Journal of Mathematical Inequalities
Volume 14, Number 1 (2020), 161–173 doi:10.7153/jmi-2020-14-12

MONOTONICITY PROPERTIES ON k–DIGAMMA FUNCTION AND ITS RELATED INEQUALITIES

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(*Communicated by G. Nemes*)

Abstract. In this work, we give some monotonicity properties of k-analogues of digamma and polygamma functions and then we obtain some inequalities related to these functions. At last, we give harmonic mean inequality for k-digamma function for all positive real values of k and x.

1. Introduction

The gamma function, which is introduced by Euler, is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

for $x > 0$. Gautschi [8] obtained a very interesting mean inequality for the function. He showed that the harmonic mean inequality

$$1 \leqslant \frac{2\Gamma(x)\Gamma(1/x)}{\Gamma(x) + \Gamma(1/x)} \tag{1}$$

is valid for all positive real values of x. Even though the function is used by many different branches, since its first derivative increases dramatically, logarithms of the function is also interested by many researchers. Such as in [28], the Binet's first formula for $\ln\Gamma(x)$ states that

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-xt}}{t}\, dt \tag{2}$$

for $x > 0$. The logarithmic derivative of gamma function is called digamma (or psi) function and its integral representation is given by

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-tx}}{1 - e^t}\right) dt \tag{3}$$

Mathematics subject classification (2010): 26A48, 33B15, 26D07.
Keywords and phrases: k-polygamma function, k-digamma function, inequality, k-Binet's first formula for $ln\Gamma_k(x)$.

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for $x>0$. In [16], authors used the equation (2) to obtain completely monotonic properties of functions involving the digamma and polygamma functions. As a corollary, they found the following double sided inequalities

$$\frac{1}{2x}-\frac{1}{12x^2}<\psi(x+1)-\ln x<\frac{1}{2x} \tag{4}$$

$$\frac{1}{2x^2}-\frac{1}{6x^3}<\frac{1}{x}-\psi'(x+1)<\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5} \tag{5}$$

for $x>0$. In [2], Alzer and Jameson presented some inequalities and concavity properties of the functions involving the digamma function and proved that the harmonic mean inequality for digamma function

$$-\gamma\leqslant\frac{2\psi(x)\psi(1/x)}{\psi(x)+\psi(1/x)} \tag{6}$$

holds true for all positive real numbers x, where $\gamma=\lim_{n\to\infty}(1+\ldots+\frac{1}{n}-\log n)$. The sign of equality holds if and only if $x=1$. We refer the interested readers to [17, 18, 4, 6, 9, 19, 10, 11, 20, 5] and references therein for more information about gamma and digamma functions.

During last decade, Díaz and Pariguan in [7] introduced k-generalized Pochhammer symbol as follows:

DEFINITION 1. [7] Let $x\in\mathbb{C}$, $k\in\mathbb{R}$ and $n\in\mathbb{N}^+$, the Pochhammer k-symbol is given by

$$(x)_{n,k}=x(x+k)(x+2k)\ldots(x+(n-1)k).$$

By using the definition, they defined k-gamma function Γ_k as the following limit expression.

DEFINITION 2. [7] For $k>0$, the k-gamma function Γ_k is given by

$$\Gamma_k(x)=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{n}{k}-1}}{(x)_{n,k}},\quad x\in\mathbb{C}\setminus k\mathbb{Z}^-.$$

Also in the paper, they obtained integral and infinite product representations of the function by

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt$$

$$\frac{1}{\Gamma_k(x)}=xk^{-\frac{x}{k}}e^{\frac{x}{k}\gamma}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{nk}\right)e^{-\frac{x}{nk}}\right)$$

for $x \in \mathbb{C}$, $Re(x) > 0$. They proved the k-generalization of Bohr-Mollerup Theorem, Stirling formula and some properties on k-gamma function such as

$$\Gamma_k(x+k) = x\Gamma_k(x) \tag{7}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)} \tag{8}$$

$$\Gamma_k(k) = 1 \tag{9}$$

$$\Gamma_k(x) \text{ is logarithmically convex for } x \in \mathbb{R} \tag{10}$$

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right). \tag{11}$$

They also defined k-zeta function as follows:

DEFINITION 3. [7] The k-zeta function is given by

$$\zeta_k(x,s) = \sum_{n=0}^{\infty} \frac{1}{(x+nk)^s} \tag{12}$$

for $k, x > 0$ and $s > 1$.

In [14], Krasniqi obtained series representation of k-digamma (or k-psi, generalized digamma) function as

$$\psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(x+nk)} \tag{13}$$

for $x, k > 0$ and showed that the function $\psi_k'(x)$ is strictly completely monotonic on $(0,\infty)$. In [25], authors gave several integral representations of k-digamma function such as

$$\psi_k(x) = \frac{\ln k}{k} + \frac{1}{k}\int_0^{\infty}\left(\frac{e^{-t}}{t} - \frac{e^{-\frac{x}{k}t}}{1-e^{-t}}\right)dt \tag{14}$$

for $x > 0$. Applying logarithmic derivative of the equation (11) leads us to the recurrence formula for k-digamma function by

$$\psi_k(x+k) = \frac{1}{x} + \psi_k(x) \tag{15}$$

and for first and second derivatives of the equation (15), we get

$$\psi_k'(x+k) = \psi_k'(x) - \frac{1}{x^2} \tag{16}$$

$$\psi_k''(x+k) = \psi_k''(x) + \frac{2}{x^3} \tag{17}$$

respectively for $x, k > 0$.

In [13], Kokologiannaki showed the following monotonicity property for k-zeta function.

THEOREM 1. [13] *The k-zeta function $\zeta_k(x,s)$ decrease with respect to s for $x > 1$ and $k > 0$, $n > 0$, $s > 1$.*

Yin et. al. in [26] showed the following completely monotonicity properties of functions related to k-digamma and k-polygamma functions.

THEOREM 2. [26] *For $k > 0$, the function $x^2\psi_k'(x)$ is strictly increasing on $(0,\infty)$.*

THEOREM 3. [26] *For $k > 0$, the function $\psi_k'(\frac{1}{x})$ is strictly concave on $(0,\infty)$.*

It is worth to mention that theorems 2 and 3 tend to the classical ones in [12]. Also authors obtained concavity of the function related to k-digamma function.

THEOREM 4. [26] *For $k \geqslant \frac{1}{\sqrt[3]{3}}$, the function $\lambda_k(x) = \psi_k(x) + \psi_k(\frac{1}{x})$ is strictly concave on $(0,\infty)$.*

Thus authors in [26] gave the following inequalities for k-digamma function.

THEOREM 5. [26] *The following inequalities*

$$\psi_k(x) + \psi_k\left(\frac{1}{x}\right) \leqslant \frac{2\ln k + 2\psi(\frac{1}{k})}{k}, \qquad x > 0, k \geqslant \frac{1}{\sqrt[3]{3}} \tag{18}$$

$$\psi_k(1+x)\psi_k(1-x) \leqslant \frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k^2}, \qquad 0 < x < 1, \frac{1}{\sqrt[3]{3}} \leqslant k \leqslant 1 \tag{19}$$

$$\psi_k(x)\psi_k\left(\frac{1}{x}\right) \leqslant \frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k^2}, \qquad 0 < x, \frac{1}{\sqrt[3]{3}} \leqslant k \leqslant 1 \tag{20}$$

hold true.

Hence they obtained harmonic mean inequality for k-digamma function as follows:

COROLLARY 1. [26] *For $x \in (0,\infty)$ and $\frac{1}{\sqrt[3]{3}} \leqslant k \leqslant 1$, we have*

$$\frac{2\psi_k(x)\psi_k(\frac{1}{x})}{\psi_k(x) + \psi_k(\frac{1}{x})} \geqslant \frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k[\ln k + \psi(\frac{1}{k})]}. \tag{21}$$

One can find more information about k-special functions and its related topic in [1, 3, 15, 21, 22, 23, 24, 25, 26, 27, 29] and references therein.
In this paper, we shall show a k-generalization of Binet's formula (2). Then we shall obtain some monotonic properties of functions related to k-digamma and k-polygamma functions. As a corollary, we shall find double sided inequalities for k-digamma and k-polygamma functions and by the aim of these results, we shall lastly obtain mean inequalities for k-digamma functions which are slightly generalizations of some results obtained in [26]. We want to note that all results in this work are valid for all positive real values of k.

2. Main Results

Before we obtain main results, we need the following properties.

LEMMA 1. *For $x > 0$ and any non-negative integer n, the integral*

$$\frac{1}{x^{n+1}} = \frac{1}{k^{n+1}n!}\int_0^\infty t^n e^{-\frac{xt}{k}}\,dt$$

holds true.

Proof. The result can be proven by using mathematical induction and integration by parts. □

Thus we can give a k-generalization of Binet's first formula (2).

LEMMA 2. *For $x > 0$, the equalities*

$$\ln\Gamma_k(x) = \left(\frac{x}{k}-\frac{1}{2}\right)\ln x - \frac{1}{2}\ln k - \frac{x}{k} + \ln\sqrt{2\pi} + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-\frac{xt}{k}}}{t}\,dt \quad (22)$$

and

$$\psi_k(x) = \frac{\ln x}{k} - \frac{1}{x} + \frac{1}{k}\int_0^\infty \left(\frac{1}{t}-\frac{1}{e^t-1}\right)e^{-\frac{xt}{k}}\,dt \quad (23)$$

are valid.

Proof. For the proof of equation (22), let us take $x+k$ instead of x in equation (14). So we have

$$\psi_k(x+k) = \frac{\ln k}{k} + \frac{1}{k}\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-\frac{x}{k}t}}{e^t-1}\right)dt.$$

By using the following integrals

$$\frac{1}{2x} = \frac{1}{2k}\int_0^\infty e^{-\frac{x}{k}t}\,dt \quad \text{and} \quad \ln\frac{x}{k} = \int_0^\infty \frac{e^{-t}-e^{-\frac{x}{k}t}}{t}\,dt,$$

we get

$$\psi_k(x+k) = \frac{\ln x}{k} + \frac{1}{2x} - \frac{1}{k}\int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)e^{-\frac{x}{k}t}\,dt.$$

The integrand is continuous as $t \to 0$ and since the term $\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}$ is bounded as $t \to \infty$, the integral converges uniformly for $x > 0$. Integrating from k to x and using the relation (7) yields

$$\begin{aligned}\ln\Gamma_k(x+k) - \ln\Gamma_k(2k) &= \frac{1}{k}(x\ln x - x) - \frac{1}{k}(k\ln k - k) + \frac{1}{2}\ln x - \frac{1}{2}\ln k \\ &\quad + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-\frac{x}{k}t}-e^{-t}}{t}\,dt\end{aligned}$$

$$\ln\Gamma_k(x) = \left(\frac{x}{k}-\frac{1}{2}\right)\ln x - \frac{x}{k} - \frac{1}{2}\ln k + 1 + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-\frac{x}{k}t}}{t}dt$$
$$-\int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-t}}{t}dt.$$

Taking $x=\frac{k}{2}$ in the last equation leads us that

$$\ln\Gamma_k(k/2) = -\frac{1}{2}-\frac{1}{2}\ln k + 1 + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-\frac{t}{2}}}{t}dt$$
$$-\int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-t}}{t}dt$$
$$\frac{1}{2}(\ln\pi - \ln k) = \frac{1}{2}-\frac{1}{2}\ln k + J - I$$

where J and I denote the integrals on the right side of the equation, respectively. Since $J=\frac{1}{2}+\frac{1}{2}\ln\frac{1}{2}$ and $I = 1-\ln\sqrt{2\pi}$ (see [28, pp. 248-250]), the proof is completed.

We want to note that using the relation between k-gamma and gamma functions (11) and replacing x by $\frac{x}{k}$ at the equation (2) lead us

$$\ln\Gamma_k(x) = \left(\frac{x}{k}-1\right)\ln k + \left(\frac{x}{k}-\frac{1}{2}\right)\ln\frac{x}{k} - \frac{x}{k} + \ln\sqrt{2\pi} + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-\frac{xt}{k}}}{t}dt$$

as desired. One can obtain equation (23) by differentiating the equation (22). □

Now we are ready to obtain monotonicity properties of functions involving k-digamma and k-polygamma functions for all $x,k>0$.

THEOREM 6. *The following functions*

$$\psi_k(x) - \frac{\ln x}{k} + \frac{1}{2x} + \frac{k}{12x^2}, \tag{24}$$
$$\frac{\ln x}{k} - \frac{1}{2x} - \psi_k(x), \tag{25}$$
$$\psi_k'(x) - \frac{1}{kx} - \frac{1}{2x^2} - \frac{k}{6x^3} + \frac{k^3}{30x^5}, \tag{26}$$
$$\frac{1}{kx} + \frac{1}{2x^2} + \frac{k}{6x^3} - \psi_k'(x) \tag{27}$$

are strictly completely monotonic for $x>0$.

Proof. By using the equation (23) and Lemma 1, the first function can be defined by

$$f_1(x) = \psi_k(x) - \frac{\ln x}{k} + \frac{1}{2x} + \frac{k}{12x^2}$$
$$= \frac{1}{k}\int_0^\infty \left(\frac{1}{t}-\frac{1}{e^t-1}-\frac{1}{2}+\frac{t}{12}\right)e^{-\frac{xt}{k}}dt$$

for all positive real values of x and k. Then we get

$$f_1(x) = \frac{1}{k}\int_0^\infty \frac{(t^2-6t+12)e^t-(t^2+6t+12)}{12t(e^t-1)}e^{-\frac{xt}{k}}\,dt.$$

Let us denote the nominator as d_1. Then we get that $d_1(t) = (t^2-6t+12)e^t-(t^2+6t+12) > 0$ and $d_1(0)=0$. Hence we obtain that $(-1)^n f_1^{(n)}(x) > 0$ as desired.
Now let us define the function f_2 by

$$f_2(x) = \psi_k(x) - \frac{\ln x}{k} + \frac{1}{2x}$$

for $x > 0$. Then we obtain

$$\begin{aligned} f_2(x) &= \frac{1}{k}\int_0^\infty \left(\frac{1}{t}-\frac{1}{e^t-1}-\frac{1}{2}\right)e^{-\frac{xt}{k}}\,dt \\ &= \frac{1}{k}\int_0^\infty \frac{(2-t)e^t-(t+2)}{2t(e^t-1)}e^{-\frac{xt}{k}}\,dt. \end{aligned}$$

Let us denote the nominator as d_2. So we find that $d_2(t) = (2-t)e^t-(t+2) < 0$ and $d_2(0)=0$ for all $t>0$. Hence, $(-1)^{n+1} f_2^{(n)}(x) > 0$ as desired.
Differentiating the equation (23) leads that

$$\psi_k'(x) = \frac{1}{kx}+\frac{1}{x^2}+\frac{1}{k^2}\int_0^\infty \left(\frac{t}{e^t-1}-1\right)e^{-\frac{xt}{k}}\,dt \tag{28}$$

Then by using the equation (28) and Lemma 1, we can define the function by

$$\begin{aligned} f_3(x) &= \frac{1}{kx}+\frac{1}{2x^2}+\frac{k}{6x^3}-\psi_k'(x) \\ &= \frac{k}{6x^3}-\frac{1}{2x^2}-\frac{1}{k^2}\int_0^\infty \left(\frac{t}{e^t-1}-1\right)e^{-\frac{xt}{k}}\,dt \\ &= \frac{k}{k^3 2!6}\int_0^\infty t^2 e^{-\frac{xt}{k}}\,dt - \frac{1}{2k^2}\int_0^\infty t e^{-\frac{xt}{k}}\,dt - \frac{1}{k^2}\int_0^\infty \left(\frac{t}{e^t-1}-1\right)e^{-\frac{xt}{k}}\,dt \\ &= \frac{1}{k^2}\int_0^\infty \left(\frac{t^2}{12}-\frac{t}{2}-\frac{t}{e^t-1}+1\right)e^{-\frac{xt}{k}}\,dt \\ &= \frac{1}{k^2}\int_0^\infty \frac{(12-6t+t^2)e^t-(t^2+6t+12)}{12(e^t-1)}e^{-\frac{xt}{k}}\,dt. \end{aligned}$$

Since nominator of the function is same as the function f_1, we get $(-1)^n f_3^{(n)}(x) > 0$ for non-negative integer values of n.

At last, let us define the function f_4 by

$$\begin{aligned} f_4(x) &= \psi_k'(x) - \frac{1}{kx} - \frac{1}{2x^2} - \frac{k}{6x^3} + \frac{k^3}{30x^5} \\ &= \frac{1}{k^2}\int_0^\infty \left(\frac{t}{2} - \frac{t^2}{12} + \frac{t^4}{720} + \frac{t}{e^t-1} - 1\right) e^{-\frac{xt}{k}}\,dt \\ &= \frac{1}{k^2}\int_0^\infty \frac{(t^4-60t^2+360t-720)e^t - (t^4-60t^2-360t-720)}{720(e^t-1)} e^{-\frac{xt}{k}}\,dt. \end{aligned}$$

Since it is easy to see that $(t^4-60t^2+360t-720)e^t - (t^4-60t^2-360t-720) > 0$ for all $t > 0$. Thus $(-1)^n f_4^{(n)}(x) < 0$ in $(0,\infty)$ for any non-negative integer n. □

As an immediate consequence of the theorem, we obtain the following double sided inequalities on k-digamma and k-polygamma functions.

COROLLARY 2. *The following inequalities*

$$\frac{\ln x}{k} - \frac{1}{2x} - \frac{k}{12x^2} < \psi_k(x) < \frac{\ln x}{k} - \frac{1}{2x}, \tag{29}$$

$$\frac{1}{kx} + \frac{1}{2x^2} + \frac{k}{6x^3} - \frac{k^3}{30x^5} < \psi_k'(x) < \frac{1}{kx} + \frac{1}{2x^2} + \frac{k}{6x^3} \tag{30}$$

and

$$-\frac{1}{kx^2} - \frac{1}{x^3} - \frac{k}{2x^4} < \psi_k''(x) < -\frac{1}{kx^2} - \frac{1}{x^3} \tag{31}$$

are valid for $x > 0$.

We want to note that the inequalities (29) and (30) tend to the inequalities (4) and (5) obtained in [16, Corollary 1] respectively as $k \to 1$ and Theorem 6 is a k-generalization of classical one in [16, Theorem 1]

The following results will help us to obtain harmonic mean inequalities for k-digamma function.

THEOREM 7. *The function*

$$P_k(x) = \psi_k(x) + \psi_k\left(\frac{1}{x}\right)$$

is strictly concave for all $k, x > 0$.

Proof. By differentiation, we get

$$P_k'(x) = \psi_k'(x) - \frac{1}{x^2}\psi_k'\left(\frac{1}{x}\right)$$

and

$$\begin{aligned} P_k''(x) &= \psi_k''(x) + \frac{2}{x^3}\psi_k'(\left(\frac{1}{x}\right) + \frac{1}{x^4}\psi_k''\left(\frac{1}{x}\right) \\ x^4 P_k''(x) &= \psi_k''\left(\frac{1}{x}\right) + 2x\psi_k'\left(\frac{1}{x}\right) + x^4\psi_k''(x). \end{aligned}$$

Applying the recurrence formulas (16) and (17) and inequalities (30) and (31) lead us that

$$\begin{aligned}
x^4P''(x) &= -2x^3+\psi_k''\left(\frac{1}{x}\right)+2x\left(x^2+\psi_k'\left(\frac{1}{x}+k\right)\right)+x^4\psi_k''(x)\\
&< -2x^3+\left(-\frac{1}{k\left(k+\frac{1}{x}\right)^2}-\frac{1}{\left(k+\frac{1}{x^3}\right)}\right)\\
&\quad +2x\left(x^2+\frac{1}{k\left(k+\frac{1}{x}\right)}+\frac{1}{2(k+1/x)^2}+\frac{k}{6(k+1/x)^3}\right)+x^4\left(-\frac{1}{kx^2}-\frac{1}{x^3}\right)\\
&= -2x^3-\frac{x^2}{k(kx+1)^2}-\frac{x^3}{(kx+1)^3}+2x^3+\frac{2x^2}{k(kx+1)}+\frac{kx^3}{(kx+1)^2}+\frac{kx^4}{3(kx+1)^3}\\
&\quad -\frac{x^2}{k}-x\\
&< -\frac{x}{3(kx+1)^3}(3k^3x^3+3k^2x^4-3k^2x^3+9k^2x^2+2kx^3-3kx^2+9kx+3x^2+3).
\end{aligned}$$

Since the last term on the right side of the inequality is a polynomial, by using associative property, it is not difficult to obtain that if $k>0$ and $0<x\leqslant 3$, then we get $3k^3x^3+3k^2x^4+3k^2x^2(3-x)+2kx^3+3kx(3-x)+3x^2+3\geqslant 0$ and also if $k>0$ and $x>3$, then we get $3k^3x^3+3k^2x^3(x-1)+9k^2x^2+2kx^2(x-\frac{3}{2})+9kx+3x^2+3>0$. This completes the proof. □

COROLLARY 3. *The inequality*

$$\psi_k(x)+\psi_k\left(\frac{1}{x}\right)<2\psi_k(1) \tag{32}$$

holds true for $x>0$ *and* $x\neq 1$.

Proof. Using the concavity of the function $P_k(x)$ from Theorem 7 leads us to

$$P_k'(x)=\psi_k'(x)-\frac{1}{x^2}\psi_k'\left(\frac{1}{x}\right)\Longrightarrow P'(1)=0.$$

So we obtain that $P_k'(x)>P_k'(1)$, for $0<x<1$ and $P_k'(x)<P_k'(1)$ for $x>1$. Hence we get that $P_k(x)<P_k(1)=2\psi_k(1)$ for $x>0$ and $x\neq 1$ as desired. □

We want to remark that since $k>0$ and $\psi_k(x)=\dfrac{\ln k}{k}+\dfrac{1}{k}\psi\left(\dfrac{x}{k}\right)$, we have $P_k(1)=\dfrac{2}{k}\left(\ln k+\psi\left(\dfrac{1}{k}\right)\right)$ and $P_k(1)<0$ for all $k>0$. This fact can also easily be seen from the right side of the inequality (29) by taking $x=1$.

LEMMA 3. *The k-digamma function can be presented by power expansion as*

$$\psi_k(1+x)=\psi_k(1)+\sum_{n=2}^{\infty}(-1)^n\zeta_k(1,n)x^{n-1} \tag{33}$$

for $|x|<1$.

LEMMA 4. *The inequality*

$$\psi_k(1+y)\psi_k(1-y)<\psi_k^2(1) \tag{34}$$

is valid for $y\in(0,1)$.

Proof. Since ψ_k is completely monotonic function on $(0,\infty)$ for all $k>0$, we can find a point $x_0=x_0(k)$ such that $\psi_k(x_0)=0$ and from the inequalities (29), we see that the point x_0 is always greater than 1. As an immediate consequence, we have $\psi_k(1-y)<0$ for all $k>0$ and $y\in(0,1)$. Since x_0 depends on k and $1<1+y<2$, we must find an interval for k where $\psi_k(x_0)=0$ for $1<x_0<2$. By using the right side of the inequalities (29), we get $0<k\lesssim 2,36$. So we have to investigate two cases:
Case 1: If $k>2,36$, then we have $\psi_k(1-y)<0$ and $\psi_k(1+y)<0$ for $y\in(0,1)$ since $x_0>2$.
Case 2: If $0<k\lesssim 2,36$, then we get the following two other cases:
Case i: If $y\in(x_0-1,1)$, then we have $\psi_k(1-y)<0<\psi_k(1+y)$. Hence we obtain as desired.
Case ii. If $y\in(0,x_0-1)$, then we have $\psi_k(1-y)<0$ and $\psi_k(1+y)<0$.
Now, we obtain the result for Case 1 and Case ii simultaneously. By using Lemma 3, we get

$$0<-\psi_k(1-y)\leqslant-\psi_k(1)-\zeta_k(1,2)y+\zeta_k(1,3)y^2$$

and

$$0\leqslant-\psi_k(1+y)\leqslant-\psi_k(1)+\zeta_k(1,2)y+2\zeta_k(1,3)y^2.$$

Hence we find the following inequality

$$\begin{aligned}\psi_k(1-y)\psi_k(1-y)\leqslant\psi_k^2(1)+(-\zeta_k^2(1,2)-3\psi_k(1)\zeta_k(1,3))y^2\\ -\zeta_k(1,2)\zeta_k(1,3)y^3+2\zeta_k(1,3)^2y^4.\end{aligned}$$

Using the relations

$$\zeta_k(1,2)=\psi_k'(1)\quad\text{and}\quad\zeta_k(1,3)=-\frac{1}{2}\psi_k''(1)$$

yield

$$\begin{aligned}\psi_k(1-y)\psi_k(1-y)\leqslant\psi_k^2(1)+\left(-\psi_k'(1)^2+\frac{3}{2}\psi_k(1)\psi_k''(1)\right)y^2\\ +\frac{1}{2}\psi_k'(1)\psi_k''(1)y^3+\frac{1}{2}\psi_k''(1)^2y^4.\end{aligned}$$

From Theorem 6 and Corollary 2, we also have

$$\begin{aligned}-\frac{1}{2}-\frac{k}{12}<\psi_k(1)<-\frac{1}{2}\\ \frac{1}{k}+\frac{1}{2}<\psi_k'(1)<\frac{1}{k}+\frac{1}{2}+\frac{k}{6}\\ -\frac{1}{k}-1-\frac{k}{2}<\psi_k''(1)<-\frac{1}{k}-1.\end{aligned}$$

Hence we get

$$-\psi_k'(1)^2+\frac{3}{2}\psi_k(1)\psi_k''(1)<0, \quad \frac{1}{2}\psi_k'(1)\psi_k''(1)<0, \quad \frac{1}{2}\psi_k''(1)>0$$

and by using the facts that $y\in(0,1)$ and previous inequalities on k-digamma and k-polygamma functions at $x=1$, we rewrite the inequality as

$$\begin{aligned}
\psi_k(1-y)\psi_k(1-y) &\leqslant \psi_k^2(1)+(-\psi_k'(1)^2+\frac{3}{2}\psi_k(1)\psi_k''(1))y^2 \\
&\quad +\frac{1}{2}\psi_k'(1)\psi_k''(1)y^3+\frac{1}{2}\psi_k''(1)^2y^4 \\
&\leqslant \psi_k^2(1)+\left[-\psi_k'(1)^2+\frac{3}{2}\psi_k(1)\psi_k''(1)+\frac{1}{2}\psi_k'(1)\psi_k''(1)+\frac{1}{2}\psi_k''(1)^2\right]y^3 \\
&\leqslant \psi_k^2(1)+\left[-\left(\frac{1}{k}+\frac{1}{2}+\frac{k}{6}\right)^2+\frac{3}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{k}-1\right)\right. \\
&\quad \left.+\frac{1}{2}\left(\frac{1}{k}+\frac{1}{2}+\frac{k}{6}\right)\left(-\frac{1}{k}-1\right)+\frac{1}{2}\left(1+\frac{1}{k}\right)^2\right]y^4.
\end{aligned}$$

Since $-(1/k+1/2+k/6)^2+3/2(-1/2)(-1/k-1)+1/2(1/k+1/2+k/6)(-1/k-1)+1/2(1+1/k)^2=-(k^4+9k^3-12k^2+36)/36k^2<0$ for $k>0$, we get desired result. □

THEOREM 8. *The inequality*

$$\psi_k(x)\psi_k\left(\frac{1}{x}\right)\leqslant\psi_k^2(1) \tag{35}$$

is valid for all positive real values of x.

Proof.

Case 1. If $x\geqslant x_0$, then we obtain $\psi_k\left(\frac{1}{x}\right)<0\leqslant\psi_k(x)$. Hence we obtain as desired.

Case 2. If $1<x<x_0$, then let us take $x=1+y$. Thus $\frac{1}{x}>\frac{1}{1-y}$ and since k-digamma function is completely monotonic on $(0,\infty)$, we get $\psi_k(1-y)<\psi_k\left(\frac{1}{x}\right)<0$. Using this fact and Lemma 4 lead us to

$$\psi_k(x)\psi_k(1/x)=\psi_k(1+y)\psi_k(1/x)<\psi_k(1+y)\psi_k(1-y)\leqslant\psi_k^2(1)$$

as desired.

Case 3. If $x\in(0,1)$, then we can take $z=1/x$ and use the method above. □

At last, for all real values of x and k, we can give harmonic mean inequality for k-digamma function.

THEOREM 9. *The inequality*

$$\psi_k(1) \leqslant \frac{2\psi_k(x)\psi_k(1/x)}{\psi_k(x)+\psi_k(1/x)} \tag{36}$$

holds true for $x, k > 0$.

Proof. By using the inequalities (32) and (35), we get

$$\begin{aligned} 2\psi_k(x)\psi_k(1/x) &< 2\psi_k^2(1) \\ \frac{2\psi_k(x)\psi_k(1/x)}{\psi_k(x)+\psi_k(1/x)} &> \frac{2\psi_k^2(1)}{\psi_k(x)+\psi_k(1/x)} \\ \frac{2\psi_k(x)\psi_k(1/x)}{\psi_k(x)+\psi_k(1/x)} &> \frac{2\psi_k^2(1)}{2\psi_k(1)} \\ &= \psi_k(1) \end{aligned}$$

for all real values of $x \neq 1$ and k. The sign of equality holds when $x = 1$. □

At last we want to note that Theorem 7, inequalities (32), (33), (35) and (36) are extensions of the results in [26].

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(Received April 26, 2019)

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