frac{s-c}{c}+\frac{s-b}{b}\right)+\frac{(s-b)(s-c)}{a}\left(\frac{1}{b}+\frac{1}{c}\right)=1, \tag{20}$$

and two analogous identities, so we have

$$Q\leqslant x^2+y^2+z^2$$

and inequality (15) is proved.

Clearly, the equality of (15) holds if and only if

$$z^2\frac{s-b}{c}=y^2\frac{s-c}{b},\ x^2\frac{s-c}{a}=z^2\frac{s-a}{c},\ y^2\frac{s-a}{b}=x^2\frac{s-b}{a},$$

i.e.,$x:y:z=\sqrt{a(s-a)}:\sqrt{b(s-b)}:\sqrt{c(s-c)}$. Again, by the following half-angle formula

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}, \tag{21}$$

we see that

$$\sqrt{a(s-a)}:\sqrt{b(s-b)}:\sqrt{c(s-c)}=\cos\frac{A}{2}:\cos\frac{B}{2}:\cos\frac{C}{2}.$$

Thus, we further conclude that the equality in (15) holds if only if $x:y:z=\cos\frac{A}{2}:\cos\frac{B}{2}:\cos\frac{C}{2}$.

When the triangle ABC is an acute triangle, we can use the substitutions $A\to\pi-2A, B\to\pi-2B, C\to\pi-2C$ in (15), and then inequality (10) follows. Also, by the equality condition of (15), we know that the equality of (10) holds if and only if $x:y:z=\sin A:\sin B:\sin C$.

Combining the arguments of the two cases above, we finish the proof of Theorem 1.

REMARK 1. For general ternary quadratic inequality,we have the following conclusion (see [7, Lemma 4]): Let p_1,p_2,p_3,q_1,q_2,q_3 be real numbers such that $p_1>0$, $p_2>0$, $p_3>0$, $4p_2p_3-q_1^2>0$, $4p_3p_1-q_2^2>0$, $4p_1p_2-q_3^2>0$ and

$$D_0\equiv 4p_1p_2p_3-(q_1q_2q_3+p_1q_1^2+p_2q_2^2+p_3q_3^2)\geqslant 0. \tag{22}$$

Then the following inequality

$$p_1x^2+p_2y^2+p_3z^2\geqslant q_1yz+q_2zx+q_3xy \tag{23}$$

holds for all real numbers x,y,z. If $x,y,z\neq 0$, then the equality in (23) holds if and only if

$$(2p_1q_1+q_2q_3)x=(2p_2q_2+q_3q_1)y=(2p_3q_3+q_1q_2)z. \tag{24}$$

Here, we point out that Theorem 1 can also be proved by using the above conclusion (omit here).

REMARK 2. In fact, inequality (10) can be extended to the following

$$\begin{aligned}&x^2+y^2+z^2\\&\geqslant(z\cos B+y\cos C)^2+(x\cos C+z\cos A)^2+(y\cos A+x\cos B)^2\\&\geqslant 2(yz\cos A+zx\cos B+xy\cos C),\end{aligned} \tag{25}$$

which gives a refinement of the Wolstenholme inequality (1) for the acute triangle. The second inequality in (25) actually holds for all triangles and can also be proved by using the conclusion given in Remark 1. Clearly, inequality chain (25) is equivalent to

$$\begin{aligned} & x^2+y^2+z^2 \\ & \geqslant \left(z\sin\frac{B}{2}+y\sin\frac{C}{2}\right)^2+\left(x\sin\frac{C}{2}+z\sin\frac{A}{2}\right)^2+\left(y\sin\frac{A}{2}+x\sin\frac{B}{2}\right)^2 \\ & \geqslant 2\left(yz\sin\frac{A}{2}+zx\sin\frac{B}{2}+xy\sin\frac{C}{2}\right), \end{aligned} \tag{26}$$

which holds for any triangle ABC.

3. Two equivalent weighted geometric inequalities

In this section, we shall prove two weighted geometric inequalities, which are equivalent with the result of Theorem 1. In what follows, we shall continue to use the previous symbols.

The famous Erdös-Mordell inequality states that

$$R_1+R_2+R_3 \geqslant 2(r_1+r_2+r_3). \tag{27}$$

(Some recent results about this inequality can be found in [2], [8-10], [12] and [13]). And, we know that D.F.Barrow obtains the following improvement of (27):

$$R_1+R_2+R_3 \geqslant 2(w_1+w_2+w_3). \tag{28}$$

In fact, it is well known that these two inequalities can be generalized to the case with weights (see [11]):

$$x^2R_1+y^2R_2+z^2R_3 \geqslant 2(yzr_1+zxr_2+xyr_3) \tag{29}$$

and

$$x^2R_1+y^2R_2+z^2R_3 \geqslant 2(yzw_1+zxw_2+xyw_3) \tag{30}$$

respectively, where x,y,z are arbitrary real numbers. Both equalities in (29) and (30) hold if and only if $x:y:z=\sin A:\sin B:\sin C$ and P is the circumcenter of the triangle ABC.

In [5], the author has proved that the Wolstenholme inequality (1) is equivalent to inequalities (29) and (30). In this section, we shall apply inequality (10) to establish two weighted inequalities involving the sharpened versions of the Erdös-Mordell inequality and Barrow's inequality, which are both equivalent with inequality (10).

The following sharped version of the Erdös-Mordell inequality

$$\frac{(r_2+r_3)^2}{R_1}+\frac{(r_3+r_1)^2}{R_2}+\frac{(r_1+r_2)^2}{R_3} \leqslant R_1+R_2+R_3 \tag{31}$$

is established by the author in [8]. Recently, when the author tried to prove the following sharpness of (31), i.e.

$$\frac{(w_2+w_3)^2}{R_1}+\frac{(w_3+w_1)^2}{R_2}+\frac{(w_1+w_2)^2}{R_3} \leqslant R_1+R_2+R_3, \tag{32}$$

the main result of this paper, i.e., inequality (10) was found. And, we obtained the weighted generalizations of (31) and (32), which are both equivalent with (10), as follows:

THEOREM 2. *Let P be an interior point P of any triangle ABC (P may lie on the boundary except the vertices of ABC), then for any real numbers x,y,z the following inequality holds:*

$$\frac{(zr_2+yr_3)^2}{R_1}+\frac{(xr_3+zr_1)^2}{R_2}+\frac{(yr_1+xr_2)^2}{R_3}\leqslant x^2R_1+y^2R_2+z^2R_3, \tag{33}$$

If ABC is an acute triangle, then the equality in (33) holds if and only if P is the circumcenter of ABC and $x:y:z=\sin A:\sin B:\sin C$; If ABC is a right triangle with $A=\pi/2$, then the equality in (33) holds if and only if $y=z, B=C=\pi/4$ and P is the midpoint of BC.

THEOREM 3. *Let P be an interior point P of any triangle ABC (P may lie on the boundary except the vertices of ABC), then for any real numbers x,y,z the following inequality holds:*

$$\frac{(zw_2+yw_3)^2}{R_1}+\frac{(xw_3+zw_1)^2}{R_2}+\frac{(yw_1+xw_2)^2}{R_3}\leqslant x^2R_1+y^2R_2+z^2R_3, \tag{34}$$

If ABC is an acute triangle, then the equality in (34) holds if and only if P is the circumcenter of ABC and $x:y:z=\sin A:\sin B:\sin C$; If ABC is a right triangle with $A=\pi/2$, then the equality in (34) holds if and only if $y=z$, $B=C=\pi/4$ and P is the midpoint of BC.

Now, we give the proof of Theorem 3 (we shall show that Theorem 2 follows from Theorem 3 easily).

Proof. In triangle ABC, we know that the lengths of bisector w_a of $\angle BAC$ is given by

$$w_a=\frac{2bc}{b+c}\cos\frac{A}{2}. \tag{35}$$

Since $b+c\geqslant 2\sqrt{bc}$, we have

$$w_a\leqslant\sqrt{bc}\cos\frac{A}{2}, \tag{36}$$

with equality if and only if $b=c$.

Applying inequality (36) to $\triangle BPC$, we know that for any interior point P of triangle ABC holds:

$$w_1\leqslant\sqrt{R_2R_3}\cos\frac{\alpha}{2}, \tag{37}$$

where $\alpha=\angle BPC$. Equality holds if and only if $R_2=R_3$. Similarly, we have

$$w_2\leqslant\sqrt{R_3R_1}\cos\frac{\beta}{2},\ w_3\leqslant\sqrt{R_1R_2}\cos\frac{\gamma}{2}, \tag{38}$$

where $\beta = \angle CPA$ and $\gamma = \angle APB$. Therefore, in the case $x,y,z > 0$, to prove (34) we need to prove that

$$\left(z\sqrt{R_3}\cos\frac{\beta}{2} + y\sqrt{R_2}\cos\frac{\gamma}{2}\right)^2 + \left(x\sqrt{R_1}\cos\frac{\gamma}{2} + z\sqrt{R_3}\cos\frac{\alpha}{2}\right)^2 + \left(y\sqrt{R_2}\cos\frac{\alpha}{2} + x\sqrt{R_1}\cos\frac{\beta}{2}\right)^2 \leqslant x^2R_1 + y^2R_2 + z^2R_3. \tag{39}$$

Note that $\alpha/2, \beta/2, \gamma/2$ can be viewed angles of a non-obtuse triangle, thus we may put $A = \alpha/2, B = \beta/2, C = \gamma/2$ in inequality (10) and make the substitutions $x \to x\sqrt{R_1}, y \to y\sqrt{R_2}, z \to z\sqrt{R_3}$ at the same time. We therefore know that inequality (39) and then (34) hold for positive real numbers x,y,z.

To show inequality (34) holds for any real numbers x,y,z, we first prove the following inequality:

$$R_1 \geqslant \frac{w_2^2}{R_2} + \frac{w_3^2}{R_3}. \tag{40}$$

By (38), we only need to prove that

$$\cos^2\frac{\beta}{2} + \cos^2\frac{\gamma}{2} \leqslant 1. \tag{41}$$

Since $\alpha/2, \beta/2, \gamma/2$ can be regarded as angles of a non-obtuse triangle. Thus, if we can show that the following inequality

$$\cos^2 B + \cos^2 C \leqslant 1 \tag{42}$$

holds for the non-obtuse triangle ABC, then (41) is proved. Clearly, to prove (42) we only need to show that $\sin B \geqslant \cos C$. Since

$$\begin{aligned}\sin B - \cos C &= \sin B - \sin\left(\frac{\pi}{2} - C\right)\\ &= 2\cos\frac{1}{2}\left(B - C + \frac{1}{2}\pi\right)\sin\frac{1}{2}\left(B + C - \frac{1}{2}\pi\right)\\ &= 2\sin\left(\frac{1}{2}\pi - A\right)\cos\left(\frac{\pi}{4} + \frac{B-C}{2}\right)\end{aligned}$$

and $B - C < \pi/2$, we conclude that $\sin B \geqslant \cos C$ holds for the non-obtuse triangle ABC. Thus, inequality (40) is proved (equality in (40) holds if and only if $A = \pi/2, B = C = \pi/4$ and P is the midpoint of BC).

Now, we notice that the following simple conclusion (which is easily shown): If $p_1, p_2, p_3, q_1, q_2, q_3 \geqslant 0$ and the ternary quadratic inequality (23) holds for positive real numbers x,y,z, then (23) holds for all real numbers x,y,z. Since (34) has been proved for $x,y,z > 0$, thus by inequality (40) we can further conclude that inequality (34) holds for all real numbers x,y,z.

Finally, according to the equality condition of (10) we easily confirm that of inequality (34). The proof of Theorem 3 is completed.

REMARK 3. Since $r_1 \leqslant w_1, r_2 \leqslant w_2, r_3 \leqslant w_3$, we conclude easily that Theorem 2 can be obtained by Theorem 3.

REMARK 4. If we let triangle ABC be a non-obtuse triangle and let P be its circumcenter, then $R_1 = R_2 = R_3 = R, r_1 = w_1 = R\cos A, r_2 = w_2 = R\cos B, r_3 = w_3 = R\cos C$ (R being the circumradius of ABC). In this case ,inequality (10) follows from inequality (33) and (34). So, Theorem 2 and 3 are actually both equivalent with Theorem 1.

REMARK 5. By inequality (34) of Theorem 3, it is easy to obtain the previous inequality (9). Indeed, since $(zw_2 + yw_3)^2 \geqslant 4yzw_2w_3$, it follows from (34) that

$$x^2R_1 + y^2R_2 + z^2R_3 \geqslant 4\left(yz\frac{w_2w_3}{R_1} + zx\frac{w_3w_1}{R_2} + xy\frac{w_1w_2}{R_3}\right).$$

Replacing $x \to x\sqrt{R_2R_3/R_1}$ etc. in the above inequality, we obtain (9) at once.

4. Some applications Theorem 1 and Theorem 2

In this section, we give some applications of Theorem 1 and Theorem 2.

Clearly, by Theorem 1 we have the following beautiful trigonometric inequality:

COROLLARY 1. *In the non-obtuse triangle ABC, the following inequality holds:*

$$(\cos B + \cos C)^2 + (\cos C + \cos A)^2 + (\cos A + \cos B)^2 \leqslant 3. \tag{43}$$

From this inequality, we can derive the following Walker's non-obtuse triangle inequality [16]:

$$s^2 \geqslant 2R^2 + 8Rr + 3r^2, \tag{44}$$

where R and r denote the circumradius and inradius of $\triangle ABC$, respectively.

Form the equivalent form (11) of inequality (10), it is easy to get

$$\frac{\sin^2 B - \cos^2 C}{\cos^2 A}x^2 + \frac{\sin^2 C - \cos^2 A}{\cos^2 B}y^2 + \frac{\sin^2 A - \cos^2 B}{\cos^2 C}z^2 \geqslant 2(yz + zx + xy). \tag{45}$$

In addition, by (2) we have the following equivalent inequality:

$$\frac{x^2}{\cos^2 A} + \frac{y^2}{\cos^2 B} + \frac{z^2}{\cos^2 C} \geqslant 4(yz + zx + xy). \tag{46}$$

Adding (45) and (46) gives

COROLLARY 2. *For the acute triangle ABC and real numbers x, y, z, the following inequality holds:*

$$\frac{\sin^2 B + \sin^2 C}{\cos^2 A}x^2 + \frac{\sin^2 C + \sin^2 A}{\cos^2 B}y^2 + \frac{\sin^2 A + \sin^2 B}{\cos^2 C}z^2 \geqslant 6(yz + zx + xy). \tag{47}$$

Noting that in the acute triangle ABC holds (see [5, p. 351]):

$$2\cos^2\frac{A}{2}\geqslant \sin^2 B+\sin^2 C, \tag{48}$$

we get the following inequality from (47).

COROLLARY 3. *For the acute triangle ABC and real numbers x,y,z, the following inequality holds:*

$$\frac{\cos^2\frac{A}{2}}{\cos^2 A}x^2+\frac{\cos^2\frac{B}{2}}{\cos^2 B}y^2+\frac{\cos^2\frac{C}{2}}{\cos^2 C}z^2\geqslant 3(yz+zx+xy). \tag{49}$$

For a point P in the plane of any triangle ABC, we have the following Hayashi inequality (see [4]):

$$\frac{R_2R_3}{bc}+\frac{R_3R_1}{ca}+\frac{R_1R_2}{ab}\geqslant 1. \tag{50}$$

Thus, putting $x=R_1/a$ in (49) etc., we get

COROLLARY 4. *For a point P in the plane of the acute triangle ABC, the following inequality holds:*

$$\frac{\sin^2 B+\sin^2 C}{\sin^2 2A}R_1^2+\frac{\sin^2 C+\sin^2 A}{\sin^2 2B}R_2^2+\frac{\sin^2 A+\sin^2 B}{\sin^2 2C}R_3^2\geqslant 12R^2. \tag{51}$$

By inequalities (48) and (51), we easily obtain

COROLLARY 5. *For a point P in the plane of the acute triangle ABC, the following inequality holds:*

$$\frac{R_1^2}{\cos^2 A\sin^2\frac{A}{2}}+\frac{R_2^2}{\cos^2 B\sin^2\frac{B}{2}}+\frac{R_3^2}{\cos^2 C\sin^2\frac{C}{2}}\geqslant 48R^2. \tag{52}$$

Let P be the incenter of $\triangle ABC$, then

$$r_1=r_2=r_3=R_1\sin\frac{A}{2}=R_2\sin\frac{B}{2}=R_3\sin\frac{C}{2}=r.$$

Thus, by Theorem 2 we have

COROLLARY 6. *For any triangle ABC and real numbers x,y,z, the following inequality holds:*

$$\frac{x^2}{\sin\frac{A}{2}}+\frac{y^2}{\sin\frac{B}{2}}+\frac{z^2}{\sin\frac{C}{2}}\geqslant (y+z)^2\sin\frac{A}{2}+(z+x)^2\sin\frac{B}{2}+(x+y)^2\sin\frac{C}{2}. \tag{53}$$

REMARK 6. Inspired by inequality (53) and the known inequality:

$$\sin\frac{A}{2} \leqslant \frac{a}{b+c}, \tag{54}$$

the author finds that the previous inequality (7) can be improved to the following

$$x^2\frac{v+w}{u}+y^2\frac{w+u}{v}+z^2\frac{u+v}{w} \geqslant \frac{u}{v+w}(y+z)^2+\frac{v}{w+u}(z+x)^2+\frac{w}{u+v}(x+y)^2, \tag{55}$$

with equality if and only if $x:y:z=u:v:w$. This inequality can also be proved by using the conclusion given in Remark 1 (we omit here).

For an interior point P of any triangle ABC, we have the following well known inequality (see [1, inequality 12.20]):

$$2R_1\sin\frac{A}{2} \geqslant r_2+r_3, \tag{56}$$

thus by (53) we have

$$\begin{aligned}&\frac{R_1}{r_2+r_3}x^2+\frac{R_2}{r_3+r_1}y^2+\frac{R_3}{r_1+r_2}z^2\\ &\geqslant \frac{r_2+r_3}{4R_1}(y+z)^2+\frac{r_3+r_1}{4R_2}(z+x)^2+\frac{r_1+r_2}{4R_3}(x+y)^2.\end{aligned} \tag{57}$$

Since $(y+z)^2 \geqslant 4yz$, then

$$\frac{R_1}{r_2+r_3}x^2+\frac{R_2}{r_3+r_1}y^2+\frac{R_3}{r_1+r_2}z^2 \geqslant yz\frac{r_2+r_3}{R_1}+zx\frac{r_3+r_1}{R_2}+xy\frac{r_1+r_2}{R_3}. \tag{58}$$

By replacing $x \to x\sqrt{R_2R_3/R_1}$ etc., we get

COROLLARY 7. *For an interior point P of any triangle ABC and real numbers x,y,z, the following inequality holds:*

$$\frac{R_2R_3}{r_2+r_3}x^2+\frac{R_3R_1}{r_3+r_1}y^2+\frac{R_1R_2}{r_1+r_2}z^2 \geqslant yz(r_2+r_3)+zx(r_3+r_1)+xy(r_1+r_2). \tag{59}$$

In particular, for $x=y=z=1$, we have

$$\frac{R_2R_3}{r_2+r_3}+\frac{R_3R_1}{r_3+r_1}+\frac{R_1R_2}{r_1+r_2} \geqslant 2(r_1+r_2+r_3), \tag{60}$$

which is given in [5,Corollary 4.59].

Putting $x=r_a, y=r_b, z=r_c$ (r_a being the corresponding radius of described circle of ABC, etc.) in (59) and then noting that $r_br_c=s(s-a)$ and

$$\begin{aligned}&(s-a)(r_2+r_3)+(s-b)(r_3+r_1)+(s-c)(r_1+r_2)\\ &=ar_1+br_2+cr_3\\ &=2rs,\end{aligned}$$

we obtain

COROLLARY 8. *For an interior point P of any triangle ABC, the following inequality holds:*

$$\frac{R_2R_3}{r_2+r_3}r_a^2+\frac{R_3R_1}{r_3+r_1}r_b^2+\frac{R_1R_2}{r_1+r_2}r_c^2\geqslant 2rs^2. \tag{61}$$

Next, we shall apply the weighted trigonometric inequality (53) to derive a geometric inequality. Firstly, we establish the following inequality:

$$a^2\sin\frac{A}{2}+b^2\sin\frac{B}{2}+c^2\sin\frac{C}{2}\geqslant\frac{1}{2}(a^2+b^2+c^2). \tag{62}$$

Using the Law of Cosines, we easily verify the following identity:

$$\begin{aligned}&a^2(\cos B+\cos C)+b^2(\cos C+\cos A)+c^2(\cos A+\cos B)-(a^2+b^2+c^2)\\&=\frac{(b+c-a)(b-c)^2}{2a}+\frac{(c+a-b)(c-a)^2}{2b}+\frac{(a+b-c)(a-b)^2}{2c},\end{aligned} \tag{63}$$

which shows that

$$a^2(\cos B+\cos C)+b^2(\cos C+\cos A)+c^2(\cos A+\cos B)\geqslant(a^2+b^2+c^2). \tag{64}$$

Since $\cos B+\cos C\leqslant 2\sin\frac{A}{2}$, inequality (62) follows from (64) immediately.

By the previous formula (21) and inequality (36), one easily can obtain that

$$4w_a^2\leqslant(b+c)^2-a^2. \tag{65}$$

Applying this inequality to $\triangle BPC$, we know that the following inequality

$$(R_2+R_3)^2\geqslant a^2+4w_1^2 \tag{66}$$

holds for any interior point P of the triangle ABC and its two analogues are valid. Putting $x=R_1,y=R_2,z=R_3$ in (53) and then using (66) and (62), we get

COROLLARY 9. *For an interior point P of any triangle ABC, the following inequality holds:*

$$\frac{R_1^2}{\sin\frac{A}{2}}+\frac{R_2^2}{\sin\frac{B}{2}}+\frac{R_3^2}{\sin\frac{C}{2}}-4\left(w_1^2\sin\frac{A}{2}+w_2^2\sin\frac{B}{2}+w_3^2\sin\frac{C}{2}\right)\geqslant\frac{1}{2}(a^2+b^2+c^2). \tag{67}$$

5. Open problems

In [5],[8],[9],and [10], the author presents some conjectures for the Erdös-Mordell inequality (27). Here, we give the following two conjectures as open problems again.

CONJECTURE 1. *For an interior point P of any triangle ABC, the following inequality holds:*

$$\frac{R_1+R_2+R_3}{r_1+r_2+r_3}\geqslant\frac{4R(m_a+m_b+m_c)}{a^2+b^2+c^2}, \tag{68}$$

where m_a,m_b,m_c are the medians of ABC.

CONJECTURE 2. *For an interior point P of any triangle ABC, the following inequality holds:*

$$\frac{R_1+R_2+R_3}{r_1+r_2+r_3} \geqslant \frac{1}{2}\left(\frac{a^2}{m_b m_c}+\frac{b^2}{m_c m_a}+\frac{c^2}{m_a m_b}\right). \tag{69}$$

It is easily proved that both values of the right hands of (68) and (69) are not less than 2 (but can not be compared with each other), thus inequalities (68) and (69) are sharpened versions of the Erdös-Mordell inequality.

For the previous geometric inequality (32), we present the following generalized conjecture with one parameter:

CONJECTURE 3. *For an interior point P of any triangle ABC and positive number k, the following inequality holds:*

$$\frac{(kw_1+w_2+w_3)^2}{R_1+kw_1}+\frac{(kw_2+w_3+w_1)^2}{R_2+kw_2}+\frac{(kw_3+w_1+w_2)^2}{R_3+kw_3} \leqslant \frac{k+2}{2}(R_1+R_2+R_3). \tag{70}$$

Considering generalizations of Theorem 3 for polygons, we put forward the following conjecture:

CONJECTURE 4. *Let P be an interior point of the convex polygon $A_1A_2\cdots A_n$ ($n>3$), and let $PA_i=R_i(i=1,2,\cdots,n)$. Denote by $w_1,w_2,\cdots w_n$ the lengths of the bisector of $\angle A_iPA_{i+1}$ ($i=1,2,\ldots,n,A_{n+1}=A_1$). Then*

$$\sum_{i=1}^{n} R_i \geqslant \frac{1}{4}\sec^2\frac{\pi}{n}\sum_{i=1}^{n}\frac{(w_i+w_{i+1})^2}{R_i}. \tag{71}$$

If the above inequality holds true, then using Cauchy inequality one can easily obtain the well known result:

$$\sum_{i=1}^{n} R_i \geqslant \sec\frac{\pi}{n}\sum_{i=1}^{n} w_i, \tag{72}$$

which is first proved by N.Ozeki in [15].

Finally, for the previous inequality (43), we give the following general conjecture with double exponents:

CONJECTURE 5. *If $m \geqslant 1$ and $n \geqslant 1$, then the following inequality:*

$$a^m(\cos B+\cos C)^n+b^m(\cos C+\cos A)^n+c^m(\cos A+\cos B)^n \geqslant a^m+b^m+c^m \tag{73}$$

holds for any triangle ABC. If $n>0\geqslant m$ and $n+m\leqslant 1$, then (73) holds reversely; If $n>0\geqslant m$ and $n+m\leqslant 2$, then (73) holds reversely for the non-obtuse triangle ABC.

In the case when $m=0$ and $n=2$, the above conjecture becomes Corollary 1.

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