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INEQUALITIES FOR GENERALIZED PARAMETRIC MARCINKIEWICZ INTEGRALS ALONG POLYNOMIAL COMPOUND CURVES

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Abstract. Under some weaker size conditions assumed on the integral kernels both on the unit sphere and in the radial directions, the sharp L^p boundedness was proved for the generalized parametric Marcinkiewicz integrals along polynomial compound curves via an extrapolation argument. As applications, the corresponding results for generalized parametric Marcinkiewicz integral operators related to the Littlewood-Paley g_λ^*-functions and area integrals are also established.

1. Introduction

The primary purpose of this paper is to present certain sharp L^p boundedness for rough generalized parametric Marcinkiewicz integrals along polynomial compound curves. Let us begin with some notations and definitions.

DEFINITION 1. (Generalized parametric Marcinkiewicz integral operators along submanifolds). Let $\mathbb{R}^n$ $(n \geqslant 2)$ be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Assume that $\Omega \in L^1(\mathrm{S}^{n-1})$ is a function of homogeneous degree zero and satisfies

$$\int_{\mathrm{S}^{n-1}} \Omega(u)d\sigma(u) = 0. \tag{1}$$

Let h be a measurable function on $\mathbb{R}_+ := [0,\infty)$ and $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ be a suitable mapping. For a suitable function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ and $1 < q < \infty$, we define the generalized parametric Marcinkiewicz integral operators $\mathfrak{M}^q_{h,\Omega,\Gamma,\rho}$ by

$$\mathfrak{M}^q_{h,\Omega,\Gamma,\rho} f(x) = \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{|y|\leqslant t} f(x-\Gamma(y))\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy\right|^q \frac{dt}{t}\right)^{1/q}, \tag{2}$$

where $y' = y/|y|$ for $y \in \mathbb{R}^n \setminus \{0\}$, $\rho = \varsigma + i\tau$ ($\varsigma,\tau \in \mathbb{R}$ with $\varsigma > 0$) and $f \in \mathscr{S}(\mathbb{R}^n)$ (the space of Schwartz functions on $\mathbb{R}^n$).

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For the sake of simplicity, we denote $\mathfrak{M}^q_{h,\Omega,\Gamma,\rho} = \mathfrak{M}_{h,\Omega,\rho}$ if $q = 2$ and $\Gamma(y) = y$. Specially, when $\rho = 1$ and $h(\cdot) \equiv 1$, the operator $\mathfrak{M}_{h,\Omega,\rho}$ reduces to the classical well-known Marcinkiewicz integral operator $\mathfrak{M}_\Omega$, which was first introduced by Stein [26] who established the $L^p\,(1 < p \leqslant 2)$ bounds for $\mathfrak{M}_\Omega$ under the condition that $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for $0 < \alpha \leqslant 1$. Stein's result was later extended and improved greatly by many authors. For example, see [6, 7] for the case $\Omega \in H^1(\mathrm{S}^{n-1})$ (the Hardy space on S^{n-1}), [3, 4] for the case $\Omega \in L(\log L)^{1/2}(\mathrm{S}^{n-1})$, [4, 8] for the case $\Omega \in B_r^{(0,-1/2)}(\mathrm{S}^{n-1})$ (the block space generated by r-blocks). For $h(\cdot) \equiv 1$, the operator $\mathfrak{M}_{h,\Omega,\rho}$ is just the classical parametric Marcinkiewicz integral operator $\mathfrak{M}_{\Omega,\rho}$. Hörmander [11] (resp., Sakamoto and Yabuta [24]) first studied the L^p bounds for $\mathfrak{M}_{\Omega,\rho}$ with real (resp., complex) number ρ. For further research on $\mathfrak{M}_{h,\Omega,\rho}$ and other extensions, we refer the readers to consult [15, 16, 17, 18, 19, 20, 22, 23], among others.

On the other hand, the investigation on the generalized Marcinkiewicz integral operator has also attracted the attention of many authors. When $\rho = 1$ and $\Gamma(y) = y$, we denote $\mathfrak{M}^q_{h,\Omega,\Gamma,\rho}$ by $\mathfrak{M}^q_{h,\Omega}$. In 2002, Chen et al. [5] first introduced the operator $\mathfrak{M}^q_{h,\Omega}$ and showed that $\mathfrak{M}^q_{h,\Omega}$ is bounded from the homogeneous Triebel-Lizorkin space $\dot F^0_{p,q}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p, q < \infty$ if $h(\cdot) \equiv 1$ and $\Omega \in L^s(\mathrm{S}^{n-1})$ for some $1 < s \leqslant \infty$. Later on, the above result was improved by Fan and Wu [9] to the case $\Omega \in L(\log L)^{1/q}(\mathrm{S}^{n-1})$ for $q \geqslant 2$ and $\Omega \in L(\log L)^{1/q+\varepsilon}(\mathrm{S}^{n-1})$ for $1 < q < 2$ and any $\varepsilon > 0$. Meanwhile, Al-Qassem et al. [2] established the bounds of $\mathfrak{M}^q_{h,\Omega} : \dot F^0_{p,q}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $p \in (2\beta/(2\beta-1), 2\beta)$ and $q \in (2\beta/(2\beta-1), 2\beta)$ under the conditions that $h(\cdot) \equiv 1$ and $\Omega \in \mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1$. Here $\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for $\beta > 0$ was introduced by Grafakos and Stefanov [10] in the study of L^p bounds for rough singular integrals. In particular, Le [13] observed that $\mathfrak{M}^q_{h,\Omega}$ is bounded from $\dot F^0_{p,q}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p, q < \infty$ provided that $h \in \Delta_{\max\{2,q'\}}(\mathbb{R}_+)$ and $\Omega \in L(\log L)(\mathrm{S}^{n-1})$. Recently, Al-Qassem et al. [1] improved the results of [9, 13] and proved the following conclusions.

THEOREM 1. ([1]) *Let Ω satisfy the condition (1) and $1 < q < \infty$. Then:*

(i) *If $\Omega \in L(\log L)(\mathrm{S}^{n-1}) \cup \big(\bigcup_{r>1} B_r^{(0,0)}(\mathrm{S}^{n-1})\big)$ and $h \in \mathscr{N}_1(\mathbb{R}_+)$, then*

$$\|\mathfrak{M}^q_{h,\Omega,\rho} f\|_{L^p(\mathbb{R}^n)} \leqslant C_p(1 + N_1(h))\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)}, \quad \text{for } 1 < p < q.$$

(ii) *If $\Omega \in L(\log L)^{1/q}(\mathrm{S}^{n-1}) \cup \big(\bigcup_{r>1} B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})\big)$ and $h \in \mathscr{N}_{1/q}(\mathbb{R}_+)$, then*

$$\|\mathfrak{M}^q_{h,\Omega,\rho} f\|_{L^p(\mathbb{R}^n)} \leqslant C_p(1 + N_{1/q}(h))\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)}, \quad \text{for } q \leqslant p < \infty.$$

(iii) *If $\Omega \in L(\log L)^{1/q}(\mathrm{S}^{n-1}) \cup \big(\bigcup_{r>1} B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})\big)$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma > 2$, then*

$$\|\mathfrak{M}^q_{h,\Omega,\rho} f\|_{L^p(\mathbb{R}^n)} \leqslant C_p\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)},$$

for $1 < p < q$ if $2 < \gamma < \infty$ and $q' \geqslant \gamma$, and for $\gamma' < p < \infty$ if $2 < \gamma \leqslant \infty$ and $q' < \gamma$.

Here the above constants $C_p > 0$ are independent of h.

This paper focuses on rough generalized parametric Marcinkiewicz integrals along polynomial compound curves. More precisely, we shall establish certain sharp L^p bounds for $\mathfrak{M}^q_{h,\Omega,\Gamma,\rho}$ with $\Gamma(y) = P_N(\varphi(|y|))y'$, where P_N is a real polynomial on $\mathbb{R}$ of degree N and satisfies $P_N(0) = 0$ and $\varphi \in \mathfrak{F}$. Here $\mathfrak{F}$ is the set of all positive increasing $\mathscr{C}^1(\mathbb{R}_+)$ functions ϕ such that there exist $C_\phi, c_\phi > 0$ such that $t\phi'(t) \geqslant C_\phi\phi(t)$ and $\phi(2t) \leqslant c_\phi\phi(t)$ for all $t > 0$.

REMARK 1. There are some model examples for the class $\mathfrak{F}$, such as t^α $(\alpha > 0)$, $t^\beta \ln(1+t)$ $(\beta \geqslant 1)$, $t\ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. Note that there exists $B_\varphi > 1$ such that $\varphi(2t) \geqslant B_\varphi\varphi(t)$ for any $\varphi \in \mathfrak{F}$ (see [21]).

For convenience, we denote $\mathfrak{M}^q_{h,\Omega,\Gamma,\rho} = \mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}$ if $\Gamma(y) = P_N(\varphi(|y|))y'$. Recently, the first author [18] established the following result.

THEOREM 2. ([18]) *Let P_N be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$ and $\varphi \in \mathfrak{F}$. Assume that $h(\cdot) \equiv 1$ and $\Omega \in \mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1/2$ and satisfies (1). Then*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \leqslant C_p\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},$$

for $p \in (1+1/(2\beta), 1+2\beta)$ and $q \in (1+1/(2\beta), 1+2\beta)$. Here the constant $C_p > 0$ is independent of the coefficients of P_N, but may depend on $p, q, n, \varphi, \rho, N$.

Based on the above, a question that arises naturally is the following.

QUESTION 1. Is the operator $\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}$ bounded from $\dot{F}^0_{p,q}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ under the same conditions h, Ω in Theorem 1 and P_N, φ in Theorem 2?

This question is the main motivation of this work, which can be addressed by the following results.

THEOREM 3. *Let P_N be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$ and $\varphi \in \mathfrak{F}$. Suppose that Ω satisfies (1). Then:*

(i) *If $\Omega \in L(\log L)(\mathrm{S}^{n-1})$ and $h \in \mathscr{N}_1(\mathbb{R}_+)$, then*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \leqslant C_p(1+\|\Omega\|_{L(\log L)(\mathrm{S}^{n-1})})(1+N_1(h))\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},$$

for $1 < p < q$.

(ii) *If $\Omega \in L(\log L)^{1/q}(\mathrm{S}^{n-1})$ and $h \in \mathscr{N}_{1/q}(\mathbb{R}_+)$ for $1 < q < \infty$, then*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \leqslant C_p(1+\|\Omega\|_{L(\log L)^{1/q}(\mathrm{S}^{n-1})})(1+N_{1/q}(h))\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},$$

for $q \leqslant p < \infty$.

(iii) *If* $\Omega\in L(\log L)^{1/q}(\mathrm{S}^{n-1})$ *and* $h\in\Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma>2$*. Then*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(1+\|\Omega\|_{L(\log L)^{1/q}(\mathrm{S}^{n-1})})\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)},$$

for $1<p<q\leqslant\gamma'$ *if* $2<\gamma<\infty$*, and for* $\gamma'<p<\infty$ *and* $\gamma'<q<\infty$ *if* $2<\gamma\leqslant\infty$*.*

Here the above constants $C_p>0$ *are independent of* h,Ω *and the coefficients of* P_N*, but may depend on* p,q,n,φ,ρ,N*.*

THEOREM 4. *Let* P_N *be a real polynomial on* $\mathbb{R}$ *of degree* N *and satisfy* $P_N(0)=0$ *and* $\varphi\in\mathfrak{F}$*. Suppose that* Ω *satisfies* (1)*. Then:*

(i) *If* $\Omega\in B_r^{(0,0)}(\mathrm{S}^{n-1})$ *for some* $r>1$ *and* $h\in\mathscr{N}_1(\mathbb{R}_+)$*, then*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(1+\|\Omega\|_{B_r^{(0,0)}(\mathrm{S}^{n-1})})(1+N_1(h))\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)},$$

for $1<p<q$*.*

(ii) *If* $\Omega\in B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})$ *for some* $r>1$ *and* $h\in\mathscr{N}_{1/q}(\mathbb{R}_+)$ *for* $1<q<\infty$*, then*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(1+\|\Omega\|_{B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})})(1+N_{1/q}(h))\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)},$$

for $q\leqslant p<\infty$*.*

(iii) *If* $\Omega\in B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})$ *for some* $r>1$ *and* $h\in\Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma>2$*. Then*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(1+\|\Omega\|_{B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})})\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)},$$

for $1<p<q\leqslant\gamma'$ *if* $2<\gamma<\infty$*, and for* $\gamma'<p<\infty$ *and* $\gamma'<q<\infty$ *if* $2<\gamma\leqslant\infty$*.*

Here the above constants $C_p>0$ *are independent of* h,Ω *and the coefficients of* P_N*, but may depend on* p,q,n,φ,ρ,N*.*

Theorems 3 and 4 can be proved by applying extrapolation arguments following from [25] and the following refined sharp results.

THEOREM 5. *Let* P_N *be a real polynomial on* $\mathbb{R}$ *of degree* N *and satisfy* $P_N(0)=0$ *and* $\varphi\in\mathfrak{F}$*. Suppose that* $\Omega\in L^s(\mathrm{S}^{n-1})$ *for some* $s\in(1,2]$ *satisfying* (1) *and* $h\in\Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma\in(1,2]$*. Then:*

(i) *For* $1<p<q$*, it holds that*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(s-1)^{-1}(\gamma-1)^{-1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)}.$$

(ii) *For* $q\leqslant p<\infty$*, it holds that*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(s-1)^{-1/q}(\gamma-1)^{-1/q}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)}.$$

Here the above constants $C_p > 0$ are independent of h, Ω, γ, s and the coefficients of P_N, but may depend on $p, q, n, \varphi, \rho, N$.

THEOREM 6. *Let P_N be a real polynomial on $\mathbb{R}$ of degree N and satisfies $P_N(0) = 0$ and $\varphi \in \mathfrak{F}$. Suppose that $\Omega \in L^s(\mathrm{S}^{n-1})$ for some $s \in (1,2]$ satisfying (1) and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (2,\infty]$. Then*

$$\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \leqslant C_p(s-1)^{-1/q}(\gamma-1)^{-1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},$$

for $1 < p < q \leqslant \gamma'$ if $2 < \gamma < \infty$, and for $\gamma' < p < \infty$ and $\gamma' < q < \infty$ if $2 < \gamma \leqslant \infty$. Here the above constants $C_p > 0$ are independent of h, Ω, γ, s and the coefficients of P_N, but may depend on p, q, φ, ρ, N.

REMARK 2. There are some remarks as follows:

(a) Theorem 3 (i) and (ii) extend [23, Corollary 2], which corresponds to the case $q = 2$. Theorems 3 and 4 generalizes Theorem 1, which corresponds to the case $P_N(t) = t$ and $\varphi(t) = t$. Moreover, Theorems 3 and 4 are essentially different from Theorem 2, even in the special case $h(t) \equiv 1$.

(b) Our main results improve and generalize the main results in [4, 5, 9, 13].

(c) It should be pointed out that all of our main results are new, even in the special case: $\rho = 1$, $q = 2$, $h(t) \equiv 1$ and $\varphi(t) \equiv t$.

The paper is organized as follows. Section 2 contains some notations and lemmas, which play key roles in our proofs. The proofs of main results will be given in Section 3. Finally, we establish the L^p bounds for generalized parametric Marcinkiewicz integral operators related to Littlewood-Paley g^*_λ-functions and area integrals in Section 4. We would like to remark that the main method employed in the proofs of Theorems 5 and 6 is a combination of ideas and arguments from [1, 14, 23, 27]. The proofs of Theorems 3 and 4 are based on Theorems 5 and 6 and some extrapolation arguments following from [4, 25].

Throughout the paper, we let p' denote the conjugate index of p which satisfies $1/p + 1/p' = 1$. The letter C will stand for positive constants not necessarily the same one at each occurrence but is independent of the essential variables.

2. Preliminary notations and lemmas

This section is devoted to presenting some definitions and lemmas, which play key roles in the proofs of main results. We start with some definitions of function spaces.

DEFINITION 2. (Function classes $L(\log L)^\alpha(\mathrm{S}^{n-1})$, $B^{(0,v)}_r(\mathrm{S}^{n-1})$ and $\mathscr{F}_\beta(\mathrm{S}^{n-1})$).

(i) For $\alpha > 0$, the class $L(\log L)^\alpha(\mathrm{S}^{n-1})$ denotes the class of all measurable functions Ω on S^{n-1} which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(\mathrm{S}^{n-1})} = \int_{\mathrm{S}^{n-1}} |\Omega(\theta)| \log^\alpha(|\Omega(\theta)| + 2)\,d\sigma(\theta) < \infty.$$

(ii) The block spaces in $\mathbb{R}^n$ originated from the work of Taibleson and Weiss on the convergence of the Fourier series in connection with developments of the real Hardy spaces. The block spaces on S^{n-1} was introduced by Jiang and Lu [12] in studying the homogeneous singular integral operators. A r-block on S^{n-1} is an $L^r(\mathrm{S}^{n-1})$ $(1<r\leqslant\infty)$ function b which satisfies $\mathrm{supp}(b)=I$ and $\|b\|_r\leqslant |I|^{1-1/r}$, where $|I|=\sigma(I)$, and $I=\{x\in\mathrm{S}^{n-1}:|x-x_0|<\alpha\}$ for some $\alpha\in(0,1]$ and $x_0\in\mathrm{S}^{n-1}$. The block $B_r^{(0,v)}(\mathrm{S}^{n-1})$ is defined by

$$B_r^{(0,v)}(\mathrm{S}^{n-1})=\Big\{\Omega\in L^1(\mathrm{S}^{n-1}):\Omega=\sum_{\mu=1}^{\infty}\lambda_\mu b_\mu,\ \ M_r^{(0,v)}(\{\lambda_\mu\})<\infty\Big\},$$

where $v>-1$, $\lambda_\mu\in\mathbb{C}$, b_μ is a r-block supported on a cap I_μ on S^{n-1} and $M_r^{(0,v)}(\{\lambda_\mu\})=\sum_{\mu=1}^{\infty}|\lambda_\mu|\big(1+\log^{(v+1)}(|I_\mu|^{-1})\big)$. The norm of $B_r^{(0,v)}(\mathrm{S}^{n-1})$ is given by $\|\Omega\|_{B_r^{(0,v)}(\mathrm{S}^{n-1})}=N_r^{(0,v)}(\Omega)=\inf\{M_r^{(0,v)}(\{\lambda_\mu\})\}$, where the infimum is taken over all r-block decompositions of Ω.

(iii) For $\beta>0$, $\mathscr{F}_\beta(\mathrm{S}^{n-1})$ is defined by

$$\mathscr{F}_\beta(\mathrm{S}^{n-1}):=\Big\{\Omega\in L^1(\mathrm{S}^{n-1}):\ \sup_{\xi\in\mathrm{S}^{n-1}}\int_{\mathrm{S}^{n-1}}|\Omega(y')|\log^{\beta}\frac{2}{|\xi\cdot y'|}d\sigma(y')<\infty\Big\}.$$

REMARK 3. The following inclusion relations are valid:

$L^r(\mathrm{S}^{n-1})\subsetneq L(\log L)^{\beta_1}(\mathrm{S}^{n-1})\subsetneq L(\log L)^{\beta_2}(\mathrm{S}^{n-1})$ for $r>1$ and $0<\beta_2<\beta_1$;

$L(\log L)^{\beta}(\mathrm{S}^{n-1})\subsetneq H^1(\mathrm{S}^{n-1})$ for $\beta\geqslant 1$;

$L(\log L)^{\beta}(\mathrm{S}^{n-1})\nsubseteq H^1(\mathrm{S}^{n-1})\nsubseteq L(\log L)^{\beta}(\mathrm{S}^{n-1})$ for $0<\beta<1$;

$\bigcup_{q>1}L^q(\mathrm{S}^{n-1})\subsetneq\bigcap_{\beta>1}\mathscr{F}_\beta(\mathrm{S}^{n-1})\nsubseteq L\log L(\mathrm{S}^{n-1})$;

$\bigcap_{\beta>1}\mathscr{F}_\beta(\mathrm{S}^{n-1})\nsubseteq H^1(\mathrm{S}^{n-1})\nsubseteq\bigcup_{\beta>1}\mathscr{F}_\beta(\mathrm{S}^{n-1})$;

$\bigcup_{r>1}L^r(\mathrm{S}^{n-1})\subsetneq B_q^{(0,v)}(\mathrm{S}^{n-1})$ for $q>1$ and $v>-1$;

$B_q^{(0,v_2)}(\mathrm{S}^{n-1})\subsetneq B_q^{(0,v_1)}(\mathrm{S}^{n-1})$ for $q>1$ and $v_2>v_1>-1$;

$\bigcup_{q>1}B_q^{(0,v)}(\mathrm{S}^{n-1})\nsubseteq\bigcup_{r>1}L^r(\mathrm{S}^{n-1})$ for $v>-1$;

$B_q^{(0,v)}(\mathrm{S}^{n-1})\subset H^1(\mathrm{S}^{n-1})+L(\log L)^{1+v}(\mathrm{S}^{n-1})$ for $q>1$, $v>-1$.

DEFINITION 3. (Function classes $\Delta_\gamma(\mathbb{R}_+)$ and $\mathscr{N}_\gamma(\mathbb{R}_+)$).

(i) For $\gamma>0$, $\Delta_\gamma(\mathbb{R}_+)$ will always be used to denote the collection of all measurable functions $h:[0,\infty)\to\mathbb{C}$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)}=\sup_{R>0}\Big(\frac{1}{R}\int_0^R|h(t)|^{\gamma}dt\Big)^{1/\gamma}<\infty.$$

(ii) For $\gamma > 0$, the set $\mathcal{N}_\gamma(\mathbb{R}_+)$ denotes the collection of all measurable functions $h : [0,\infty) \to \mathbb{C}$ satisfying

$$N_\gamma(h) = \sum_{m=1} m^\gamma 2^m d_m(h) < \infty \text{ with } d_m(h) = \sup_{k\in\mathbb{Z}} 2^{-k}|E(k,m)|,$$

where $E(k,1) = \{t \in (2^k, 2^{k+1}] : |h(t)| \leqslant 2\}$ and

$$E(k,m) = \{t \in (2^k, 2^{k+1}] : 2^{m-1} < |h(t)| \leqslant 2^m\}, \text{ for } m \geqslant 2.$$

REMARK 4. The function class $\mathcal{N}_\gamma(\mathbb{R}_+)$ was first introduced by Sato in [25]. It is well-known that

$$\Delta_{\gamma_2}(\mathbb{R}_+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}_+), \text{ for } 0 < \gamma_1 < \gamma_2 \leqslant \infty,$$

$$\mathcal{N}_{\gamma_2}(\mathbb{R}_+) \subsetneq \mathcal{N}_{\gamma_1}(\mathbb{R}_+), \text{ for } 0 < \gamma_1 < \gamma_2 < \infty,$$

$$\Delta_\gamma(\mathbb{R}_+) \subsetneq \mathcal{N}_\alpha(\mathbb{R}_+), \text{ for any } \gamma \geqslant 1 \text{ and } \alpha > 0.$$

We now recall the definition of the Homogeneous Triebel-Lizorkin spaces.

DEFINITION 4. (Homogeneous Triebel-Lizorkin spaces). Let $\mathcal{S}'(\mathbb{R}^n)$ be the tempered distribution class on $\mathbb{R}^n$. For $\alpha \in \mathbb{R}$ and $0 < p, q \leqslant \infty$, $(p \neq \infty)$, the homogeneous Triebel-Lizorkin spaces $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{F}^\alpha_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} = \left\| \left(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\},$$

where $\widehat{\Psi_i}(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ satisfies the conditions: $0 \leqslant \phi(x) \leqslant 1$; $\text{supp}(\phi) \subset \{x \in \mathbb{R}^n : 1/2 \leqslant |x| \leqslant 2\}$; $\phi(x) \geqslant c > 0$ if $3/5 \leqslant |x| \leqslant 5/3$; $\sum_{j\in\mathbb{Z}} \phi(2^j\xi) = 1$ for $\xi \neq 0$.

REMARK 5. It is well-known that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ and also the following hold:

(a) $\dot{F}^0_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, for $1 < p < \infty$.

(b) $(\dot{F}^\alpha_{p,q}(\mathbb{R}^n))^* = \dot{F}^{-\alpha}_{p',q'}(\mathbb{R}^n)$, for $\alpha \in \mathbb{R}$ and $1 < p, q < \infty$.

(c) $\dot{F}^\alpha_{p,q_1}(\mathbb{R}^n) \subset \dot{F}^\alpha_{p,q_2}(\mathbb{R}^n)$, for $\alpha \in \mathbb{R}$, $0 < p \leqslant \infty$ and $q_1 \leqslant q_2$.

Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence with satisfying $\inf_{k\in\mathbb{Z}} a_{k+1}/a_k \geqslant a > 1$. A sequence $\{\Phi_k\}_{k\in\mathbb{Z}}$ is said to be a partition of unity adapted to $\{a_k\}_{k\in\mathbb{Z}}$ if Φ_k satisfies the following conditions:

$$\text{supp}\widehat{\Phi_k} \subset \{a_{k-1} \leqslant |\xi| \leqslant a_{k+1}\}; \ \sum_{k\in\mathbb{Z}} \widehat{\Phi_k}(\xi) = 1 \text{ for } \xi \in \mathbb{R}^n \setminus \{0\}; \ |\xi^\alpha \partial^\beta \widehat{\Phi_k}(\xi)| \leqslant C_\beta,$$

for any multi-index β. Let $\mathscr{A}_n$ denote the set of all polynomials on $\mathbb{R}^n$. Let $1 < p, q < \infty$ and $\alpha \in \mathbb{R}$. For $f \in \mathscr{S}(\mathbb{R}^n)/\mathscr{A}_n$, we define the norm $\|f\|_{\dot{F}^{\alpha}_{p,q}(\{\Phi_k\}_{k\in\mathbb{Z}},\mathbb{R}^n)}$ by

$$\|f\|_{\dot{F}^{\alpha}_{p,q}(\{\Phi_k\}_{k\in\mathbb{Z}},\mathbb{R}^n)} = \Big\|\Big(\sum_{k\in\mathbb{Z}} a_k^{\alpha q}|\Phi_k * f|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

The following is a well-known characterization of homogeneous Triebel-Lizorkin spaces, which plays a key role in our proofs.

LEMMA 1. ([27]) *Let $\alpha \in \mathbb{R}$ and $1 < p,q < \infty$. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence of positive numbers with $1 < a \leqslant \frac{a_{k+1}}{a_k} \leqslant b$ for all $k \in \mathbb{Z}$. Then $\|f\|_{\dot{F}^{\alpha}_{p,q}(\{\Phi_k\}_{k\in\mathbb{Z}},\mathbb{R}^n)}$ is equivalent to $\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$.*

In what follows, we set $P_N(t) = \sum_{i=1}^{N} b_i t^i$ with each $b_i \neq 0$. Let $P_0(t) = 0$ and $P_\lambda(t) = \sum_{i=1}^{\lambda} b_i t^i$ for $\lambda \in \{1,2,\dots,N\}$. Let h,Ω,ρ be given as in (2) and $\varphi \in \mathfrak{F}$. For $\lambda \in \{0,1,\dots,N\}$, we define the family of measures $\{\sigma^{\lambda}_{h,\Omega,t}\}_{t>0}$ by

$$\widehat{\sigma^{\lambda}_{h,\Omega,t}}(\xi) = \frac{1}{t^\rho}\int_{t/2<|y|\leqslant t} e^{-2\pi i \xi\cdot P_\lambda(\varphi(|y|))y'}\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy.$$

The related maximal operators $\sigma^{\lambda,*}_{h,\Omega}$ and $M^{\lambda,*}_{h,\Omega,\theta}$ are defined by

$$\sigma^{\lambda,*}_{h,\Omega}(f)(x) = \sup_{t>0}\big||\sigma^{\lambda}_{h,\Omega,t}| * f(x)\big|,$$

$$M^{\lambda,*}_{h,\Omega,\theta}(f)(x) = \sup_{k\in\mathbb{Z}}\int_{\theta^k}^{\theta^{k+1}}\big||\sigma^{\lambda}_{h,\Omega,t}| * f(x)\big|\frac{dt}{t},$$

where $\theta \geqslant 2$ and $|\sigma^{\lambda}_{h,\Omega,t}|$ is defined in the same way as $\sigma^{\lambda}_{h,\Omega,t}$, but with Ω and h replaced by $|\Omega|$ and $|h|$, respectively.

The following result follows from [14, Lemma 2.2].

LEMMA 2. ([14]) *Let $N \in \mathbb{N}\setminus\{0\}$ and $P_N(t) = \sum_{i=1}^{\lambda} b_i t^i$ with each $b_i \neq 0$. If $\varphi \in \mathfrak{F}$ and $\Omega \in L^s(\mathrm{S}^{n-1})$ for some $s > 1$. Then for any $0 < \varepsilon < \min\{1/s', 1/N\}$, $r > 0$ and $\xi \in \mathbb{R}^n$, it holds that*

$$\int_{r/2}^{r}\Big|\int_{\mathrm{S}^{n-1}}\Omega(u')e^{-iP_N(\varphi(t))\xi\cdot u'}d\sigma(u')\Big|^2\frac{dt}{t} \leqslant C(\varphi)\|\Omega\|^2_{L^s(\mathrm{S}^{n-1})}|\varphi(r)^N b_N\xi|^{-\varepsilon},$$

where the constant $C(\varphi) > 0$ is independent of Ω, s and the coefficients of P_N, but depends on φ.

LEMMA 3. *Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for some $s \in (1,2]$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (1,2]$. Suppose that $\varphi \in \mathfrak{F}$. Then, for $1 \leqslant \lambda \leqslant N$, $t > 0$ and $\xi \in \mathbb{R}^n$, the following estimates hold:*

$$\max\big\{\|\sigma^{\lambda}_{h,\Omega,t}\|, |\widehat{\sigma^{\lambda}_{h,\Omega,t}}(\xi)|, |\widehat{|\sigma^{\lambda}_{h,\Omega,t}|}(\xi)|\big\} \leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}, \quad (3)$$

$$\max\left\{|\widehat{\sigma^{\lambda}_{h,\Omega,t}}(\xi)-\widehat{\sigma^{\lambda-1}_{h,\Omega,t}}(\xi)|,\left||\widehat{\sigma^{\lambda}_{h,\Omega,t}}|(\xi)-|\widehat{\sigma^{\lambda-1}_{h,\Omega,t}}|(\xi)\right|\right\} \leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}(\varphi(t)^\lambda|b_\lambda\xi|)^{\frac{1}{2\lambda\gamma's'}}, \tag{4}$$

$$\max\left\{|\widehat{\sigma^{\lambda}_{h,\Omega,t}}(\xi)|,||\widehat{\sigma^{\lambda}_{h,\Omega,t}}|(\xi)|\right\}\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}(\varphi(t)^\lambda|b_\lambda\xi|)^{-\frac{1}{2\lambda\gamma's'}}. \tag{5}$$

The above $C>0$ are independent of h,Ω,s,γ and the coefficients of P_λ, but may depend on φ.

Proof. By a change of variable, one has

$$\widehat{\sigma^{\lambda}_{h,\Omega,t}}(\xi)=\frac{1}{t^\rho}\int_{t/2}^{t}\int_{\mathrm{S}^{n-1}}\Omega(x')e^{-2\pi iP_\lambda(\varphi(r))\xi\cdot x'}d\sigma(x')\frac{h(r)}{r^{1-\rho}}dr. \tag{6}$$

It follows easily that

$$|\widehat{\sigma^{\lambda}_{h,\Omega,t}}(\xi)|\leqslant 2\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|h\|_{\Delta_1(\mathbb{R}_+)}\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}.$$

Similarly one may get

$$||\widehat{\sigma^{\lambda}_{h,\Omega,t}}|(\xi)|\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}.$$

Then (3) holds. By Hölder's inequality and (6), one finds

$$\begin{aligned}&\left|\widehat{\sigma^{\lambda}_{h,\Omega,t}}(\xi)-\widehat{\sigma^{\lambda-1}_{h,\Omega,t}}(\xi)\right|\\&=\left|\frac{1}{t^\rho}\int_{t/2}^{t}\int_{\mathrm{S}^{n-1}}\Omega(x')(e^{-2\pi iP_\lambda(\varphi(r))\xi\cdot x'}-e^{-2\pi iP_{\lambda-1}(\varphi(r))\xi\cdot x'})d\sigma(x')\frac{h(r)}{r^{1-\rho}}dr\right|\\&\leqslant C\varphi(t)^\lambda|b_\lambda\xi|\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\varphi(t)^\lambda|b_\lambda\xi|.\end{aligned}$$

Similarly, it holds that

$$\left||\widehat{\sigma^{\lambda}_{h,\Omega,t}}|(\xi)-|\widehat{\sigma^{\lambda-1}_{h,\Omega,t}}|(\xi)\right|\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\varphi(t)^\lambda|b_\lambda\xi|.$$

These inequalities together with (3) yield (4).

On the other hand, by Hölder's inequality, Lemma 2 and (6), one has

$$\begin{aligned}|\widehat{\sigma^{\lambda}_{h,\Omega,t}}(\xi)|&\leqslant\int_{t/2}^{t}\left|\int_{\mathrm{S}^{n-1}}\Omega(x')e^{-2\pi iP_\lambda(\varphi(r))\xi\cdot x'}d\sigma(x')\right||h(r)|\frac{dr}{r}\\&\leqslant 2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|^{1-\frac{2}{\gamma'}}_{L^1(\mathrm{S}^{n-1})}\left(\int_{t/2}^{t}\left|\int_{\mathrm{S}^{n-1}}\Omega(y')e^{-2\pi iP_\lambda(\varphi(r))\xi\cdot x'}\right|^2\frac{dr}{r}\right)^{1/\gamma'}\\&\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}(\varphi(t)^\lambda|b_\lambda\xi|)^{-\frac{1}{2\lambda s'\gamma'}}.\end{aligned}$$

Similarly, we get

$$\left||\widehat{\sigma^{\lambda}_{h,\Omega,t}}|(\xi)\right|\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}(\varphi(t)^\lambda|b_\lambda\xi|)^{-\frac{1}{2\lambda s'\gamma'}}.$$

This proves (5) and finishes the proof of Lemma 3. □

By Lemma 3 and the arguments similar to those used to derive [23, Lemma 2.3], one can get the following result. The details are omitted.

LEMMA 4. *Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for some $s \in (1,2]$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (1,2]$. Let $\theta = 2^{s'\gamma'}$ and $\varphi \in \mathfrak{F}$. Then for $\lambda \in \{1,2,\dots,N\}$ and $1<p<\infty$, the following inequalities hold:*

$$\|\sigma_{h,\Omega}^{\lambda,*}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C(s-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},$$

$$\|M_{h,\Omega,\theta}^{\lambda,*}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C(s-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}.$$

The above constants $C>0$ are independent of h,Ω,s,γ and the coefficients of P_λ.

Applying Lemma 4, we can obtain:

LEMMA 5. *Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for some $s \in (1,2]$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (1,2]$. Let $\lambda \in \{1,2,\dots,N\}$, $1<q<\infty$ and $\varphi \in \mathfrak{F}$. Then:*

(i) *For $1<p<q$, it holds that*

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}|\sigma_{h,\Omega,t}^{\lambda}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C(s-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.\end{aligned}\tag{7}$$

(ii) *For $q \leqslant p<\infty$, it holds that*

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}|\sigma_{h,\Omega,t}^{\lambda}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C(s-1)^{-1/q}(\gamma-1)^{-1/q}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.\end{aligned}\tag{8}$$

Here the above constants $C>0$ are independent of h,Ω,s,γ and the coefficients of P_λ.

Proof. This lemma is a variant of [23, Lemma 2.4]. We first prove (7). Let $1<q<\infty$ and $1<p<q$. By duality, there exists a sequence of functions $\{f_k(x,t)\}$ defined on $\mathbb{R}^n\times\mathbb{R}_+$ with

$$\|\{f_k(\cdot,\cdot)\}\|_{L^{p'}(\mathbb{R}^n,\ell^{q'}(L^{q'}([2^{s'\gamma'k},2^{s'\gamma'(k+1)}],dt/t)))} \leqslant 1$$

such that

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}|\sigma_{h,\Omega,t}^{\lambda}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}\sigma_{h,\Omega,t}^{\lambda}*g_k(x)f_k(x,t)\frac{dt}{t}dx\\ &\leqslant C(s-1)^{-1/q}(\gamma-1)^{-1/q}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\|H\|_{L^{p'/q'}(\mathbb{R}^n)}^{1/q'},\end{aligned}\tag{9}$$

where

$$H(x)=\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}|\sigma^{\lambda}_{h,\Omega,t}*\tilde{f}_k(x,t)|^{q'}\frac{dt}{t}\ \text{ and }\ \tilde{f}_k(x,t)=f(-x,t).$$

Since $p'/q'>1$, there exists a nonnegative function $u\in L^{(p'/q')'}(\mathbb{R}^n)$ with $\|u\|_{L^{(p'/q')'}(\mathbb{R}^n)}=1$ such that

$$\|H\|_{L^{p'/q'}(\mathbb{R}^n)}=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}|\sigma^{\lambda}_{h,\Omega,t}*\tilde{f}_k(x,t)|^{q'}\frac{dt}{t}u(x)dx.$$

A change of variable together with the Hölder's inequality yields easily that

$$\begin{aligned}\int_{t/2<|y|\leqslant t}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy&=\int_{t/2}^{t}|h(r)|\frac{dr}{r}\int_{S^{n-1}}|\Omega(y')|d\sigma(y')\\&\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})}.\end{aligned}\tag{10}$$

By (10) and the Hölder's inequality, one has

$$\begin{aligned}&|\sigma^{\lambda}_{h,\Omega,t}*g_k(x)|^{q'}\\&=\Big(\int_{t/2<|y|\leqslant t}|g_k(x-P_\lambda(\varphi(|y|))y')|\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\Big)^{q'}\\&\leqslant\Big(\int_{t/2<|y|\leqslant t}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\Big)^{q'/q}\\&\quad\times\int_{t/2<|y|\leqslant t}|g_k(x-P_\lambda(\varphi(|y|))y')|^{q'}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\\&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q'/q}\\&\quad\times\int_{t/2<|y|\leqslant t}|g_k(x-P_\lambda(\varphi(|y|))y')|^{q'}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy.\end{aligned}\tag{11}$$

Estimates (11) together with the Hölder's inequality and Lemma 4 yield that

$$\begin{aligned}&\|H\|_{L^{p'/q'}(\mathbb{R}^n)}\\&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q'/q}\\&\quad\times\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}\int_{t/2<|y|\leqslant t}|\tilde{f}_k(x-P_\lambda(\varphi(|y|))y',t)|^{q'}\times\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\frac{dt}{t}u(x)dx\\&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q'/q}\\&\quad\times\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}|f_k(z,t)|^{q'}\int_{t/2<|y|\leqslant t}u(P_\lambda(\varphi(|y|))y'-z)\times\frac{|h(|y|)\Omega(y)|}{|y|^n}dydz\frac{dt}{t}\\&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q'/q}\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}|f_k(z,t)|^{q'}\frac{dt}{t}\Big)\tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{u})(z)dz\\&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q'/q}\\&\quad\times\Big\|\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}|f_k(\cdot,t)|^{q'}\frac{dt}{t}\Big\|_{L^{p'/q'}(\mathbb{R}^n)}\|\tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{u})\|_{L^{(p'/q')'}(\mathbb{R}^n)}\\&\leqslant C(s-1)^{-1}(\gamma-1)^{-1}(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q'},\end{aligned}\tag{12}$$

where $\tilde{u}(x) = u(-x)$ and $\tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{u}) = \sigma^{\lambda,*}_{h,\Omega}(\tilde{u})$ with $\rho = 1$. Combining (12) with (9) leads to

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k|^q \frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\leqslant C(s-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

This proves (7).

We now prove (8). Let $q \leqslant p < \infty$. By duality, there exists a nonnegative function f in $L^{(p/q)'}(\mathbb{R}^n)$ with $\|f\|_{L^{(p/q)'}(\mathbb{R}^n)} \leqslant 1$ such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k|^q \frac{dt}{t}\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k|^q \frac{dt}{t} f(x)dx. \tag{13}$$

Similar arguments to those as in deriving (11) may yield that

$$|\sigma^{\lambda}_{h,\Omega,t} * g_k(x)|^q \leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q/q'} \times \int_{t/2<|y|\leqslant t} |g_k(x - P_\lambda(\varphi(|y|))y')|^q \frac{|h(|y|)\Omega(y)|}{|y|^n}dy. \tag{14}$$

By change of variable and (14), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k|^q \frac{dt}{t} f(x)dx\\
&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q/q'}\\
&\quad\times \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} \int_{t/2<|y|\leqslant t} |g_k(x - P_\lambda(\varphi(|y|))y'))|^q \times \frac{|h(|y|)\Omega(y)|}{|y|^n}dy\frac{dt}{t} f(x)dx\\
&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q/q'}\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}|g_k(x)|^q\Big)\tilde{M}^{\lambda,*}_{h,\Omega,\theta}(\tilde{f})(-x)dx.
\end{aligned} \tag{15}$$

Here $\tilde{f}(x) = f(-x)$ and $\tilde{M}^{\lambda,*}_{h,\Omega,\theta} = M^{\lambda,*}_{h,\Omega,\theta}$ with $\rho = 1$ and $\theta = 2^{s'\gamma'}$. By Lemma 4, (13), (15) and the Hölder's inequality, we obtain

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k|^q \frac{dt}{t}\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^n)}\\
&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^{q/q'}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k(x)|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^n)}\|\tilde{M}^{\lambda,*}_{h,\Omega,\theta}(\tilde{f})(-\cdot)\|_{L^{(p/q)'}(\mathbb{R}^n)}\\
&\leqslant C(s-1)^{-1}(\gamma-1)^{-1}(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})})^q\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^n)},
\end{aligned}$$

which leads to (8) for $q < p < \infty$. When $p = q$, then inequality (10) gives that

$$\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}}\int_{t/2<|y|\leqslant t}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\frac{dt}{t} \leqslant C(s-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})}. \tag{16}$$

By (16), the Hölder's inequality and the Fubini's theorem, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}}|\sigma_{h,\Omega,t}^{\lambda}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}^q\\
&=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}}|\sigma_{h,\Omega,t}^{\lambda}*g_k|^q\frac{dt}{t}dx\\
&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})})^{q/q'}\\
&\quad\times\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}}\int_{t/2<|y|\leqslant t}|g_k(x-P_\lambda(\varphi(|y|))y')|^q\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\frac{dt}{t}dx\\
&\leqslant C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})})^{q/q'}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|^q dx\\
&\quad\times\sup_{k\in\mathbb{Z}}\int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}}\int_{t/2<|y|\leqslant t}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\frac{dt}{t}\\
&\leqslant C(s-1)^{-1}(\gamma-1)^{-1}(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})})^q\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}^q,
\end{aligned}$$

which gives (8) for the case $p=q$. This completes the proof of Lemma 5. □

LEMMA 6. *Let $\Omega\in L^s(\mathrm{S}^{n-1})$ for some $1<s\leqslant\infty$ and $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $1<\gamma\leqslant\infty$. Let $\varphi\in\mathfrak{F}$. Then, for $\lambda\in\{1,2,\dots,N\}$ and $\gamma'<p<\infty$, it holds that*

$$\|\sigma_{h,\Omega}^{\lambda,*}(f)\|_{L^p(\mathbb{R}^n)}\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}.$$

The constants $C>0$ are independent of h, Ω, s, γ and the coefficients of P_λ.

Proof. By change of variable and the Hölder's inequality, one has

$$\begin{aligned}
&\big||\sigma_{h,\Omega,t}^{\lambda}|*f(x)\big|\\
&\leqslant\int_{t/2}^{t}\int_{\mathrm{S}^{n-1}}|\Omega(y')||f(x-P_\lambda(\varphi(r))y')|d\sigma(y')|h(r)|\frac{dr}{r}\\
&\leqslant\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big(\int_{t/2}^{t}\Big|\int_{\mathrm{S}^{n-1}}|\Omega(y')||f(x-P_\lambda(\varphi(r))y')|d\sigma(y')\Big|^{\gamma'}\frac{dr}{r}\Big)^{1/\gamma'}\\
&\leqslant\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/\gamma}\Big(\int_{\mathrm{S}^{n-1}}|\Omega(y')|\int_{t/2}^{t}|f(x-P_\lambda(\varphi(r))y')|^{\gamma'}\frac{dr}{r}d\sigma(y')\Big)^{1/\gamma'},
\end{aligned}$$

which implies

$$\begin{aligned}
\sigma_{h,\Omega}^{\lambda,*}(f)(x)\leqslant{}&\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/\gamma}\\
&\times\Big(\int_{\mathrm{S}^{n-1}}|\Omega(y')|\Big(\sup_{t>0}\int_{t/2}^{t}|f(x-P_\lambda(\varphi(r))y')|^{\gamma'}\frac{dr}{r}\Big)d\sigma(y')\Big)^{1/\gamma'}.
\end{aligned}\tag{17}$$

By the fact that $\varphi\in\mathfrak{F}$ and [21, Lemma 3.2], we can get

$$\Big\|\Big(\sup_{t>0}\int_{t/2}^{t}|f(\cdot-P_\lambda(\varphi(r))y')|\frac{dr}{r}\Big)\Big\|_{L^v(\mathbb{R}^n)}\leqslant C\|f\|_{L^v(\mathbb{R}^n)}\tag{18}$$

for any $1 < v < \infty$. Here $C > 0$ is independent of y' and the coefficients of P_λ. By (17), (18) and the Minkowski's inequality, we have

$$\|\sigma_{h,\Omega}^{\lambda,*}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^n)} \leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},$$

for $\gamma' < p < \infty$. This proves Lemma 6. □

Applying Lemma 6, we can get the following result.

LEMMA 7. *Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for some $1 < s \leqslant 2$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \geqslant 2$. Then*

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}}|\sigma_{h,\Omega,t}^{\lambda}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C_p(s-1)^{-1/q}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\end{aligned} \tag{19}$$

holds for $1 < p < q \leqslant \gamma'$ if $2 \leqslant \gamma < \infty$, and for $\gamma' < p < \infty$ if $2 \leqslant \gamma \leqslant \infty$ and $q > \gamma'$. The above constant $C > 0$ is independent of h, Ω, γ, s and the coefficients of P_λ.

Proof. We first prove (19) for $1 < p < q \leqslant \gamma'$ if $2 \leqslant \gamma < \infty$. Let $1 < p < q \leqslant \gamma'$. By the similar argument as in getting (9), there exists a sequence of functions $\{f_k(x,t)\}$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ with

$$\|\{f_k(\cdot,\cdot)\}\|_{L^{p'}(\mathbb{R}^n,\ell^{q'}(L^{q'}([2^{s'k},2^{s'(k+1)}],dt/t)))} \leqslant 1$$

such that

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}}|\sigma_{h,\Omega,t}^{\lambda}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C(s-1)^{-1/q}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\|G\|_{L^{p'/q'}(\mathbb{R}^n)}^{1/q'},\end{aligned} \tag{20}$$

where

$$G(x) = \sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}}|\sigma_{h,\Omega,t}^{\lambda}*\tilde{f}_k(x,t)|^{q'}\frac{dt}{t} \text{ and } \tilde{f}_k(x,t) = f(-x,t).$$

Since $p' > q'$, there exists a nonnegative function $u \in L^{(p'/q')'}(\mathbb{R}^n)$ with $\|u\|_{L^{(p'/q')'}(\mathbb{R}^n)} = 1$ such that

$$\|G\|_{L^{p'/q'}(\mathbb{R}^n)} = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{2^{s'k}}^{2^{s'(k+1)}}|\sigma_{h,\Omega,t}^{\lambda}*\tilde{f}_k(x,t)|^{q'}\frac{dt}{t}u(x)dx.$$

By a change of variable and Hölder's inequality, we obtain

$$|\sigma_{h,\Omega,t}^{\lambda}*\tilde{f}_k(x,t)|$$

$$
\begin{aligned}
&\leqslant \int_{t/2<|y|\leqslant t} |\tilde{f}_k(x-P_\lambda(\varphi(|y|))y',t)| \frac{|h(|y|)\Omega(y)|}{|y|^n}dy \\
&= \int_{t/2}^{t}\int_{S^{n-1}} |\tilde{f}_k(x-P_\lambda(\varphi(r))y',t)||\Omega(y')|d\sigma(y')|h(r)|\frac{dr}{r} \\
&\leqslant C\|h\|_{\Delta_q(\mathbb{R}_+)}(\|\Omega\|_{L^1(S^{n-1})})^{1/q} \\
&\quad \times\Big(\int_{t/2}^{t}\int_{S^{n-1}} |\tilde{f}_k(x-P_\lambda(\varphi(r))y',t)|^{q'}|\Omega(y')|d\sigma(y')\frac{dr}{r}\Big)^{1/q'} \\
&= C\|h\|_{\Delta_q(\mathbb{R}_+)}(\|\Omega\|_{L^1(S^{n-1})})^{1/q}\Big(\int_{t/2<|y|\leqslant t} |\tilde{f}_k(x-P_\lambda(\varphi(|y|))y',t)|^{q'}\frac{|\Omega(y)|}{|y|^n}\Big)^{1/q'}.
\end{aligned}
$$

This together with the arguments similar to those used in deriving (12) yields that

$$
\begin{aligned}
\|G\|_{L^{p'/q'}(\mathbb{R}^n)} &\leqslant C\Big(\|h\|_{\Delta_q(\mathbb{R}_+)}(\|\Omega\|_{L^1(S^{n-1})})^{1/q}\Big)^{q'} \\
&\quad\times\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}} |f_k(z,t)|^{q'}\frac{dt}{t}\Big)\tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{u})(z)dz \\
&\leqslant C\Big(\|h\|_{\Delta_q(\mathbb{R}_+)}(\|\Omega\|_{L^1(S^{n-1})})^{1/q}\Big)^{q'} \\
&\quad\times\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}} |f_k(z,t)|^{q'}\frac{dt}{t}\Big)\Big\|_{L^{p'/q'}(\mathbb{R}^n)}\|\tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{u})\|_{L^{(p'/q')'}(\mathbb{R}^n)},
\end{aligned}
\tag{21}
$$

where $\tilde{u}(x)=u(-x)$ and $\tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{u})=\sigma^{\lambda,*}_{h,\Omega}(\tilde{u})$ with $\rho=1$ and $h=1$. Note that $1<(p'/q')'<\infty$. Invoking Lemma 6, we have

$$
\|\tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{u})\|_{L^{(p'/q')'}(\mathbb{R}^n)} \leqslant C\|\Omega\|_{L^1(S^{n-1})}\|u\|_{L^{(p'/q')'}(\mathbb{R}^n)}. \tag{22}
$$

Here the constant $C>0$ is independent of the coefficients of P_λ. Since $q\leqslant\gamma$. Then $\|h\|_{\Delta_q(\mathbb{R}_+)}\leqslant\|h\|_{\Delta_\gamma(\mathbb{R}_+)}$. This together with (21) and (22) implies that

$$
\|G\|_{L^{(p'/q')}(\mathbb{R}^n)} \leqslant C\|h\|^{q'}_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|^{q'}_{L^s(S^{n-1})}. \tag{23}
$$

Combining (23) with (20) yields that

$$
\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1/q}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

This gives (19) for $1<p<q\leqslant\gamma'$ if $2\leqslant\gamma<\infty$.

We now prove (19) for $\gamma'<p<\infty$ if $2\leqslant\gamma\leqslant\infty$ and $q>\gamma'$. Let $2\leqslant\gamma\leqslant\infty$, $q>\gamma'$ and $\gamma'<p<\infty$. We first prove the following inequality

$$
\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t}*g_k|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'}\Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1/\gamma'}\|\Omega\|_{L^s(S^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^p(\mathbb{R}^n)},
\end{aligned}
\tag{24}
$$

when $2 \leqslant \gamma \leqslant \infty$.

When $2 \leqslant \gamma < \infty$. We get easily by Hölder's inequality that

$$\begin{aligned} &|\sigma^{\lambda}_{h,\Omega,t} * g_k(x)|^{\gamma'} \\ &= \Big(\int_{t/2<|y|\leqslant t} |g_k(x-P_\lambda(\varphi(|y|)y'))| \frac{|h(|y|)\Omega(y)|}{|y|^n} dy\Big)^{\gamma'} \\ &\leqslant \Big(\int_{t/2<|y|\leqslant t} \frac{|h(|y|)|^{\gamma}|\Omega(y)|}{|y|^n} dy\Big)^{\gamma'/\gamma} \\ &\quad \times \int_{2^{k-1}t<|y|\leqslant 2^k t} |g_k(x-P_\lambda(\varphi(|y|)y'))|^{\gamma'} \frac{|\Omega(y)|}{|y|^n} dy \\ &\leqslant C(\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{\gamma'/\gamma} \int_{t/2<|y|\leqslant t} |g_k(x-P_\lambda(\varphi(|y|)y'))|^{\gamma'} \frac{|\Omega(y)|}{|y|^n} dy. \end{aligned} \tag{25}$$

Let $\gamma' < p < \infty$. It is clear that $p/\gamma' > 1$. By duality, there is a nonnegative function $f \in L^{(p/\gamma')'}(\mathbb{R}^d)$ with $\|f\|_{L^{(p/\gamma')'}(\mathbb{R}^d)} \leqslant 1$ such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} \int_{2^{s'k}}^{2^{s'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k|^{\gamma'} dt\Big)^{1/\gamma'}\Big\|^{\gamma'}_{L^p(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} \int_{2^{s'k}}^{2^{s'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k|^{\gamma'} dt f(x) dx. \tag{26}$$

By some changes of variables and Hölder's inequality, (25) and (26) may yield that

$$\begin{aligned} &\Big\|\Big(\sum_{k\in\mathbb{Z}} \int_{2^{s'k}}^{2^{s'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k|^{\gamma'} dt\Big)^{1/\gamma'}\Big\|^{\gamma'}_{L^p(\mathbb{R}^d)} \\ &\leqslant C(\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{\gamma'/\gamma} \\ &\quad \times \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} \int_{2^{s'k}}^{2^{s'(k+1)}} \int_{t/2<|y|\leqslant t} |g_k(x-P_\lambda(\varphi(|y|)y'))|^{\gamma'} \frac{|\Omega(y)|}{|y|^n} dy dt f(x) dx \\ &\leqslant C(\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{\gamma'/\gamma} \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} |g_k(x)|^{\gamma'} \tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{f})(-x) dx \\ &\leqslant C(\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{\gamma'/\gamma} \\ &\quad \times \Big\|\Big(\sum_{k\in\mathbb{Z}} |g_k(x)|^{\gamma'}\Big)^{1/\gamma'}\Big\|^{\gamma'}_{L^{p/\gamma'}(\mathbb{R}^d)} \|\tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{f})(-\cdot)\|_{L^{(p/\gamma')'}(\mathbb{R}^d)}. \end{aligned} \tag{27}$$

Here $\tilde{f}(x) = f(-x)$ and $\tilde{\sigma}^{\lambda,*}_{h,\Omega}(\tilde{f}) = \sigma^{\lambda,*}_{h,\Omega}(\tilde{f})$ with $\rho = 1$ and $h(t) \equiv 1$. (27) together with Lemma 6 yields (24) for $2 \leqslant \gamma < \infty$.

When $\gamma = \infty$. By duality, there exists a nonnegative function $f \in L^{p'}(\mathbb{R}^n)$ with $\|f\|_{L^{p'}(\mathbb{R}^n)} = 1$ such that

$$\Big\|\sum_{k\in\mathbb{Z}} \int_{2^{s'k}}^{2^{s'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k| \frac{dt}{t}\Big\|_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \int_{2^{s'k}}^{2^{s'(k+1)}} |\sigma^{\lambda}_{h,\Omega,t} * g_k| \frac{dt}{t} f(x) dx.$$

One can easily check that

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}}|\sigma^{\lambda}_{h,\Omega,t}*g_k|\frac{dt}{t}f(x)dx &\leqslant \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|\int_{2^{s'k}}^{2^{s'(k+1)}}|\sigma^{\lambda}_{h,\Omega,t}|*\tilde{f}(-x)\frac{dt}{t}dx\\ &\leqslant 2(s-1)^{-1}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|\sigma^{\lambda,*}_{h,\Omega}(\tilde{f})(-x)dx.\end{aligned}$$

Invoking Lemma 6, one may get

$$\Big\|\sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}}|\sigma^{\lambda}_{h,\Omega,t}*g_k|\frac{dt}{t}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C(s-1)^{-1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big\|\sum_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\mathbb{R}^n)}.$$

This yields (24) for $\gamma=\infty$. By change of variable again, it holds that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'k}}^{2^{s'(k+1)}}|\sigma^{\lambda}_{h,\Omega,t}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}=\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{1}^{2^{s'}}|\sigma^{\lambda}_{h,\Omega,2^{s'k}t}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}. \tag{28}$$

(28) and (24) will lead to

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{1}^{2^{s'}}|\sigma^{\lambda}_{h,\Omega,2^{s'k}t}*g_k|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C(s-1)^{-1/\gamma'}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{29}$$

On the other hand, by Lemma 6, one can easily check that

$$\begin{aligned}\Big\|\sup_{k\in\mathbb{Z}}\sup_{t\in[1,2^{s'}]}|\sigma^{\lambda}_{h,\Omega,2^{s'k}t}*g_k|\Big\|_{L^p(\mathbb{R}^n)}&\leqslant\Big\|\sigma^{\lambda,*}_{h,\Omega}\Big(\sup_{k\in\mathbb{Z}}|g_k|\Big)\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Big\|\sup_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{30}$$

Note that $\gamma'<q\leqslant\infty$. Interpolation between (29) and (30) implies that

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{1}^{2^{s'}}|\sigma^{\lambda}_{h,\Omega,2^{s'k}t}*g_k|^{q}\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C(s-1)^{-1/q}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

This together with (28) yields (19) for $\gamma'<p<\infty$ if $2\leqslant\gamma\leqslant\infty$ and $q>\gamma'$. This finishes the proof of Lemma 7. □

3. Proofs of Theorems 3-6

Let P_N be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0)=0$. We may assume without loss of generality that $P_N(t)=\sum_{i=1}^{N}b_it^i$ with each $b_i\neq0$. Let $\sigma^{\lambda}_{h,\Omega,t}$ be given as in Section 2. We start with proving Theorem 5.

Proof of Theorem 5. By Minkowski's inequality, we can write

$$\begin{aligned}
&\mathfrak{M}^{q}_{h,\Omega,P_N,\varphi,\rho}f(x)\\
&=\Big(\int_0^\infty\Big|\sum_{k=-\infty}^{0}\frac{1}{t^\rho}\int_{2^{k-1}t<|y|\leqslant 2^k t}f(x-P_N(\varphi(|y|)y'))\frac{\Omega(y)h(|y|)}{|y|^{n-\rho}}dy\Big|^q\frac{dt}{t}\Big)^{1/q}\\
&\leqslant\sum_{k=-\infty}^{0}\Big(\int_0^\infty\Big|\frac{1}{t^\rho}\int_{2^{k-1}t<|y|\leqslant 2^k t}f(x-P_N(\varphi(|y|))y')\frac{\Omega(y)h(|y|)}{|y|^{n-\rho}}dy\Big|^q\frac{dt}{t}\Big)^{1/q}\\
&\leqslant\frac{1}{1-2^{-\zeta}}\Big(\int_0^\infty|\sigma^N_{h,\Omega,t}*f(x)|^q\frac{dt}{t}\Big)^{1/q}.
\end{aligned}\tag{31}$$

Let ψ be a $\mathscr{C}_0^\infty(\mathbb{R})$ function such that $\psi(t)\equiv 1$ for $|t|\leqslant 1/2$ and $\psi(t)\equiv 0$ for $|t|>1$. For $1\leqslant\lambda\leqslant N$ and $\xi\in\mathbb{R}^n$, we define the family of measures $\{\nu_{t,\lambda}\}_{t>0}$ by

$$\widehat{\nu_{t,\lambda}}(\xi)=\widehat{\sigma^{\lambda}_{h,\Omega,t}}(\xi)\prod_{j=\lambda+1}^{N}\psi(\varphi(t)^j|b_j\xi|)-\widehat{\sigma^{\lambda-1}_{h,\Omega,t}}(\xi)\prod_{j=\lambda}^{N}\psi(\varphi(t)^j|b_j\xi|).\tag{32}$$

Clearly,

$$\sigma^N_{h,\Omega,t}=\sum_{\lambda=1}^{N}\nu_{t,\lambda}.\tag{33}$$

Here we use the convention $\Pi_{j\in\emptyset}a_j=1$ and the fact that $\sigma^0_{h,\Omega,t}=0$ because of (1).

By (31) and (33), one has

$$\mathfrak{M}^{q}_{h,\Omega,P_N,\varphi,\rho}f(x)\leqslant C(\zeta)\sum_{\lambda=1}^{N}\Big(\int_0^\infty|\nu_{t,\lambda}*f(x)|^q\frac{dt}{t}\Big)^{1/q}=:C(\zeta)\sum_{\lambda=1}^{N}\mathscr{D}_\lambda(f)(x).\tag{34}$$

For $1\leqslant\lambda\leqslant N$. Define $\Psi_{k,\lambda}$ by $\Psi_{k,\lambda}(\xi)=\Phi_k(\xi)$, where Φ_k is given as in Lemma 1 with $a_k=\varphi(2^{-s'\gamma'k})^{-\lambda}|b_\lambda^{-1}|$. By the properties of φ, it holds that

$$1<B_\varphi^{s'\gamma'\lambda}\leqslant\frac{a_{k+1}}{a_k}\leqslant c_\varphi^{s'\gamma'\lambda},\ \ \forall k\in\mathbb{Z}.\tag{35}$$

By the Minkowski's inequality and the definition of $\Psi_{k,\lambda}$, we can write

$$\begin{aligned}
\mathscr{D}_\lambda(f)(x)&=\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}\Big|\sum_{j\in\mathbb{Z}}\nu_{t,\lambda}*\Psi_{j-k,\lambda}*f(x)\Big|^q\frac{dt}{t}\Big)^{1/q}\\
&\leqslant\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{s'\gamma'k}}^{2^{s'\gamma'(k+1)}}|\nu_{t,\lambda}*\Psi_{j-k,\lambda}*f(x)|^q\frac{dt}{t}\Big)^{1/q}\\
&=:\sum_{j\in\mathbb{Z}}\mathscr{D}_{\lambda,j,q}(f)(x).
\end{aligned}\tag{36}$$

Applying Lemma 3 and (32), there exists a constant $C>0$ independent of h,Ω,s,γ and the coefficients of P_N such that

$$|\widehat{\nu_{t,\lambda}}(\xi)|\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(S^{n-1})}\min\{1,\varphi(t)^\lambda|b_\lambda\xi|,(\varphi(t)^\lambda|b_\lambda\xi|)^{-1}\}^{\frac{1}{2\lambda\gamma's'}},\tag{37}$$

for $1 \leqslant \lambda \leqslant N$. Combining (37) with the Plancherel's theorem implies that

$$\begin{aligned}
&\|\mathscr{D}_{\lambda,j,2}(f)\|^2_{L^2(\mathbb{R}^n)} \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} |\nu_{t,\lambda} * \Psi_{j-k,\lambda} * f(x)|^2 \frac{dt}{t} dx \\
&\leqslant \sum_{k\in\mathbb{Z}} \int_{E_{j-k}} \int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} |\widehat{\nu_{t,\lambda}}(x)|^2 \frac{dt}{t} |\hat{f}(x)|^2 dx \\
&\leqslant C s'\gamma' B_\varphi^{-|j|} (\|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})})^2 \sum_{k\in\mathbb{Z}} \int_{E_{j-k}} |\hat{f}(x)|^2 dx \\
&\leqslant C(s-1)^{-1}(\gamma-1)^{-1} (\|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})})^2 B_\varphi^{-|j|} \|f\|^2_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{38}$$

where

$$E_{j-k} = \{x \in \mathbb{R}^n : \varphi(2^{s'\gamma'(k-j+1)})^{-\lambda} \leqslant |b_\lambda x| \leqslant \varphi(2^{s'\gamma'(k-j-1)})^{-\lambda}\}.$$

(38) together with the fact that $\dot{F}^0_{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ yields that

$$\|\mathscr{D}_{\lambda,j,2}(f)\|_{L^2(\mathbb{R}^n)} C(s-1)^{-1/2}(\gamma-1)^{-1/2} \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} B_\varphi^{-\frac{|j|}{2}} \|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}. \tag{39}$$

On the other hand, invoking Lemma 5 and (32), there exists $C > 0$ independent of h, Ω, s, γ and the coefficients of P_λ such that

$$\begin{aligned}
&\Big\| \Big(\sum_{k\in\mathbb{Z}} \int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} |\nu_{t,\lambda} * g_k|^q \frac{dt}{t} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1}(\gamma-1)^{-1} \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \Big\| \Big(\sum_{k\in\mathbb{Z}} |g_k|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)},
\end{aligned} \tag{40}$$

for all $p \in (1, q)$ and

$$\begin{aligned}
&\Big\| \Big(\sum_{k\in\mathbb{Z}} \int_{2^{s'\gamma' k}}^{2^{s'\gamma'(k+1)}} |\nu_{t,\lambda} * g_k|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1/q}(\gamma-1)^{-1/q} \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \Big\| \Big(\sum_{k\in\mathbb{Z}} |g_k|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)},
\end{aligned} \tag{41}$$

for all $p \in [q, \infty)$. By (35), (40), (41) and Lemma 1 we obtain

$$\begin{aligned}
&\|\mathscr{D}_{\lambda,j,q}(f)\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1}(\gamma-1)^{-1} \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \Big\| \Big(\sum_{k\in\mathbb{Z}} |\Psi_{j-k,\lambda} * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1}(\gamma-1)^{-1} \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \ \text{ for } 1 < p < q,
\end{aligned} \tag{42}$$

for all $p \in (1,q)$ and

$$\begin{aligned}
&\|\mathscr{D}_{\lambda,j,q}(f)\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1/q}(\gamma-1)^{-1/q}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \times \Big\|\Big(\sum_{k\in\mathbb{Z}}|\Psi_{j-k,\lambda}*f|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1/q}(\gamma-1)^{-1/q}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)},
\end{aligned} \tag{43}$$

for all $p \in [q,\infty)$. A change of variable leads to the following

$$\mathscr{D}_{\lambda,j,q}(f)(x) = \Big(\sum_{k\in\mathbb{Z}}\int_1^{2^{s'\gamma'}}|\nu_{2^{s'\gamma'k}t,\lambda}*\Psi_{j-k,\lambda}*f(x)|^q\frac{dt}{t}\Big)^{1/q}. \tag{44}$$

By interpolation among (39) and (42)-(44), one may get

$$\|\mathscr{D}_{\lambda,j,q}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C(s-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}B_\varphi^{-\beta|j|}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)}, \tag{45}$$

for all $1<p<q$,

$$\|\mathscr{D}_{\lambda,j,q}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C(s-1)^{-1/q}(\gamma-1)^{-1/q}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}B_\varphi^{-\beta|j|}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)}, \tag{46}$$

for all $q \leqslant p < \infty$. Here $\beta > 0$ depends only on p,q. (34) together with (36), (45) and (46) yields the conclusion of Theorem 5. □

Proof of Theorem 6. By Lemmas 1, 3 and 7 and the arguments similar to those used in proving Theorem 5, we can get the conclusion of Theorem 6. The details are omitted. □

Proofs of Theorems 3 and 4. By Theorem 5 and extrapolation arguments following from [4, 25], we can get (i)-(ii) of Theorems 3 and 4. Similarly, we can get (iii) of Theorems 3 and 4 by Theorem 6 and extrapolation arguments. □

4. Further results

As applications of Theorems 3 and 4, certain L^p bounds for the parametric Marcinkiewicz integral operators $\mathscr{M}^{\lambda,q,*}_{h,\Omega,P_N,\varphi,\rho}$ and $\mathscr{M}^{q}_{h,\Omega,P_N,\varphi,\rho,S}$ related to the Littlewood-Paley g^*_λ -function and the area integral S, respectively, will be established. Here

$$\begin{aligned}
\mathscr{M}^{\lambda,q,*}_{h,\Omega,P_N,\varphi,\rho}f(x) := \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} \\
\times\Big|\frac{1}{t^\rho}\int_{|y|\leqslant t}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-P_N(\varphi(|y|))y')dy\Big|^q\frac{dydt}{t^{n+1}}\Big)^{1/q},
\end{aligned}$$

where $\lambda > 0$ and $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$;

$$\mathscr{M}^{q}_{h,\Omega,P_N,\varphi,\rho,S}f(x) := \Big(\iint_{\Gamma(x)}\Big|\frac{1}{t^\rho}\int_{|y|\leqslant t}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-P_N(\varphi(|y|))y')dy\Big|^q\frac{dydt}{t^{n+1}}\Big)^{1/q},$$

where $\Gamma(x)=\{(y,t)\in\mathbb{R}_+^{n+1}:\ |x-y|<t\}$ and h, Ω, P_N, φ, ρ be given as in (2).

Similar arguments to those in getting [20, Lemma 4.2] may give that:

LEMMA 8. *Let $\lambda>1$ and $1<q<\infty$. Then there exists a constant $C(n,\lambda)>0$ such that for any nonnegative locally integrable function g on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n}(\mathscr{M}_{h,\Omega,P_N,\varphi,\rho}^{\lambda,q,*}f(x))^q g(x)dx\leqslant C(n,\lambda)\int_{\mathbb{R}^n}(\mathfrak{M}_{h,\Omega,P_N,\varphi,\rho}^{q}f(x))^q M(g)(x)dx,$$

where M is the usual Hardy-Littlewood maximal operator on $\mathbb{R}^n$.

As applications of Theorems 3 and 4, we can get:

THEOREM 7. *Let P_N be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0)=0$ and $\varphi\in\mathfrak{F}$. Let Ω satisfy (1) and $1<q<\infty$.*

(i) *If $\Omega\in L(\log L)^{1/q}(\mathrm{S}^{n-1})$ and $h\in\mathscr{N}_{1/q}(\mathbb{R}_+)$ for $1<q<\infty$, then*

$$\|\mathscr{M}_{h,\Omega,P_N,\varphi,\rho}^{\lambda,q,*}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(1+\|\Omega\|_{L(\log L)^{1/q}(\mathrm{S}^{n-1})})(1+N_{1/q}(h))\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},$$

for $q\leqslant p<\infty$.

(ii) *If $\Omega\in L(\log L)^{1/q}(\mathrm{S}^{n-1})$ and $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $2<\gamma\leqslant\infty$. Then*

$$\|\mathscr{M}_{h,\Omega,P_N,\varphi,\rho}^{\lambda,q,*}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(1+\|\Omega\|_{L(\log L)^{1/q}(\mathrm{S}^{n-1})})\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},$$

for $\gamma'<q\leqslant p<\infty$.

(iii) *If $\Omega\in B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})$ for some $r>1$ and $h\in\mathscr{N}_{1/q}(\mathbb{R}_+)$ for $1<q<\infty$, then*

$$\|\mathscr{M}_{h,\Omega,P_N,\varphi,\rho}^{\lambda,q,*}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(1+\|\Omega\|_{B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})})(1+N_{1/q}(h))\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},$$

for $q\leqslant p<\infty$.

(iv) *If $\Omega\in B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})$ for some $r>1$ and $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $2<\gamma\leqslant\infty$. Then*

$$\|\mathscr{M}_{h,\Omega,P_N,\varphi,\rho}^{\lambda,q,*}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p(1+\|\Omega\|_{B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})})\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},$$

for $\gamma'<q\leqslant p<\infty$.

Here the above constants $C_p>0$ are independent of h,Ω and the coefficients of P_N, but may depend on $p,q,n,\lambda,\varphi,\rho,N$. The same results hold for $\mathfrak{M}_{h,\Omega,P_N,\varphi,\rho,S}^{q}$.

Proof. Fix $1 < q \leqslant p < \infty$, by the duality, L^p bounds for M, Hölder's inequality and Lemma 7, one has

$$\begin{aligned}\|\mathscr{M}^{\lambda,q,*}_{h,\Omega,P_N,\varphi,\rho}f\|^q_{L^p(\mathbb{R}^n)} &= \sup_{\|g\|_{L^{(p/q)'}(\mathbb{R}^n)}\leqslant 1}\int_{\mathbb{R}^n}(\mathscr{M}^{\lambda,q,*}_{h,\Omega,P_N,\varphi,\rho}f(x))^q g(x)dx \\ &\leqslant C(n,\lambda)\sup_{\|g\|_{L^{(p/q)'}(\mathbb{R}^n)}\leqslant 1}\int_{\mathbb{R}^n}(\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}f(x))^q M(g)(x)dx \\ &\leqslant C(n,\lambda,p,q)\|\mathfrak{M}^q_{h,\Omega,P_N,\varphi,\rho}f\|^q_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Combining this with Theorems 3 and 4 implies the conclusions of Theorem 7 for $\mathscr{M}^{\lambda,q,*}_{h,\Omega,P_N,\varphi,\rho}$.

On the other hand, it is clear that

$$\mathscr{M}^q_{h,\Omega,P_N,\varphi,\rho,S}f(x) \leqslant 2^{n\lambda/q}\mathscr{M}^{\lambda,q,*}_{h,\Omega,P_N,\varphi,\rho}f(x),$$

which together with the bounds for $\mathscr{M}^{\lambda,q,*}_{h,\Omega,P_N,\varphi,\rho}$ implies the bounds for $\mathscr{M}^q_{h,\Omega,P_N,\varphi,\rho,S}$. □

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