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q–GENERALIZED BERNSTEIN–DURRMEYER POLYNOMIALS

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Abstract. The purpose of the present paper is to introduce a q-Durrmeyer variant of generalized-Bernstein operators proposed by Chen et al. (2017). The convergence rate of these operators is examined by means of the Lipschitz class and the Peetre's K-functional. Also, we define the bivariate case of these operators and study the degree of approximation with the aid of the partial moduli of continuity and higher order modulus of continuity via Peetre's K-functional approach. Finally, we introduce the GBS (Generalized Boolean Sum) of the considered operators and investigate the approximation of the Bögel continuous and Bögel differentiable functions with the aid of the Lipschitz class and the mixed modulus of smoothness. Some numerical examples with illustrative graphics have been added to validate the theoretical results and also compare the rate of convergence by using Matlab algorithms.

1. Introduction

Chen et al. [10] introduced a new generalization of the Bernstein polynomials depending on a parameter $0 \leqslant \alpha \leqslant 1$, as follows:

$$T_{n,\alpha}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\alpha}(x) f\left(\frac{k}{n}\right), \ x \in I, I \text{ being } [0,1] \tag{1.1}$$

where $p_{n,k}^{\alpha}(x) = \left[\binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha x(1-x)\right] x^{k-1}(1-x)^{n-k-1}$ and gave a new proof of the Weierstrass theorem. The rate of convergence by means of the modulus of continuity and the shape preserving properties of the above operators were also established. We note that the operator (1.1) includes the well known Bernstein polynomials for $\alpha = 0$. Durrmeyer [13] introduced an integral modification of the Bernstein polynomials called as Bernstein Durrmeyer polynomials to approximate Lebesgue integrable functions in [0,1]. We remark that, over the last thirty years, the authors discuss about different type of Durrmeyer operators. Gupta [16] introduced a q-analogue of the Bernstein-Durrmeyer polynomials and studied the rate of convergence of these operators. Neer et al. [19] and Gupta et al. [15] studied the properties of approximation of Durrmeyer type operators based on Polya distribution. The extension

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of above researches in the direction of simultaneous approximation, differences of positive linear operators and quantum calculus, we can find in [1], [2], [3], [14], [18] and [20].

Motivated by the above work, for $f \in C(I)$ the space of continuous functions with $||f|| = \sup_{x\in I}|f(x)|$, we define the Durrmeyer analogue of the generalized Bernstein polynomials given by (1.1), based on q-integers as

$$D_{n,q}^{\alpha}(f;x) = [n+1]_q \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) \int_0^1 p_{n,k}^{q,\alpha}(qt) f(t) d_q t, \tag{1.2}$$

where,

$$p_{n,k}^{q,\alpha}(x) = \binom{n-2}{k}_q (1-\alpha)x^k(1-x)_q^{n-k-1} + q^{-2}\binom{n-2}{k-2}_q (1-\alpha)x^{k-1}(1-qx)_q^{n-k} + \binom{n}{k}_q \alpha x^k (1-qx)_q^{n-k}$$

and study the rate of convergence of the operators (1.2) given by means of the modulus of continuity and Ditzian Totik modulus of smoothness. The bivariate generalization of the operators (1.2) is introduced and the degree of approximation is investigated with the aid of the modulus of continuity and the Peetre's K-functional. Finally, we define the associated GBS (Generalized Boolean Sum) operators and examine the rate of convergence in terms of the mixed modulus of smoothness for functions in the Bögel space.

2. Preliminaries

LEMMA 1. *The operators defined in (1.2) satisfy the following inequalities:*

i) $D_{n,q}^{\alpha}(e_0;x) = 1$;

ii) $$D_{n,q}^{\alpha}(e_1;x) = \frac{1}{[n+2]_q}\left\{\left[(1-\alpha)(q^2-1)x^2 + (\alpha q^2 - \alpha + 1)x\right][n]_q + (q+1)^2(1-q)(1-\alpha)x^2 + (1-\alpha)(q^2-1)x + 1\right\};$$

iii) $$\begin{aligned} D_{n,q}^{\alpha}(e_2;x) = -\frac{1}{[n+2]_q[n+3]_q} & \left\{x^2\left[q(1-\alpha)(1-q^4)x - q(\alpha q^4 - \alpha + 1)\right][n]_q^2 \right.\\ & + \left[q(1-\alpha)(q^4-1)(q^2+2q+2)x^2\right.\\ & + (\alpha q^6 + 3\alpha q^5 - q^6 + 2\alpha q^4 - 2q^5 + \alpha q^3 - 2q^4 - 2\alpha q^2\\ & \left. - q^3 - 4\alpha q + 2q^2 - \alpha + 4q + 1)x - (q+1)^2(\alpha q^2 - \alpha + 1)\right] x[n]\\ & + q(1-q)(1-\alpha)(q^2+1)(q^2+q+1)(q+1)^2x^3\\ & + (1-\alpha)(q+1)(q^6 + 2q^5 + 2q^4 + q^3 - q^2 - 3q - 1)x^2\\ & \left. - (1-\alpha)(q+1)(q^4+q^3+q^2-q-1)x - q - 1\right\}. \end{aligned}$$

In what follows let $q := q_n$ be a sequence such that $\lim_{n\to\infty} q_n = 1$ and

$$\lim_{n\to\infty} q_n^n = c,\ 0 \leqslant c < 1. \tag{2.1}$$

Consequently simple computations we obtain

$$\lim_{n\to\infty} D^{\alpha}_{n,q_n}(e_i;x) = x^i,\ \text{for } i = 0,1,2.$$

LEMMA 2. *The order of convergence for the central moments of the operators (1.2) are given as:*

i) $\lim_{n\to\infty} [n]_{q_n} D^{\alpha}_{n,q_n}(t-x;x) = 2x(1-c)(1-x)(1-\alpha) - 2x + 1$;

ii) $\lim_{n\to\infty} [n]_{q_n} D^{\alpha}_{n,q_n}((t-x)^2;x) = 2x(1-x)$;

iii) $\lim_{n\to\infty} [n]^2_{q_n} D^{\alpha}_{n,q_n}((t-x)^4;x) = x^2(1-x)(7x^2-7x+5)$;

iv) $\lim_{n\to\infty} [n]^2_{q_n} D^{\alpha}_{n,q_n}((t-x)^6;x) = 15x^4(1-x)(7x^2+5x-7)$.

LEMMA 3. *For $f \in C(I)$ and every $x \in I$, we have $|D^{\alpha}_{n,q_n}(f;x)| \leqslant \| f \|$.*

Proof. Using Lemma 1, the proof of the theorem is straightforward, hence the details are omitted.

3. Convergence properties

THEOREM 1. *For any $f \in C(I)$, the sequence $\{D^{\alpha}_{n,q_n}(f;x)\}$ converges to f uniformly in I.*

Proof. The proof follows from the well known Korovkin theorem regarding the convergence of a sequence of linear positive operators from $C[a,b]$ *into* $C[a,b]$. So, it is enough to prove the conditions

$$\lim_{n\to\infty} D^{\alpha}_{n,q_n}(e_i;x) = x^i,\ i = 0,1,2,\ \text{uniformly in } [0,1], \tag{3.1}$$

thus (3.1) follows from Lemma 1.

EXAMPLE 1. Let $f(x) = x^4 - x^2 + 6$, $\alpha = 0.8$, $q_n = 1 - \frac{1}{n}$ and $n \in \{20,100,200\}$. Denote $E^{\alpha}_{n,q_n}(f;x) = \left|D^{\alpha}_{n,q_n}(f;x) - f(x)\right|$ the error function of approximation by D^{α}_{n,q_n} operators. The convergence of the operators D^{α}_{n,q_n} to the function f is illustrated in Figure 1. The error of approximation E^{α}_{n,q_n} is given in Figure 2. Also, in Table 1 we computed the error of approximation for D^{α}_{20,q_n}, D^{α}_{100,q_n} and D^{α}_{200,q_n} at certain points.

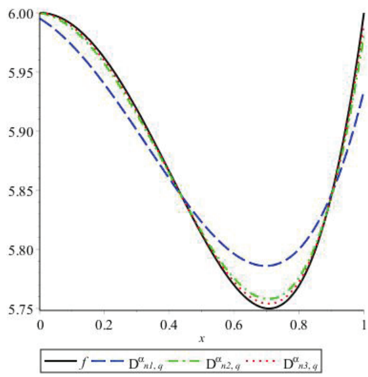

Figure 1: Approximation process by D^{α}_{n,q_n}

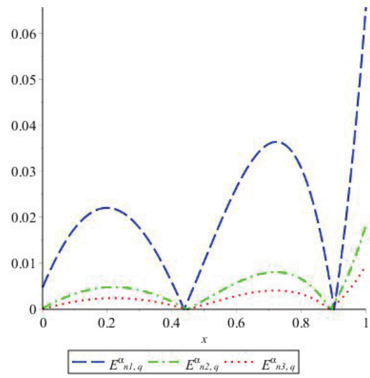

Figure 2: Error of approximation

TABLE 1. *Error of approximation* E^{α}_{n,q_n} *for* $n = 20, 100, 200$

x	$E^{\alpha}_{20,q_n}(f;x)$	$E^{\alpha}_{100,q_n}(f;x)$	$E^{\alpha}_{200,q_n}(f;x)$
0.1	0.01765832375	0.003388315926	0.001680325372
0.2	0.02204370705	0.004791097396	0.002417024901
0.3	0.01764738485	0.004155347677	0.002117187117
0.4	0.00581769107	0.001671392090	0.000868353694
0.5	0.01045630839	0.002012485915	0.001002894290
0.6	0.02650758822	0.005774182437	0.002922211230
0.7	0.03595431558	0.008001917418	0.004058762744
0.8	0.03066338047	0.006578385637	0.003316502430
0.9	0.00071430617	0.001135085683	0.000674550541

EXAMPLE 2. Let $f(x) = 8x^4 - 10x^3 + 3x^2$, $\alpha = 0.8$ and $n = 50$. The Figure 3 illustrates the variation of q-Bernstein Durrmeyer operators when $q \in \{0.7, 0.8, 0.9, 1\}$. Note that for $q = 1$ the Durrmeyer analogue of the generalized Bernstein operators becomes the classical ones. The error of approximation E^{α}_{n,q_n} is given in Figure 4. Also, in Table 2 we computed the error of approximation for $D^{\alpha}_{n,0.7}$, $D^{\alpha}_{n,0.8}$, $D^{\alpha}_{n,0.9}$ and $D^{\alpha}_{n,1}$ at certain points.

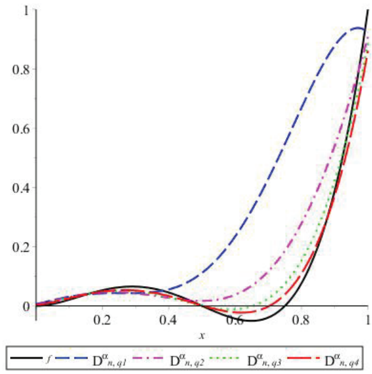

Figure 3: Approximation process by D_{n,q_n}^{α}

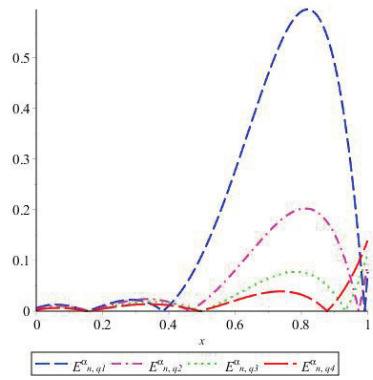

Figure 4: Error of approximation

TABLE 2. *Error of approximation* E_{n,q_n}^{α} *for* $q_n = 0.7, 0.8, 0.9, 1$

x	$E_{n,0.7}^{\alpha}(f;x)$	$E_{n,0.8}^{\alpha}(f;x)$	$E_{n,0.9}^{\alpha}(f;x)$	$E_{n,1}^{\alpha}(f;x)$
0.1	0.0100197411	0.0086608274	0.00688593452	0.00510281418
0.2	0.0106085652	0.0106085652	0.00511524501	0.00463610996
0.3	0.0221498585	0.0223135796	0.01681621488	0.01295395084
0.4	0.0106575181	0.0180686515	0.01670990510	0.01166186502
0.5	0.1111379954	0.0166779634	0.00026700912	0.00113217336
0.6	0.2772910142	0.0807648356	0.03132480448	0.02100317065
0.7	0.4684066650	0.1559209078	0.06441294020	0.03718891032
0.8	0.5916847230	0.2019583020	0.07681381890	0.03256975059
0.9	0.4888570850	0.1519677590	0.03373808520	0.01635157580

EXAMPLE 3. Let $f(x) = 144x^3 - 200x^2 + 81x - 12$, $n = 100$, $q_n = 0.8$ and $\alpha \in \{0.5, 0.8, 0.98\}$. In Figure 5 we present graph of the operators D_{n,q_n}^{α} for different values of α. The error of approximation by E_{n,q_n}^{α} is given in Figure 6. Also, in Table 3 we computed the error of approximation for $D_{n,q_n}^{0.5}$, $D_{n,q_n}^{0.8}$ and $D_{n,q_n}^{0.98}$ at certain points.

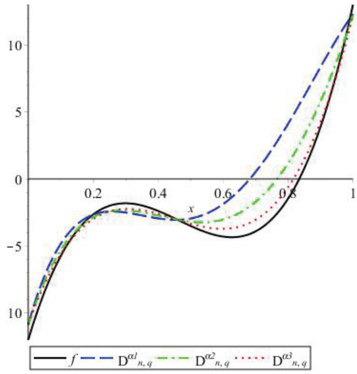

Figure 5: Approximation process by D^{α}_{n,q_n}

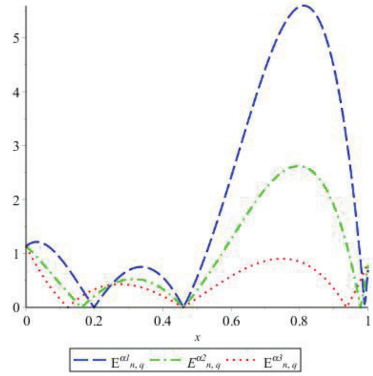

Figure 6: Error of approximation

TABLE 3. *Error of approximation* E^{α}_{n,q_n} *for* $\alpha = 0.5, 0.8, 0.98$

x	$E^{0.5}_{n,q_n}(f;x)$	$E^{0.8}_{n,q_n}(f;x)$	$E^{0.98}_{n,q_n}(f;x)$
0.10	0.9141798400	0.4368722242	0.1504876554
0.15	0.4713685026	0.0833543074	0.1494542082
0.20	0.0103794200	0.2141100930	0.3363484952
0.25	0.4241821610	0.4230133815	0.4223121098
0.30	0.6876337711	0.5206382888	0.4204409894
0.35	0.7428041520	0.4940578774	0.3448101012
0.40	0.5562389790	0.3401355400	0.2104734543
0.45	0.1189598290	0.0655249880	0.0334640770
0.50	0.5535360200	0.3133297270	0.1692059567

3.1. Rate of convergence

In this section we will give the error estimate of the operator $D^{\alpha}_{n,q_n}(.;x)$ by virtue of the first and second order modulus of continuity and K-functional. The first order modulus of continuity of $f \in C(I)$, is given by

$$\omega(f;\delta) = \sup_{0<|h|<\delta} \sup_{x,x+h\in I} |f(x+h)-f(x)|,\ \delta > 0$$

It is well known that for all $f \in C(I)$, we have

$$\lim_{\delta\to 0^+} \omega(f;\delta) = 0,$$

and for any $\delta > 0$,

$$|f(x)-f(y)| \leqslant \omega(f;\delta)\left(\frac{|x-y|}{\delta}+1\right). \tag{3.2}$$

In what follows, $\delta_n(x) := D^{\alpha}_{n,q_n}((t-x)^2;x)$ and $\lambda_n(x) = D^{\alpha}_{n,q_n}((t-x);x)$.

THEOREM 2. *Let $f \in C(I)$, then*

$$|D^{\alpha}_{n,q_n}(f;x) - f(x)| \leqslant 2\omega\left(f;\sqrt{\delta_n(x)}\right).$$

Proof. Using (3.2), we have

$$\begin{aligned}|D^{\alpha}_{n,q_n}(f;x) - f(x)| &\leqslant [n+1]_q \sum_{k=0}^{n} p^{q_n,\alpha}_{n,k}(x) \int_0^1 p^{q_n,\alpha}_{n,k}(qt)|f(t)-f(x)|d_{q_n}t \\ &= \left(\frac{1}{\delta} D^{\alpha}_{n,q_n}(|t-x|;x) + D^{\alpha}_{n,q_n}(1;x)\right)\omega(f;\delta).\end{aligned}$$

Applying Cauchy-Schwarz inequality and Lemma 1, we are led to

$$\begin{aligned}|D^{\alpha}_{n,q_n}(f;x) - f(x)| &\leqslant \left(\frac{1}{\delta}\sqrt{D^{\alpha}_{n,q_n}((t-x)^2);x)} + D^{\alpha}_{n,q_n}(1;x)\right)\omega(f;\delta) \\ &= \left(\frac{1}{\delta}\sqrt{\delta_n(x)} + 1\right)\omega(f;\delta).\end{aligned}$$

Now choosing $\delta = \sqrt{\delta_n(x)}$, the desired result is obtained.
Let $C^1(I) = \{f :\ f, f' \in C(I)\}$.

THEOREM 3. *For $f \in C^1(I)$ and $x \in I$, we have*

$$|D^{\alpha}_{n,q_n}(f;x) - f(x)| \leqslant |\lambda_n(x)||f'(x)| + 2\sqrt{\delta_n(x)}\,\omega(f';\sqrt{\delta_n(x)}).$$

Proof. Applying Taylor's formula, we have

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t (f'(u) - f'(x))du.$$

Operating on the above equality by $D^{\alpha}_{n,q_n}(.;x)$ and considering (3.2), we find

$$\begin{aligned}|D^{\alpha}_{n,q_n}(f;x) - f(x)| \leqslant& |f'(x)||D^{\alpha}_{n,q_n}((t-x);x)| \\ &+ \omega(f';\delta)\left\{\delta^{-1} D^{\alpha}_{n,q_n}((t-x)^2;x) + D^{\alpha}_{n,q_n}(|t-x|;x)\right\}.\end{aligned}$$

Now, applying the Cauchy-Schwarz inequality and choosing $\delta = \sqrt{\delta_n(x)}$, we reach the required result.
Next, we study the rate of convergence of operator $D^{\alpha}_{n,q_n}(;x)$ by means of the second order modulus of continuity and K-functional. First we recall the definition of K-functional. For any $f \in C(I)$,

$$K_2(f;\delta) = \inf\{\|f-g\| + \delta\|g''\|; g \in W^2\},$$

where

$$W^2 = \{g \in C[0,1]; g'' \in C(I)\}.$$

The second order modulus of continuity is defined as

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<|h|<\delta} \sup_{x,x+2h\in I} |f(x+2h)-2f(x+h)+f(x)|.$$

It is well known [11] that K-functional $K_2(f;\delta)$ and the second order modulus of continuity $\omega_2(f;\sqrt{\delta})$ are related by:

$$K_2(f;\delta) \leqslant C\omega_2(f;\sqrt{\delta}), \tag{3.3}$$

for some constant $C>0$ and $\delta>0$.

THEOREM 4. *For all $f \in C(I)$ and $n \in N$ there exists a constant $C>0$ such that*

$$|D^{\alpha}_{n,q_n}(f;x)| \leqslant C\omega_2\left(f;\frac{\sqrt{\phi^{\alpha}_{n,q_n}(x)}}{2\sqrt{2}}\right) + \omega\Big(f;|\lambda_n(x)|\Big),$$

where $\phi^{\alpha}_{n,q_n}(x) = \delta_n(x) + (\lambda_n(x))^2$.

Proof. For $x \in I$, we define the auxiliary operators as

$$D^{\alpha *}_{n,q_n}(f;x) = D^{\alpha}_{n,q_n}(f;x) + f(x) - f\Big(D^{\alpha}_{n,q_n}(t;x)\Big). \tag{3.4}$$

Consequently, the operator $D^{\alpha *}_{n,q_n}(;x)$ is linear, $D^{\alpha *}_{n,q_n}(1;x)=1$ and

$$D^{\alpha *}_{n,q_n}(t-x;x) = 0.$$

For every $g \in W^2$ and $x \in I$, from the Taylor's theorem we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Operating by $D^{\alpha *}_{n,q_n}(.;x)$ on the above equation and using (3.4), we have

$$\begin{aligned}
|D^{\alpha *}_{n,q_n}(g;x) - g(x)| \leqslant & \left|D^{\alpha}_{n,q_n}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\
& + \left|\int_x^{D^{\alpha}_{n,q_n}(t;x)} \Big(D^{\alpha}_{n,q_n}(t;x) - u\Big)g''(u)du\right| \\
\leqslant & \frac{1}{2}\left\{\|g''\| D^{\alpha}_{n,q_n}((t-x)^2;x) + \|g''\|\Big(D^{\alpha}_{n,q_n}(t;x) - x\Big)^2\right\}
\end{aligned}$$

$$=\frac{1}{2}\|g''\|\left(\delta_n(x)+(\lambda_n(x))^2\right). \tag{3.5}$$

For all $f\in C(I)$, in view of Lemma 3 and (3.4), we get

$$|D^{\alpha*}_{n,q_n}(f;x)|\leqslant|D^{\alpha}_{n,q_n}(f;x)|+|f(x)|+|f(D^{\alpha}_{n,q_n}(t;x))|\leqslant 3\|f\|. \tag{3.6}$$

Now, for any $g\in W^2$, using (3.5) and (3.6)

$$\begin{aligned}|D^{\alpha}_{n,q_n}(f;x)-f(x)|\leqslant&|D^{\alpha*}_{n,q_n}(f-g;x)|+|f(x)-g(x)|+|f(x)-f(D^{\alpha}_{n,q_n}(t;x)|\\&+|D^{\alpha*}_{n,q_n}(g;x)-g(x)|\\ \leqslant&4\|f-g\|+\omega(f;|\lambda_n(x)|)+\frac{1}{2}\phi^{\alpha}_{n,q_n}(x)\|g''\|.\end{aligned}$$

Now, taking the infimum on the right hand side over all $g\in W^2$, and using the relation (3.3), the required result is obtained.

THEOREM 5. *For all $f\in C^2(I)$, then*

$$\lim_{n\to\infty}[n]_{q_n}(D^{\alpha}_{n,q_n}(f;x)-f(x))=(2x(1-c)(1-x)(1-\alpha)-2x+1)f'(x)+x(1-x)f''(x)$$

converges uniformly in I.

Proof. By the Taylor's expansion of f

$$f(t)=\sum_{i=0}^{2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\xi(t,x)(t-x)^2, \tag{3.7}$$

where $\xi(t,x)\in C[0,1]$ and $\xi(t,x)\to 0$, as $t\to x$. Now applying operator $D^{\alpha}_{n,q_n}(;x)$ to both sides of the above equality (3.7), we get

$$\begin{aligned}\lim_{n\to\infty}[n]_{q_n}(D^{\alpha}_{n,q_n}(f;x)-f(x))=&\lim_{n\to\infty}[n]_{q_n}D^{\alpha}_{n,q_n}((t-x);x)f'(x)\\&+\lim_{n\to\infty}[n]_{q_n}D^{\alpha}_{n,q_n}((t-x)^2;x)\frac{f''(x)}{2!}\\&+\lim_{n\to\infty}[n]_{q_n}D^{\alpha}_{n,q_n}(\xi(t,x)(t-x)^2;x)\\=&(2x(1-c)(1-x)(1-\alpha)-2x+1)f'(x)+x(1-x)f''(x)\\&+\lim_{n\to\infty}[n]_{q_n}D^{\alpha}_{n,q_n}(\xi(t,x)(t-x)^2;x),\end{aligned} \tag{3.8}$$

uniformly in I. Applying Cauchy-Schwarz inequality, we have

$$|D^{\alpha}_{n,q_n}(\xi(t,x)(t-x)^2;x)|\leqslant\sqrt{D^{\alpha}_{n,q_n}(\xi^2(t,x);x)}\sqrt{D^{\alpha}_{n,q_n}((t-x)^4;x)}. \tag{3.9}$$

Since $\xi(t,x)\in C(I)$, applying Theorem 1

$$\lim_{n\to\infty}D^{\alpha}_{n,q_n}(\xi^2(t,x);x)=0, \tag{3.10}$$

uniformly in I. Further, in view of Lemma 2

$$\lim_{n\to\infty} D^{\alpha}_{n,q_n}((t-x)^4;x) = O\left(\frac{1}{[n]^2_{q_n}}\right), \tag{3.11}$$

uniformly in $x \in I$. Hence, from (3.9)-(3.11)

$$\lim_{n\to\infty}[n]_{q_n} D^{\alpha}_{n,q_n}(\xi(t,x)(t-x)^2;x) = 0,$$

uniformly in I. Finally, combining (3.8) and (3.11), we obtain the desired result.

4. Construction of bivariate operator

For $C(I^2)$, $I^2 = I \times I$ the space of continuous functions on I^2 with the norm $||f||_{C(I)^2} = \sup_{(x,y)\in I^2} |f(x,y)|$ and $0 \leqslant \alpha_1, \alpha_2 \leqslant 1$, the bivariate generalization of the operators given by (1.2) is defined as

$$D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(f(t,s);x,y) = [n_1+1]_{q_1}[n_2+1]_{q_2} \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p^{q_1,q_2,\alpha_1,\alpha_2}_{n_1,n_2,k_1,k_2}(x,y)$$
$$\int_0^1\int_0^1 p^{q_1,\alpha_1}_{n_1,k_1}(q_1t)p^{q_2,\alpha_2}_{n_2,k_2}(q_2s)f(t,s)d_{q_1}td_{q_2}s, \tag{4.1}$$

where

$$p^{q_1,q_2,\alpha_1,\alpha_2}_{n_1,n_2,k_1,k_2}(x,y) = p^{q_1,\alpha_1}_{n_1,k_1}(x)p^{q_2,\alpha_2}_{n_2,k_2}(y).$$

LEMMA 4. *Let $e_{ij}(t,s) = t^is^j, (t,s) \in I^2, i,j \in N\cup\{0\}$ with $i+j \leqslant 2$ be the two dimensional test functions. Then, the following equalities hold for the operators (4.1):*

i) $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(e_{00};x,y) = 1;$

ii) $$D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(e_{10};x,y) = \frac{1}{[n_1+2]_{q_1}}\left\{\left[(1-\alpha_1)(q_1^2-1)x^2+(\alpha_1q_1^2-\alpha_1+1)x\right][n_1]_{q_1}\right.$$
$$\left.+(q_1+1)^2(1-q_1)(1-\alpha_1)x^2+(1-\alpha_1)(q_1^2-1)x+1\right\};$$

iii) $$D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(e_{01};x,y) = \frac{1}{[n_2+2]_{q_2}}\left\{\left[(1-\alpha_2)(q_2^2-1)y^2+(\alpha_2q_2^2-\alpha_2+1)y\right][n_2]_{q_2}\right.$$
$$\left.+(q_2+1)^2(1-q_2)(1-\alpha_2)y^2+(1-\alpha_2)(q_2^2-1)y+1\right\};$$

iv) $$D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(e_{20};x,y) = -\frac{1}{[n_1+2]_{q_1}[n_1+3]_{q_1}}$$
$$\left\{x^2\left[q_1(1-\alpha_1)(1-q_1^4)x-q_1(\alpha_1q_1^4-\alpha_1+1)\right][n_1]^2_{q_1}\right.$$
$$+\left[q_1(1-\alpha_1)(q_1^4-1)(q_1^2+2q_1+2)x^2\right.$$
$$+(\alpha_1q_1^6+3\alpha_1q_1^5-q_1^6+2\alpha_1q_1^4-2q_1^5+\alpha_1q_1^3-2q_1^4-2\alpha_1q_1^2$$
$$-q_1^3-4\alpha_1q_1+2q_1^2-\alpha_1+4q_1+1)x$$
$$\left.-(q_1+1)^2(\alpha_1q_1^2-\alpha_1+1)\right]x[n_1]_{q_1}$$

$$+q_1(1-q_1)(1-\alpha_1)(q_1^2+1)(q_1^2+q_1+1)(q_1+1)^2x^3$$
$$+(1-\alpha_1)(q_1+1)(q_1^6+2q_1^5+2q_1^4+q_1^3-q_1^2-3q_1-1)x^2$$
$$-(1-\alpha_1)(q_1+1)(q_1^4+q_1^3+q_1^2-q_1-1)x-q_1-1\,\};$$

v) $$D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(e_{02};x,y)=-\frac{1}{[n_2+2]_{q_2}[n_2+3]_{q_2}}$$
$$\{y^2\left[q_2(1-\alpha_2)(1-q_2^4)y-q_2(\alpha_2q_2^4-\alpha_2+1)\right][n_2]^2_{q_2}$$
$$+\left[q_2(1-\alpha_2)(q_2^4-1)(q_2^2+2q_2+2)y^2\right.$$
$$+(\alpha_2q_2^6+3\alpha_2q_2^5-q_2^6+2\alpha_2q_2^4-2q_2^5+\alpha_2q_2^3-2q_2^4-2\alpha_2q_2^2$$
$$-q_2^3-4\alpha_2q_2+2q_2^2-\alpha_2+4q_2+1)y$$
$$\left.-(q_2+1)^2(\alpha_2q_2^2-\alpha_2+1)\right]y[n_2]_{q_2}$$
$$+q_2(1-q_2)(1-\alpha_2)(q_2^2+1)(q_2^2+q_2+1)(q_2+1)^2y^3$$
$$+(1-\alpha_2)(q_2+1)(q_2^6+2q_2^5+2q_2^4+q_2^3-q_2^2-3q_2-1)y^2$$
$$-(1-\alpha_2)(q_2+1)(q_2^4+q_2^3+q_2^2-q_2-1)y-q_2-1\,\}.$$

Proof. Since $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(t^is^j;x,y)=D^{\alpha_1}_{n_1,q_1}(t^i;x)\,D^{\alpha_2}_{n_2,q_2}(s^j;y)$, for $0\leqslant i,j\leqslant 2$. By using Lemma 1, the proof of the lemma is straightforward. Hence the details are omitted.

For $f\in C(I^2)$ and $\delta>0$, the first order complete modulus of continuity for the bivariate case is defined as follows:

$$\bar{\omega}(f;\delta_1,\delta_2)=\sup\left\{|f(t,s)-f(x,y)|:\ |t-x|\leqslant\delta_1,|s-y|\leqslant\delta_2\right\},$$

where $\delta_1,\delta_2>0$. Further $\bar{\omega}(f;\delta_1,\delta_2)$ satisfies the following properties:

a) $\bar{\omega}(f;\delta_1,\delta_2)\to 0$ if $\delta_1\to 0$ and $\delta_2\to 0$;

b) $|f(t,s)-f(x,y)|\leqslant\bar{\omega}(f;\delta_1,\delta_2)\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)$.

Now, we give an estimate of the rate of convergence of the bivariate operators. In what follows, let $0<q_{n_i}<1$ be a sequence in $(0,1)$ such that $q_{n_i}\to 1$ and $q_{n_i}^{n_i}\to a_i$, $(0\leqslant a_i<1)$ *as* $n_i\to\infty$ *for* $i=1,2$. Further, let $\delta_{n_1}(x)=D^{\alpha_1}_{n_1,q_{n_1}}((t-x)^2;x)$ and $\delta_{n_2}(y)=D^{\alpha_2}_{n_2,q_{n_2}}((s-y)^2;y)$.

THEOREM 6. *Let* $f\in C(I^2)$. *Then for all* $(x,y)\in I^2$, *we have*

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)|\leqslant 4\bar{\omega}(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(y)}).$$

Proof. By using the linearity and positivity of the operator $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)$ and taking into account the property b) of the modulus of continuity,

$$\begin{aligned}&|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)|\\ \leqslant& D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|f(t,s)-f(x,y)|;x,y)\end{aligned}$$

$$\leqslant \bar{\omega}(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(y)})\left(D^{\alpha_1}_{n_1,q_{n_1}}(1;x)+\frac{1}{\sqrt{\delta_{n_1}(x)}}(D^{\alpha_1}_{n_1,q_{n_1}}|t-x|;x)\right)$$
$$\times\left(D^{\alpha_2}_{n_2,q_{n_2}}(1;y)+\frac{1}{\sqrt{\delta_{n_2}(y)}}D^{\alpha_2}_{n_2,q_{n_2}}(|s-y|;y)\right).$$

Applying Cauchy-Schwarz inequality and Lemma 1

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)|$$
$$\leqslant \bar{\omega}(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(y)})\left(1+\frac{1}{\sqrt{\delta_{n_1}(x)}}\sqrt{(D^{\alpha_1}_{n_1,q_{n_1}}((t-x)^2;x)}\right)$$
$$\left(1+\frac{1}{\sqrt{\delta_{n_2}(y)}}\sqrt{(D^{\alpha_2}_{n_2,q_{n_2}}((s-y)^2;y)}\right),$$

we get the desired result.

EXAMPLE 4. Let $f(x,y)=x^2y^2+x^3y-3x^4$, $\alpha_i=0.9$ and $q_{n_i}=1-\dfrac{1}{n_i}$, $i=1,2$. The convergence of the operator $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}f$ to the function f is illustrated in Figure 7 for $n_1=n_2=20$ and $n_1=n_2=100$. We have seen that the operators $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}f$ converge towards to function f by increasing the values of n_i, $i=1,2$. Denote $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)=\left|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)\right|$ the error function of approximation by $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ operators. The error of approximation $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ is given in Figure 8. Also, in Table 4 we computed the error of approximation for $D^{\alpha_1,\alpha_2}_{20,20,q_{n_1},q_{n_2}}$ and $D^{\alpha_1,\alpha_2}_{100,100,q_{n_1},q_{n_2}}$ at certain points.

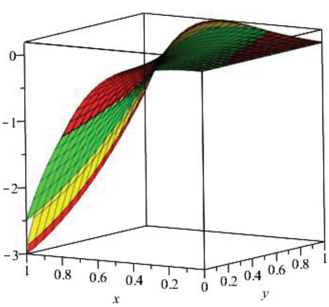

Figure 7: The convergence of $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ (green for $n_1=n_2=20$, yellow for $n_1=n_2=100$) to the function f (red)

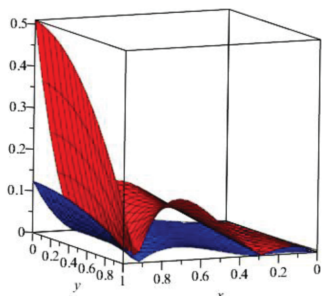

Figure 8: Error of approximation $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ (red for $n_1=n_2=20$, blue for $n_1=n_2=100$)

TABLE 4. *Error of approximation* $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ *for* $n_1 = n_2 = 20$ *and* $n_1 = n_2 = 100$

x	y	$\lvert E^{\alpha_1,\alpha_2}_{20,20,q_{n_1},q_{n_2}}(f;x,y)$	$E^{\alpha_1,\alpha_2}_{100,100,q_{n_1},q_{n_2}}(f;x,y)$
0.1	0.1	0.00518284294	0.00038101731
0.1	0.5	0.00223559373	0.00084107650
0.2	0.5	0.00636267698	0.00015775411
0.2	0.9	0.01016987500	0.00330427435
0.5	0.5	0.08099562148	0.01641607104
0.5	0.9	0.07953072238	0.01577152682
0.7	0.1	0.07599379948	0.01696266219
0.7	0.3	0.07232618048	0.01599518499
0.9	0.1	0.19629338340	0.04697646473
0.9	0.5	0.14082510840	0.03477276793

EXAMPLE 5. Let $f(x,y) = x^4y^2 + x^3y - 4xy^4$, $\alpha_i = 0.9$, $i = 1,2$ and $n = 50$. The Figure 9 illustrates the variation of q-Bernstein Durrmeyer operators when $(q_{n_1}, q_{n_2}) \in \{(0.7,0.7),(0.9,0.9),(1,1)\}$. Note that for $q_{n_1} = q_{n_2} = 1$ the Durrmeyer analogue of the generalized Bernstein operators becomes the classical ones. The error of approximation $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ is given in Figure 10. Also, in Table 5 we computed the error of approximation for $D^{\alpha_1,\alpha_2}_{n,n,0.7,0.7}$, $D^{\alpha_1,\alpha_2}_{n,n,09,0.9}$ and $D^{\alpha_1,\alpha_2}_{n,n,1,1}$ at certain points.

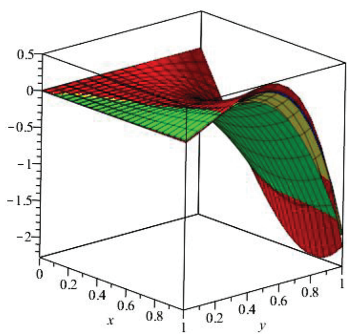

Figure 9: The convergence of $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ (green for $q_{n_1} = q_{n_2} = 0.7$, yellow for $q_{n_1} = q_{n_2} = 0.9$ and blue for $q_{n_1} = q_{n_2} = 1$) to the function f (red)

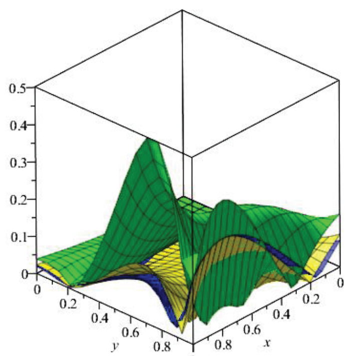

Figure 10: Error of approximation $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ (green for $q_{n_1} = q_{n_2} = 0.7$, yellow for $q_{n_1} = q_{n_2} = 0.9$ and blue for $q_{n_1} = q_{n_2} = 1$)

TABLE 5. *Error of approximation* $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ *for* $q_{n_1}=q_{n_2}=0.7$, $q_{n_1}=q_{n_2}=0.9$ *and* $q_{n_1}=q_{n_2}=1$

x	y	$E^{\alpha_1,\alpha_2}_{n,n,0.7,0.7}(f;x,y)$	$E^{\alpha_1,\alpha_2}_{n,n,0.9,0.9}(f;x,y)$	$E^{\alpha_1,\alpha_2}_{n,n,1,1}(f;x,y)$
0.1	0.1	0.00040220482	0.00001012353	0.000027808918
0.1	0.5	0.05098691644	0.01474544732	0.009249326326
0.2	0.5	0.07388756643	0.01949922728	0.011418243340
0.2	0.9	0.14713196780	0.02676797844	0.000626017335
0.5	0.5	0.10594640070	0.02768594730	0.016917484210
0.5	0.9	0.09393829230	0.04494964205	0.074709065710
0.7	0.1	0.02724483109	0.01021206781	0.006434295004
0.7	0.3	0.00375544959	0.00476985690	0.006907919030
0.9	0.1	0.03827959836	0.01642237071	0.010486283470
0.9	0.5	0.18768672610	0.07268702480	0.060592106800

4.1. Degree of approximation

Now, we study the degree of approximation for the bivariate operators by virtue of Lipschitz class.

For $0<\beta_1\leqslant 1$ and $0<\beta_2\leqslant 1$, we define the Lipschitz class $Lip_M(\beta_1,\beta_2)$ for the bivariate case as follows:

$$|f(t,s)-f(x,y)|\leqslant M|t-x|^{\beta_1}|s-y|^{\beta_2},$$

where $(t,s),(x,y)\in I^2$ are arbitrary.

THEOREM 7. *Let* $f\in Lip_M(\beta_1,\beta_2)$. *Then for all* $(x,y)\in I^2$, *we have*

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)|\leqslant M(\delta_{n_1}(x))^{\frac{\beta_1}{2}}(\delta_{n_2}(y))^{\frac{\beta_2}{2}}.$$

Proof. By our hypothesis, we may write

$$\begin{aligned}|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)| &\leqslant MD^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|t-x|^{\beta_1}|s-y|^{\beta_2};x,y)\\ &= MD^{\alpha_1}_{n_1,q_{n_1}}(|t-x|^{\beta_1};q_{n_1},x)D^{\alpha_2}_{n_2,q_{n_2}}(|s-y|^{\beta_2};y).\end{aligned}$$

Now, considering the Hölder's inequality with $p_1=\frac{2}{\beta_1}$, $q_1=\frac{2}{2-\beta_1}$ and $p_2=\frac{2}{\beta_2}$, $q_2=\frac{2}{2-\beta_2}$, respectively, we have

$$\begin{aligned}&|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)|\\ &\leqslant M(D^{\alpha_1}_{n_1,q_{n_1}}((t-x)^2;x))^{\frac{\beta_1}{2}}(D^{\alpha_1}_{n_1,q_{n_1}}(1;x))^{\frac{2-\beta_1}{2}}(D^{\alpha_2}_{n_2,q_{n_2}}((s-y)^2;y))^{\frac{\beta_2}{2}}(D^{\alpha_2}_{n_2,q_2}(1;q_{n_2},y))^{\frac{2-\beta_2}{2}}\\ &=M(\delta_{n_1}(x))^{\frac{\beta_1}{2}}(\delta_{n_2}(y))^{\frac{\beta_2}{2}}.\end{aligned}$$

In what follows, we shall use the following notation:

$$C^1(I^2) = \{f \in C(I^2) : f'_x, f'_y \in C(I^2)\}.$$

THEOREM 8. *Let $f \in C^1(I^2)$. Then, for each $(x,y) \in I^2$, we have*

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) - f(x,y)| \leqslant \| f'_x \|_{C(I^2)} \sqrt{\delta_{n_1}(x)} + \|f'_y\|_{C(I^2)} \sqrt{\delta_{n_2}(y)}.$$

Proof. Let $(x,y) \in I^2$ be a fixed point. Then we can write

$$f(t,s) - f(x,y) = \int_x^t f'_u(u,s) d_q u + \int_y^s f'_v(v)(x,v) d_q v.$$

Now, applying the operator $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ on both sides of the above equation, in view of the inequalities,
$|\int_t^x |f'_u(u,s)| d_q u| \leqslant \| f'_x \|_{C(I^2)} |t-x|$ and $|\int_y^s |f'_v(x,v)| d_q v| \leqslant \| f'_y \|_{C(I^2)} |s-y|$, we have

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) - f(x,y)| \leqslant \|f'_x\|_{C(I^2)} D^{\alpha_1}_{n_1,q_{n_1}}(|t-x|;x) + \|f'_y\|_{C(I^2)} D^{\alpha_2}_{n_2,q_{n_2}}(|s-y|;y).$$

Then considering the Cauchy-Schwarz inequality, we are lead to the required result.

For $f \in C(I^2)$ and $\delta > 0$, the partial moduli of continuity with respect to x and y is given by

$$\omega_1(f;\delta) = \sup\left\{ |f(x_1,y) - f(x_2,y)| : y \in I_2 \text{ and } |x_1 - x_2| \leqslant \delta \right\}$$

and

$$\omega_2(f;\delta) = \sup\left\{ |f(x,y_1) - f(x,y_2)| : x \in I_1 \text{ and } |y_1 - y_2| \leqslant \delta \right\}.$$

It is well known that they satisfy the properties of the usual modulus of continuity.

THEOREM 9. *Let $f \in C(I^2)$ and $(x,y) \in I^2$ be arbitrary. Then there holds the inequality*

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) - f(x,y)| \leqslant 2\{\omega_2(f;\sqrt{\delta_{n_2}(y)}) + \omega_1(f;\sqrt{\delta_{n_1}(x)})\}.$$

Proof. Using the definition of partial moduli of continuity, we have

$$\begin{aligned}&|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) - f(x,y)|\\ \leqslant & D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|f(t,s) - f(t,y)|;x,y) + D^{\alpha_1,\alpha_2}_{n_1,q_1,n_2,q_2}(|f(t,y) - f(x,y)|;x,y)\end{aligned}$$

$$\leqslant \omega_2(f;\sqrt{\delta_{n_2}(y)})\left(D^{\alpha_2}_{n_2,q_{n_2}}(1;y)+\frac{1}{\sqrt{\delta_{n_2}(y)}}(D^{\alpha_2}_{n_2,q_{n_2}}|s-y|;y)\right)$$
$$+\omega_1(f;\sqrt{\delta_{n_1}(x)})\left(D^{\alpha_1}_{n_1,q_{n_1}}(1;x)+\frac{1}{\sqrt{\delta_{n_1}(x)}}D^{\alpha_1}_{n_1,q_{n_1}}(|t-x|;x)\right).$$

Now, applying the Cauchy-Schwarz inequality

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)| \leqslant \omega_2(f;\sqrt{\delta_{n_2}(y)})\left(1+\frac{1}{\sqrt{\delta_{n_2}(y)}}\sqrt{D^{\alpha_2}_{n_2,q_{n_2}}((s-y)^2;y)}\right)$$
$$+\omega_1(f;\sqrt{\delta_{n_1}(y)})\left(1+\frac{1}{\sqrt{\delta_{n_1}(x)}}\sqrt{D^{\alpha_1,}_{n_1,q_{n_1}}((t-x)^2;x)}\right)$$
$$=2\{\omega_2(f;\sqrt{\delta_{n_2}(y)})+\omega_1(f;\sqrt{\delta_{n_1}(x)})\}.$$

This completes the proof of the theorem.

Let $f\in C^2(I^2)$ be the space of all functions $f\in C(I^2)$ such that $\frac{\partial^i f}{\partial x^i},\frac{\partial^i f}{\partial y^i}$, for i=1,2 belong to $C(I^2)$. Further, let the norm on the space $C^2(I^2)$ be defined as:

$$\|f\|_{C^2(I^2)}=\|f\|_{C(I^2)}+\sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|_{C(I^2)}+\left\|\frac{\partial^i f}{\partial y^i}\right\|_{C(I^2)}\right).$$

The Peetre's K-functional of the function $f\in C(I^2)$ is defined as:

$$\mathscr{K}(f;\delta)=\inf_{g\in C^2(I^2)}\{||f-g||_{C(I^2)}+\delta||g||_{C^2(I^2)}\},\delta>0,$$

where $\|\cdot\|_{C(I^2)}$ is the sup-norm. Also by [9], it follows that

$$\mathscr{K}(f;\delta)\leqslant M\left\{\tilde{\omega}_2(f;\sqrt{\delta})+\min(1,\delta)||f||_{C(I^2)}\right\} \tag{4.2}$$

holds for all $\delta>0$ and M is independent of δ and f, where $\tilde{\omega}_2(f;\sqrt{\delta})$ is the second order modulus of continuity in the bivariate case.

THEOREM 10. *For the function $f\in C(I^2)$, we have the following inequality*

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)|$$
$$\leqslant M\left\{\tilde{\omega}_2(f;\sqrt{A^{\alpha_1,\alpha_2}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)})+\min\{1,A^{\alpha_1,\alpha_2}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)\}||f||_{C(I^2)}\right\}$$
$$+\omega\left(f;\sqrt{\left(D^{\alpha_1}_{n_1,q_{n_1}}(t;x)-x\right)^2+\left(D^{\alpha_2}_{n_2,q_{n_2}}(s;y)-y\right)^2}\right),$$

where

$$A^{\alpha_1,\alpha_2}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)=\frac{1}{2}\left\{\delta^2_{n_1}(x)+\delta^2_{n_2}(y)+\left(D^{\alpha_1}_{n_1,q_{n_1}}(t;x)-x\right)^2+\left(D^{\alpha_2}_{n_2,q_{n_2}}(s;y)-y\right)^2\right\},$$

and the constant $M>0$, is independent of f and $A^{\alpha_1,\alpha_2}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)$.

Proof. We introduce the auxiliary operators as follows:

$$D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) = D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) - f\left(D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y), D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(s;x,y)\right) + f(x,y), \quad (4.3)$$

Using Lemma 4, we get

$$D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(1;x,y)=1,\ D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x);x,y)=0 \text{ and } D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((s-y);x,y)=0.$$

Let $g \in C^2(I^2)$ and $x,y \in I$ be arbitrary. Using the Taylor's theorem, we may write

$$\begin{aligned} g(t,s)-g(x,y) =& g(t,y)-g(x,y)+g(t,s)-g(t,y) \\ =& \frac{\partial g(x,y)}{\partial x}(t-x)+\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du \\ &+ \frac{\partial g(x,y)}{\partial y}(s-y)+\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv. \end{aligned}$$

Applying the operator $D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(.;x,y)$ on both sides of the above equation and using (4.3) we find

$$\begin{aligned} &D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(g;x,y)-g(x,y) \\ =& D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right)+D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\right) \\ =& D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right) \\ &- \int_x^{D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y)} (D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y)-u)\frac{\partial^2 g(u,y)}{\partial u^2}du \\ &+ D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\right) \\ &- \int_y^{D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(s;x,y)} (D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(s;x,y)-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv. \end{aligned}$$

Hence,

$$\begin{aligned} &|D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(g;x,y)-g(x,y)| \\ \leqslant& D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}\left(\left|\int_x^t |t-u|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|;x,y\right) \\ &+\left|\int_x^{D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y)} \left|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y)-u\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right| \\ &+D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}\left(\left|\int_y^s |s-v|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|;x,y\right) \end{aligned}$$

$$+\left|\int_y^{D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2},(s;x,y)}}\left|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(s;x,y)-v\right|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|$$
$$\leqslant \frac{1}{2}\left\{D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x)^2;x,y)+\left(D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y)-x\right)^2\right.$$
$$\left.+D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((s-y)^2;x,y)+\left(D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(s;x,y)-y\right)^2\right\}||g||_{C^2(I^2)}$$
$$=A^{\alpha_1,\alpha_2}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I^2)}. \tag{4.4}$$

Also, using Lemma 3

$$|D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)|$$
$$\leqslant|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)|+|f(D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y),D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}})(s;x,y))|+|f(x,y)|$$
$$\leqslant 3||f||_{C(I^2)}. \tag{4.5}$$

Hence, using (4.5) and (4.4), for any $g\in C^2(I^2)$

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2},}(f;x,y)-f(x,y)|$$
$$\leqslant |D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f-g;x,y)|+|D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2},}(g;x,y)-g(x,y)|+|g(x,y)-f(x,y)|$$
$$+\left|f(D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y),D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2},}(s;x,y))-f(x,y)\right|$$
$$\leqslant 4||f-g||_{C(I^2))}+|D^{*\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(g;x,y)-g(x,y)|$$
$$+\left|f(D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y),D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(s;x,y))-f(x,y)\right|$$
$$\leqslant \left(4||f-g||_{C(I^2)}+A^{\alpha_1,\alpha_2}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I^2)}\right)$$
$$+\omega\left(f;\sqrt{\left(D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(t;x,y)-x\right)^2+\left(D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2},}(s;x,y)-y\right)^2}\right).$$

Taking the infimum on the right side of above inequality over all $g\in C^2(I^2)$ and using the relation (4.2), we obtain the desired result.

5. Construction of GBS operators

Firstly, the idea to approximate B-continuous and B-differentiable functions introduced by Bögel in [7] and [8]. Subsequently, Dobrescu and Matei [12] introduced the bivariate Bernstein polynomial and established the uniform convergence of the associated GBS(Generalized Boolean Sum) operators for Bögel continuous functions. Badea et al. [5] proved the very famous "Test function theorem" for Bögel continuous functions. A quantitative variant of the Korovkin-type theorem for these functions was established by Badea and Badea in [6]. Siddharth et al. [23] established the rate of convergence of the B-continuous and B-differentiable functions by the GBS operators of Bernstein-Schurer-Kantorovich type. Agrawal and Ispir [4] investigated the degree of approximation for the GBS case of the bivariate Chlodowsky-Szász-Charlier type operators. For other relative papers we refer the readers to (c.f. [17], [21], [22] etc.). A real valued function f defined on I is called B-continuous at (x_0,y_0) if

$$\lim_{(t,s)\to(x_0,y_0)} \Delta_{(t,s)} f(x_0,y_0) = 0$$

where, $\Delta_{(t,s)} f(x,y) = f(t,s) - f(t,y) - f(x,s) + f(x,y)$.
Let $C_b(I) := \{f : f \text{ is } B-\text{continuous on } I\}$ and $B_b(I)$ be the set of all bounded functions on I, equipped with the norm $||f||_B = \sup_{(t,s)(x,y)\in I} |\Delta_{(t,s)}(x,y)|$. A function f is said to be B-differentiable at (x_0,y_0) if

$$\lim_{(t,s)\to(x_0,y_0)} \frac{\Delta_{(t,s)}(x_0,y_0)}{(t-x_0)(s-y_0)} = D_B f(x_0,y_0) < \infty.$$

Here, $D_B f$ is called B-derivative of f and the space of all B-differentiable functions is denoted by $D_b(I)$. As usual $B(I^2)$, denotes the space of bounded functions on I^2 endowed with the sup-norm $\| \,.\, \|_\infty$.
We define the GBS operator of the operator $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}$ given by (4.1), for any $f \in C_b(I^2)$ and $0 \leqslant \alpha_1, \alpha_2 \leqslant 1$ by

$$T^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(f(t,s);x,y) := D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}\left(f(t,y) + f(x,s) - f(t,s);x,y\right), \tag{5.1}$$

for all $(x,y) \in I^2$.
More precisely for any $f \in C_b(I^2)$, the GBS operator associated with $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}$ is given by

$$\begin{aligned} T^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(f;x,y) = {} & [n_1+1]_{q_1}[n_2+1]_{q_2} \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p^{q_1,q_2,\alpha_1,\alpha_2}_{n_1,n_2,k_1,k_2}(x,y) \\ & \int_0^1\int_0^1 p^{q_1,\alpha_1}_{n_1,k_1}(q_1 t)\, p^{q_2,\alpha_2}_{n_2,k_2}(q_2 s)[f(x,s)+f(t,y)-f(t,s)]\, d_{q_1}t\, d_{q_2}s. \end{aligned} \tag{5.2}$$

Hence the operator $T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ is a linear operator and is well defined from the space $C_b(I^2)$ into $C(I^2)$.

EXAMPLE 6. Let $f(x,y) = y^2 - 2\sqrt{2}(1-x-y)^2 - 8xy$, $n = 20$, $\alpha = 0.9$, $q = 1 - \frac{1}{n}$. In Figure 11 we compare the Durrmeyer analogue of the generalized Bernstein operators and its GBS type operator. We note that $T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ gives a better approximation than the operator $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$. Denote $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(f;x,y) = \left|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}(f;x,y) - f(x,y)\right|$ and $\tilde{E}^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) = \left|T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) - f(x,y)\right|$ the error functions of approximation by $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ and $T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$. In Figure 12 we compare the error of approximation for the Durrmeyer analogue of the generalized Bernstein operators and its GBS type operator. Also, in Table 6 we computed the error of approximation for $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ and $T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ at certain points.

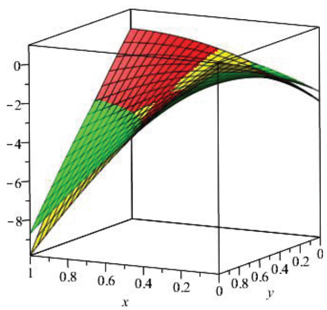

Figure 11: The convergence of $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ (green) and $T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ (yellow) to f (red)

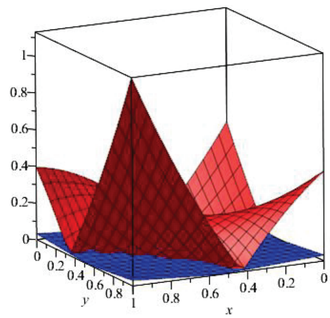

Figure 12: The errors of approximation $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ (red) and $\tilde{E}^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ (blue)

TABLE 6. *Error of approximation* $E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ *and* $\tilde{E}^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$

x	y	$E^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)$	$\tilde{E}^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)$
0.1	0.1	0.2377910218	0.023056112790
0.1	0.2	0.1300423062	0.017956657530
0.1	0.4	0.0586651127	0.007461486106
0.1	0.6	0.2106552538	0.003428698187
0.1	0.8	0.3247314057	0.014713895100
0.1	0.9	0.3671777596	0.020504618750
0.2	0.6	0.2074495036	0.002670351459
0.2	0.8	0.2717339605	0.011459534120
0.4	0.2	0.1080500269	0.005811185210
0.4	0.4	0.1513192015	0.002414708370
0.4	0.8	0.1186581676	0.004761750745
0.5	0.2	0.1581683216	0.001608873311
0.5	0.5	0.1340443680	0.000185085668
0.5	0.9	0.0416242799	0.001837162438
0.8	0.2	0.2175461275	0.011459534120
0.8	0.9	0.5209414993	0.013085584620
0.9	0.2	0.2062416947	0.015969491720
0.9	0.8	0.5494686834	0.013085587450

The mixed modulus of smoothness of $f \in C_b(I^2)$ is defined as

$$\omega_{mixed}(f;\delta_1,\delta_2) := \sup\{|\Delta f[(t,s);(x,y)]| : |x-t| < \delta_1, |y-s| < \delta_2\},$$

for all $(x,y),(t,s) \in (I^2)$ and for any $(\delta_1,\delta_2) \in (0,\infty)\times(0,\infty)$.

THEOREM 11. *For every* $f \in C_b(I^2)$, *at each point* $(x,y) \in I^2$, *the operator* (5.2) *verifies the following inequality*

$$|T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) - f(x,y)| \leqslant 4\,\omega_{mixed}(f;\delta_{n_1}(x),\delta_{n_2}(y)).$$

Proof. By the definition of $\omega_{mixed}(f,\delta_{n_1},\delta_{n_2})$ and using the elementary inequality

$$\omega_{mixed}(f;\lambda_1\delta_{n_1},\lambda_{n_2}\delta_{n_2}) \leqslant (1+\lambda_1)(1+\lambda_2)\,\omega_{mixed}(f,\delta_{n_1},\delta_{n_2});\ \lambda_1,\lambda_2>0,$$

we may write,

$$\begin{aligned} |\Delta_{(t,s)}f(x,y)| &\leqslant \omega_{mixed}(f;|t-x|,|s-y|) \\ &\leqslant \left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)\omega_{mixed}(f;\delta_1,\delta_2), \end{aligned} \tag{5.3}$$

for every $(t,s) \in I^2$, $(x,y) \in I^2$ and for any $\delta_1,\delta_2 > 0$. From the definition of $\Delta_{(t,s)}f(x,y)$, we get

$$f(x,s)+f(t,y)-f(t,s) = f(x,y) - \Delta_{(t,s)}f(x,y).$$

On applying the linear positive operator $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ to this equality and by the definition of operator $T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$, we can write

$$T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) = f(x,y)\,D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(e_{00};x,y) - D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(\Delta_{(t,s)}f(x,y);x,y).$$

Since $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(e_{00};x,y)=1$, considering the inequality (5.3), we obtain,

$$\begin{aligned} &|T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) - f(x,y)| \\ &\leqslant D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|\Delta_{(t,s)}f(x,y)|;x,y) \\ &\leqslant \Bigg(D^{\alpha_1}_{n_1,q_{n_1}}(e_{00};x) + \frac{1}{\sqrt{\delta_{n_1}}}D^{\alpha_1}_{n_1,q_{n_1}}(|t-x|;x) + \frac{1}{\sqrt{\delta_{n_2}}}D^{\alpha_2}_{n_2,q_{n_2}}(|s-y|;y) \\ &\quad + \frac{1}{\sqrt{\delta_{n_1}}\sqrt{\delta_{n_2}}}D^{\alpha_1}_{n_1,q_{n_1}}(|t-x|;x)\,D^{\alpha_2}_{n_2,q_{n_2}}(|s-y|;y)\Bigg)\omega_{mixed}(f;\sqrt{\delta_{n_1}},\sqrt{\delta_{n_2}}). \end{aligned}$$

Now, applying the Cauchy-Schwarz inequality and choosing $\delta_{n_1} = \delta_{n_1}(x)$ and $\delta_{n_2} = \delta_{n_2}(y)$, we reach the required result.

Now, let us define the Lipschitz class for B-continuous functions. For $f \in C_b(I^2)$, the Lipschitz class $Lip_M(\xi,\eta)$ with $\xi,\eta \in (0,1]$ is defined by

$$Lip_M(\xi,\eta) = \left\{ f \in C_b\left(I^2\right) : \left|\Delta_{(t,s)}f(x,y)\right| \leqslant M\,|t-x|^{\xi}\,|s-y|^{\eta},\ \text{for}\,(t,s),(x,y)\in I^2 \right\}.$$

Our next theorem gives the degree of approximation for the operators $T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ by means of the Lipschitz class of Bögel continuous functions.

THEOREM 12. *For $f \in Lip_M(\xi,\eta)$, we have*

$$|T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)| \leqslant M(\delta_{n_1}(x))^{\frac{\xi}{2}}(\delta_{n_2}(y))^{\frac{\eta}{2}},$$

for $M>0,\ \xi,\eta\in(0,1]$.

Proof. By the definition of the operator $T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)$ and by linearity of the operator $D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}$ given by (4.1), we can write

$$\begin{aligned} T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y) &= D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f(x,s)+f(t,y)-f(t,s);x,y)\\ &= D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f(x,y)-\Delta_{(t,s)}f(x,y);x,y)\\ &= f(x,y)D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(e_{00};x,y)-D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(\Delta_{(t,s)}f(x,y);x,y). \end{aligned}$$

By the hypothesis, we get

$$\begin{aligned} \left|T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)\right| &\leqslant D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}\left(\left|\Delta_{(t,s)}f(x,y)\right|;x,y\right)\\ &\leqslant MD^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}\left(|t-x|^{\xi}|s-y|^{\eta};x,y\right)\\ &= MD^{\alpha_1}_{n_1,q_{n_1}}\left(|t-x|^{\xi};x\right)D^{\alpha_2}_{n_2,q_{n_2}}\left(|s-y|^{\eta};y\right). \end{aligned}$$

Now, using the Hölder's inequality with $p_1=2/\xi, q_1=2/(2-\xi)$ and $p_2=2/\eta, q_2=2/(2-\eta)$,
we get the desired result.

THEOREM 13. *Let the function $f\in D_b(I^2)$ with $D_Bf\in B(I^2)$. Then, for each $(x,y)\in I^2$, we have*

$$\begin{aligned} &|T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)|\\ &\leqslant \frac{M}{[n_1]^{1/2}_{q_{n_1}}[n_2]^{1/2}_{q_{n_2}}}\left(||D_Bf||_\infty+\omega_{mixed}(D_Bf;[n_1]^{-1/2}_{q_{n_1}},[n_2]^{-1/2}_{q_{n_2}})\right), \end{aligned}$$

for some constant $M>0$.

Proof. Since $f\in D_b(I^2)$, we have the identity

$$\Delta_{(t,s)}f(x,y)=(t-x)(s-y)D_Bf(\xi,\eta),\ with\ x<\xi<t;\ y<\eta<s.$$

Evidently,

$$D_Bf(\xi,\eta)=\Delta D_Bf(\xi,\eta)+D_Bf(\xi,y)+D_Bf(x,\eta)-D_Bf(x,y).$$

Since $D_Bf\in B(I^2)$, by above relations, we can write

$$|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(\Delta_{(t,s)}f(x,y);x,y)|$$

$$
\begin{aligned}
&=|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x)(s-y)D_Bf(\xi,\eta);x,y)|\\
&\leqslant D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|t-x||s-y||\Delta D_Bf(\xi,\eta)|;x,y)+D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_2},q_{n_2}}(|t-x||s-y|(|D_Bf(\xi,y)|\\
&\quad+|D_Bf(x,\eta)|+|D_Bf(x,y)|);x,y)\\
&\leqslant D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|t-x||s-y|\omega_{mixed}(D_Bf;|\xi-x|,|\eta-y|);x,y)\\
&\quad+3||D_Bf||_\infty D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|t-x||s-y|;x,y).
\end{aligned}
$$

Considering the inequality (5.3) and applying the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&|T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)| \qquad (5.4)\\
&=|D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}\Delta_{(t,s)}(x,y);x,y|\\
&\leqslant 3||D_Bf||_\infty D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|t-x||s-y|;x,y)+\Big(D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|t-x||s-y|;x,y)\\
&\quad+\delta_{n_1}^{-1}D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x)^2|s-y|;x,y)+\delta_{n_2}^{-1}D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(|t-x|(s-y)^2;x,y)\\
&\quad+\delta_{n_1}^{-1}\delta_{n_2}^{-1}D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x)^2(s-y)^2;x,y)\Big)\omega_{mixed}(D_Bf;\delta_{n_1},\delta_{n_2})\\
&\leqslant 3||D_Bf||_\infty\sqrt{D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x)^2(s-y)^2;x,y)}+\Big(\sqrt{D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x)^2(s-y)^2;x,y)}\\
&\quad+\delta_{n_1}^{-1}\sqrt{D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x)^4(s-y)^2;x,y)}+\delta_{n_2}^{-1}\sqrt{D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x)^2(s-y)^4;x,y)}\\
&\quad+\delta_{n_1}^{-1}\delta_{n_2}^{-1}D^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}((t-x)^2(s-y)^2;x,y)\Big)\omega_{mixed}(D_Bf;\delta_{n_1},\delta_{n_2}). \qquad (5.5)
\end{aligned}
$$

In view of Lemma 2, for $(t,s)\in I^2,\ (x,y)\in I^2$ and $i,j=1,2$

$$
\begin{aligned}
D^{\alpha_1}_{n_1,q_{n_1}}((t-x)^{2i}(s-y)^{2j};x,y) &= D^{\alpha_1}_{n_1,q_{n_1}}((t-x)^{2i};x,y)D^{\alpha_2}_{n_2,q_{n_2}}((s-y)^{2j};x,y).\\
&\leqslant \frac{M_1}{[n_1]^i_{q_{n_1}}}\frac{M_2}{[n_2]^j_{q_{n_2}}}, \qquad (5.6)
\end{aligned}
$$

for some constants $M_1,M_2>0$. Let $\delta_{n_1}=\frac{1}{[n_1]^{1/2}_{q_{n_1}}}$, and $\delta_{n_2}=\frac{1}{[n_2]^{1/2}_{q_{n_2}}}$.

Thus, combining (5.4)-(5.6)

$$
\begin{aligned}
&|T^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(f;x,y)-f(x,y)|\\
&=3||D_Bf||_\infty O\left(\frac{1}{[n_1]^{1/2}_{q_{n_1}}}\right)O\left(\frac{1}{[n_2]^{1/2}_{q_{n_2}}}\right)\\
&\quad+O\left(\frac{1}{[n_1]^{1/2}_{q_{n_1}}}\right)O\left(\frac{1}{[n_2]^{1/2}_{q_{n_2}}}\right)\omega_{mixed}(D_Bf;[n_1]^{-1/2}_{q_{n_1}},[n_2]^{-1/2}_{q_{n_2}}).
\end{aligned}
$$

This completes the proof.

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