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A NONLINEAR VERSION OF HALANAY INEQUALITY AND APPLICATION TO NEURAL NETWORKS THEORY

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Abstract. We establish an exponential stability result for a delayed Hopfield neural network. This is proved in case one or more of the activation functions fails to satisfy the standard Lipschitz continuity condition. We use a nonlinear version of Halanay inequality, which we prove here.

1. Introduction

The most important and fundamental reservoir properties are porosity and permeability. They give a clear insight on the amount of fluid contained in a reservoir and its ability to migrate. These properties are determined by well-logging and wireline measurement which are considered expensive processes and may lead to erroneous interpretations. In the last three decades, it has been found that Artificial Neural Networks (ANNs) are capable of overcoming these difficulties. They become an extremely attractive and powerful tool for prediction, pattern recognition, forecasting, diagnosis, identification, drill bit selection, classification and optimization of field operations.

ANNs have been designed to forecast reservoir properties using data from geophysical well logs with a high accuracy [3,20,22]. They have proved their efficiency even when limited data is available [1,27]. ANNs when combined with fuzzy logic form an excellent tool for the reservoir characterization [15,17]. They are also used in history matching [21,23].

In this paper we consider a neural network of Hopfield type

$$x_i'(t) = -d_i x_i(t) + \sum_{j=1}^{m} a_{ij} f_{ij}(x_j(t)) + \sum_{j=1}^{m} b_{ij} g_{ij}(x_j(t-\tau_j)) + c_i,$$

$i = 1,\ldots,m$, for $t > 0$, with $x_j(t) = x_{0j}(t)$, $t \in [-\tau_j, 0]$ where τ_j and $x_{0j}(t)$ are given continuous real-valued functions. Here a_{ij}, b_{ij}, c_i and d_i are nonnegative constants, $i, j = 1,\ldots,m$, and f_{ij}, g_{ij} are the activation functions of the signal transmission which will be specified below. Our arguments work even for more general models, for instance, for variable delays (time-dependent delays) $\tau_j(t)$. To avoid distracting the attention of the reader from the main contribution in this paper, we shall treat a simpler model.

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The most important question addressed for this type of problems is the stability (after, of course, the well-posedness). In the beginning, explicit nice activation functions have been considered like the sigmoid function, hyperbolic tangent, Gaussian radial basis function and linear functions. Then, these functions have been enlarged to bounded, monotone and differentiable functions. In turn, the latter conditions have been dropped later one by one. Nowadays, most of the papers in the market assume the Lipschitz condition [2,5,14,19,24,25]. The case of discontinuous activation functions has been also investigated in [2,4,13,18,36] (see also references therein). In many papers, it is rather the conditions on the different parameters which are improved. The necessity of considering non-Lipschitz continuous activation functions has been highlighted, for instance, in the book of Kosko [16] and in [10]. Some attempts have been made to weaken the Lipschitz condition (without appealing to the boundedness and monotonicity) in [26,29-35,40,41,37-39]. In particular, Hölder continuous activation functions were studied in [9,28,29,32,33]. Bihari-type inequalities have been used there and therefore, unless we make some restrictive conditions, the stability is only of 'local' character. It is also worthy mentioning the QUAD condition (see, for instance, [8]): a function is said to be QUAD (Δ,ω) (Δ is an $n\times n$ diagonal matrix and ω is a positive scalar) if for any $x,y\in R^n$, we have

$$(x-y)^T\left[f(x,t)-f(y,t)\right]-(x-y)^T\Delta(x-y)\leqslant-\omega(x-y)^T(x-y).$$

In this work, we consider the case where one of the components is not Lipschitz continuous. Namely, we treat the situation where the functions f_{i_0} and g_{i_0} corresponding to component number i_0, are not necessarily Lipschitz continuous. They satisfy a relation of the form

$$|h(x)-h(y)|\leqslant\varphi(|x-y|)=|x-y|\tilde{h}(|x-y|),\ t>0 \tag{1}$$

for some (non-decreasing) function $\tilde{h}$. We prove an exponential stability result.

Writing the QUAD condition in the form

$$(x-y)^T\left[f(x,t)-f(y,t)\right]\leqslant(x-y)^T\left(\Delta-\omega I_n\right)(x-y),$$

it appears that this class does not cover functions satisfying (1) as our coefficients are nonlinear functions of the (difference between the) variables.

Existence and uniqueness:

We recall that, for the problem

$$x'(t)=f(t,x(t)),\ x(t_0)=x_0, \tag{2}$$

Peano's theorem assures the local existence of at least one solution under the only condition of continuity of f (in a neighborhood of (t_0,x_0)). There are several generalizations of Lipschitz condition ensuring uniqueness of solutions: Nagumo, Perron, Osgood, Kamke, Tonelli, etc (which we may find in classical ordinary differential equations books). For instance, uniqueness of solutions has been proved (by Nagumo and Osgood) under the condition

$$|f(t,x)-f(t,y)|\leqslant\phi(|x-y|),\ t>0,$$

(or even only a one-sided relation) where $\phi(u)$ is a function of 'continuity-modulus' type satisfying

$$\int_0^{\delta} \frac{du}{\phi(u)} = \infty, \ \delta > 0. \tag{3}$$

DEFINITION 1. A function $\phi : [0,+\infty) \to [0,+\infty)$ such that $\phi(0) = 0,\ \phi(u) > 0$ for $u > 0,$ is said to satisfy Osgood criterion if (3) holds for some $\delta > 0.$

Some researchers require that ϕ be nondecreasing. This result is generalized to the case

$$|f(t,x)| \leqslant \frac{\omega'(t)}{\omega(t)} \phi(|x|),$$

where $\omega'(t) > 0,$ a.e. on $[0,a]$ and $\frac{f(t,x)}{\omega'(t)} \to 0$ as $t \to \infty$ uniformly in $|x| < M$ (See [7]).

We also have other generalizations (due to Kamke and Perron, see [11]), to the case

$$|f(t,x) - f(t,y)| \leqslant \phi(|t-t_0|, |x-y|).$$

The function ϕ is such that the only solution of

$$z'(t) = \phi(t, z(t)),$$

on any interval $(0,a)$ satisfying $z(t) = o(t)$ as $t \to 0$ is the zero solution.

Most of the existing results hold also for the case of a system of equations.

The question now is: What if the nonlinearity f fails to be Lipschitz or satisfy the established conditions with respect to one of the components?

In [6], the author Cid proves that we have uniqueness if the function f fails to be Lipschitz continuous with respect to one of the components provided that it is Lipschitz with respect to all the other remaining components including the first one "t". More precisely, he introduced the following definition.

DEFINITION 2. The function $f : U \subset R^{n+1} \to R^n$, where U is an open set, is said to be Lipschitz continuous when fixing component $i_0 \in \{0,1,...,n\}$ if there exists a constant $L > 0$ such that

$$\begin{aligned} &\left\| f(v_0,...,w_{i_0},...,v_n) - f(\bar{v}_0,...,w_{i_0},..., \bar{v}_n) \right\|_\infty \\ &\leqslant L \left\| (v_0,...,v_{i_0-1}, v_{i_0+1},...,v_n) - (\bar{v}_0,..., \bar{v}_{i_0-1}, \bar{v}_{i_0+1},..., \bar{v}_n) \right\|_\infty \end{aligned}$$

for all $(v_0,...,w_{i_0},...,v_n), (\bar{v}_0,...,w_{i_0},..., \bar{v}_n) \in U.$

The theorem below was shown

THEOREM 1. *Assume that f is continuous and locally Lipschitz continuous when fixing a component $i_0 \in \{0,1,...,n\}$. Then, for $(t_0,x_0) \in U$, there exists $a > 0$ such that (2) has a unique solution in $[t_0-a, t_0+a]$ provided that either $i_0 = 0$ or $f_{i_0}(t_0,x_0) \neq 0.$*

Notice that this theorem ensures the uniqueness only when we do not start from the equilibrium. For the uniqueness of the equilibrium, we appeal to the Osgood condition.

REMARK 1. In case $i_0 = 0$, we recover the well-known standard Lipschitz condition with respect to x (that is with respect to all the components except the first one which is t).

REMARK 2. For the one-dimensional case (one equation), the condition $f(t_0, x_0) \neq 0$ has been relaxed in [12].

REMARK 3. In [6], the author claims that his result can be generalized to other criteria such as the ones by Kamke, Osgood, Perron, Nagumo, etc, including, of course, the autonomous case (therefore, this will be also the case for our present result).

We shall provide a proof based on a nonlinear version of Halanay inequality.

2. A nonlinear version of Halanay inequality

In this section we prove the following lemma which generalizes the standard Halanay inequality from the linear case to the nonlinear case. Indeed, the first Halanay inequality was proved for the linear case $\xi(z) \equiv z$.

LEMMA 1. *Let $z(t)$ be a nonnegative differentiable function satisfying*

$$\begin{cases} z'(t) \leqslant -az(t) + \sup_{t-\tau \leqslant s \leqslant t} [\xi(z(s))], \; t \geqslant 0, \\ z(t) = \varphi(t), \; -\tau \leqslant t \leqslant 0, \end{cases}$$

where $\xi(z) = z\eta(z)$ for some continuous function $\eta(z)$ for $z \geqslant 0$ with $\eta(0) < a$. Then, there exists $\bar{y} > 0$ and $\alpha > 0$ such that

$$z(t) \leqslant \bar{y} e^{-\alpha t}, \; t \geqslant 0$$

when

$$\sup_{-\tau \leqslant s \leqslant 0} |\varphi(s) e^{\alpha s}| \leqslant \bar{y}.$$

Notice that we can always assume w.l.o.g. that $\eta(z)$ is non-decreasing. We refer to the next section for a remark on the singular case nearby zero.

Proof. Let

$$\bar{y} := \sup\{y \geqslant 0 : \eta(y) < a\}.$$

As $\eta(0) < a$, it is clear that $\bar{y} > 0$. Consider, $\bar{y}_\delta := \delta \bar{y}$, $0 < \delta < 1$ and define

$$\psi(\alpha) := -a + \alpha + e^{\alpha\tau} \eta(\bar{y}_\delta).$$

Notice that $\psi(0) = -a + \eta(\bar{y}_\delta) < 0$ because $\bar{y}_\delta < \bar{y}$.

On the other hand, $\lim_{\alpha \to \infty} \psi(\alpha) = \infty$. Therefore, for any number b such that $0 < b < a - \eta(\bar{y}_\delta)$, there exists an $\alpha_b > 0$ such that

$$\psi(\alpha_b) := -a + \alpha_b + e^{\alpha_b \tau} \eta(\bar{y}_\delta) = -b < 0.$$

We claim that, if $\sup_{-\tau \leqslant s \leqslant 0} |\varphi(s) e^{\alpha_b s}| \leqslant \bar{y}_\delta$, then $z_{\alpha_b}(t) < \bar{y}_\delta$ where

$$z_{\alpha_b}(t) := \begin{cases} z(t) e^{\alpha_b t}, \; t > 0, \\ \varphi(t) e^{\alpha_b t}, \; -\tau \leqslant t \leqslant 0. \end{cases}$$

We argue by contradiction. Assume that $t_* > 0$ is the first time

$$z_{\alpha_b}(t) < \bar{y}_\delta, \; -\tau \leqslant t < t_*,$$

$z_{\alpha_b}(t_*) = \bar{y}_\delta$ and $z'_{\alpha_b}(t_*) \geqslant 0$. As η may be assumed non-decreasing, we obtain

$$\begin{aligned} 0 \leqslant z'_{\alpha_b}(t_*) &= z'(t_*) e^{\alpha_b t_*} + \alpha_b z(t_*) e^{\alpha_b t_*} \\ &\leqslant e^{\alpha_b t_*} \left\{ -a z(t_*) + \sup_{t_* - \tau \leqslant s \leqslant t_*} [z(s) \eta(z(s))] \right\} + \alpha_b z(t_*) e^{\alpha_b t_*} \\ &\leqslant (-a + \alpha_b) z(t_*) e^{\alpha_b t_*} + e^{\alpha_b t_*} \sup_{t_* - \tau \leqslant s \leqslant t_*} [z(s) \eta(z(s))] \\ &\leqslant (-a + \alpha_b) z_{\alpha_b}(t_*) + e^{\alpha_b \tau} \sup_{t_* - \tau \leqslant s \leqslant t_*} \left[z_{\alpha_b}(s) \eta(z_{\alpha_b}(s)) \right] \end{aligned}$$

and therefore

$$0 \leqslant z'_{\alpha_b}(t_*) < (-a + \alpha_b) \bar{y}_\delta + e^{\alpha_b \tau} \sup_{t_* - \tau \leqslant s \leqslant t_*} [\bar{y}_\delta \eta(\bar{y}_\delta)]$$

or

$$0 \leqslant z'_{\alpha_b}(t_*) < [(-a + \alpha_b) + e^{\alpha_b \tau} \eta(\bar{y}_\delta)] \bar{y}_\delta < 0.$$

This is a contradiction and therefore $z_{\alpha_b}(t) \leqslant \bar{y}_\delta, \; \forall t > 0$. Hence, in view of the definition of $z_{\alpha_b}(t,)$, we obtain

$$z(t) \leqslant \bar{y}_\delta e^{-\alpha_b t}, \; t > 0.$$

This finishes the proof. □

We shall combine this nonlinear version of Halanay inequality with a Lyapunov type functional to get local exponential stability of the system (see the result in next section).

3. Exponential stability

In this section we prove that the system

$$x'_i(t) = -d_i x_i(t) + \sum_{j=1}^{m} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} b_{ij} g_j(x_j(t - \tau)) + c_i, \tag{4}$$

$i = 1, \ldots, m, \; t > 0$ with $x_j(t) = x_{0j}(t), \; t \in [-\tau, 0], \; x_{0j}(t)$ are given continuous real-valued functions, is locally asymptotically stable in an exponential manner. We assume that f_i and g_i are (locally) Lipschitz continuous *when fixing component* $i_0 \in \{1, \ldots, n\}$

in an open set $U \subset R^n$. By the Theorem 1 we have local existence and uniqueness. After *a priori* estimates, the solution may be continued to a global one.

For f_{i_0} and g_{i_0} we assume that

$$\left|f_{i_0}\left(x_{i_0}\right)-f_{i_0}\left(\bar{x}_{i_0}\right)\right| \leqslant \Phi_{f_{i_0}}\left(\left|x_{i_0}-\bar{x}_{i_0}\right|\right)=\left|x_{i_0}-\bar{x}_{i_0}\right| \tilde{f}_{i_0}\left(\left|x_{i_0}-\bar{x}_{i_0}\right|\right),\ t \geqslant 0 \tag{5}$$

and

$$\left|g_{i_0}\left(x_{i_0}\right)-g_{i_0}\left(\bar{x}_{i_0}\right)\right| \leqslant \Phi_{g_{i_0}}\left(\left|x_{i_0}-\bar{x}_{i_0}\right|\right)=\left|x_{i_0}-\bar{x}_{i_0}\right| \tilde{g}_{i_0}\left(\left|x_{i_0}-\bar{x}_{i_0}\right|\right),\ t \geqslant 0. \tag{6}$$

Notice that w.l.o.g. $\bar{f}_{i_0}$ and $\tilde{g}_{i_0}$ may be assumed non-decreasing functions. Some clarification is in order here. This claim is not quite correct if $\bar{f}_{i_0}$ or $\tilde{g}_{i_0}$ presents some singularity at 0. This is, for instance, the case if we consider Log-Lipschitz functions (like $x|\ln x|$), which are not Lipschitz but satisfy the Osgood condition (without the nondecreasingness assumptions). Indeed, the $\ln x$ is unbounded nearby zero and therefore we cannot pass to the $\sup$ to have non-decreasing functions. Fortunately, here in our argument below, we need rather the functions $x\tilde{f}_{i_0}(x)$ and $x\tilde{g}_{i_0}(x),\ x \geqslant 0$ to be non-decreasing and this is clear in our case. Moreover, here we need $\tilde{f}_{i_0}(x)$ and $\tilde{g}_{i_0}(x)$ to be defined at zero (see the statement of the theorem below or the end of its proof).

We denote by L_i and K_i the Lipschitz constants of f_i and g_i respectively (with respect to the other components, $i \neq i_0$ and including t).

DEFINITION 3. We say that the system (4) is globally asymptotically stable if, for any two solutions $x_j(t)$ *and* $\bar{x}_j(t)$ *(with* x_{0j} *and* $\bar{x}_{0j}$ *as initial data)*, there is $\delta > 0$ such that

$$\lim_{t \to \infty} \left\|x_j(t)-\bar{x}_j(t)\right\| = 0$$

provided that $\left\|x_{0j}-\bar{x}_{0j}\right\| < \delta$ for a certain norm (of the initial data, see relation (8) for our case). It is said to be exponentially asymptotically stable if there exist two positive constants M and ν such that

$$\left\|x_j(t)-\bar{x}_j(t)\right\| \leqslant Me^{-\nu t},\ t > 0.$$

If $\bar{x}_i(t)=x_i^*,\ i=1,\dots,m$ where $x_i^*,\ i=1,\dots,m$ is the equilibrium, that is solution of

$$0=-d_i x_i^*+\sum\nolimits_{j=1}^m a_{ij} f_j\left(x_j^*\right)+\sum\nolimits_{j=1}^m b_{ij} g_j\left(x_j^*\right)+c_i,\ i=1,\dots,m \tag{7}$$

then we get the usual (local) exponential stability of this equilibrium.

We denote by

$$\left\|x_j(t)-\bar{x}_j(t)\right\| := \sum\nolimits_{i=1}^m \left|x_i(t)-\bar{x}_i(t)\right|, \tag{8}$$

$y_i(t)=x_i(t)-\bar{x}_i(t)$, $y(t)=\sum_{i=1}^m |y_i(t)|$ and $d := \min_{1\leqslant i\leqslant m} d_i$.

THEOREM 2. *Assume that* f_i *and* g_i *are (locally) Lipschitz continuous when fixing component* $i_0 \in \{1,\dots,m\}$ *and* f_{i_0} *and* g_{i_0} *satisfy (5) and (6), resp., if*

$$a := d^* = \min\{d,\beta\}-\sum\nolimits_{i=1}^m \sum\nolimits_{j\neq i_0} L_j\left|a_{ij}\right|+e^{\beta\tau}\sum\nolimits_{i=1}^m \sum\nolimits_{j\neq i_0} K_j\left|b_{ij}\right| > 0$$

and

$$2\left[\left(\sum_{i=1}^{m}\left|a_{ii_0}\right|\right)\ \tilde{f}_{i_0}(0)+\left(\sum_{i=1}^{m}\left|b_{ii_0}\right|\right)\tilde{g}_{i_0}(0)\right]<a,$$

then, solutions of (4), not starting (both) from the equilibrium, are exponentially locally stable, for some $\beta>0$.

The theorem states that, the difference of any two solutions converges to zero exponentially provided that their initial data are close enough and both not starting from the equilibrium.

REMARK 4. In this theorem, the "Lipschitz continuity when fixing a component" condition may be generalized to at least the condition of type (1). This will be clear from our argument.

Proof of Theorem 2. Let $x_i(t)$ and $\bar{x}_i(t)$, $i=1,...,m$, be two solutions of (4) with x_{0j} and $\bar{x}_{0j}$ as initial data, resp. Using our system (4), we see that

$$D^+|y_i(t)|\leqslant -d_i|y_i(t)|+\sum_{j\neq i_0}L_j\left|a_{ij}\right|\left|y_j(t)\right|+\sum_{j\neq i_0}K_j\left|b_{ij}\right|\left|y_j(t-\tau)\right| \\ +\left|a_{ii_0}\right|\left|y_{i_0}(t)\right|\ \tilde{f}_{i_0}\left(\left|y_{i_0}(t)\right|\right)+\left|b_{ii_0}\right|\left|y_{i_0}(t-\tau)\right|\ \tilde{g}_{i_0}\left(\left|y_{i_0}(t-\tau)\right|\right),\ t\geqslant 0, \tag{9}$$

$i=1,...,m,$ where D^+ denotes the right Dini derivative and we designate by $\sum_{j\neq i_0}$ the summation $\sum_{j=1,j\neq i_0}^{m}$. If

$$V(t):=e^{-\beta t}\int_{t-\tau}^{t}e^{\beta(s+\tau)}\sum_{i=1}^{m}\left[\sum_{j\neq i_0}K_j\left|b_{ij}\right|\left|y_j(s)\right|\right]ds,\ t\geqslant 0, \tag{10}$$

$$W(t):=y(t)+V(t),\ t\geqslant 0, \tag{11}$$

for some $\beta>0$ to be determined later, then from (9)-(11), we have

$$\begin{aligned}&D^+W(t)\\ &\leqslant -dy(t)-\beta V(t)+\left(\sum_{i=1}^{m}\sum_{j\neq i_0}L_j\left|a_{ij}\right|\right)y(t)\\ &\quad+e^{\beta\tau}\sum_{i=1}^{m}\sum_{j\neq i_0}K_j\left|b_{ij}\right|\left|y_j(t)\right|\\ &\quad+\sum_{i=1}^{m}\left[\left|a_{ii_0}\right|\left|y_{i_0}(t)\right|\ \tilde{f}_{i_0}\left(\left|y_{i_0}(t)\right|\right)+\left|b_{ii_0}\right|\left|y_{i_0}(t-\tau)\right|\ \tilde{g}_{i_0}\left(\left|y_{i_0}(t-\tau)\right|\right)\right],\ t\geqslant 0\end{aligned}$$

or

$$\begin{aligned}D^+W(t)\leqslant &-d^*W(t)+\left(\sum_{i=1}^{m}\sum_{j\neq i_0}L_j\left|a_{ij}\right|\right)W(t)\\ &+e^{\beta\tau}\left(\sum_{i=1}^{m}\sum_{j\neq i_0}K_j\left|b_{ij}\right|\right)W(t)+\left(\sum_{i=1}^{m}\left|a_{ii_0}\right|\right)W(t)\ \tilde{f}_{i_0}(W(t))\\ &+\left(\sum_{i=1}^{m}\left|b_{ii_0}\right|\right)W(t-\tau)\ \tilde{g}_{i_0}(W(t-\tau)),\ t\geqslant 0,\end{aligned} \tag{12}$$

where $d^*=\min\{d,\beta\}$. Arranging terms in (12), we find

$$\begin{aligned}D^+W(t)\leqslant &-\left[d^*-\sum_{i=1}^{m}\sum_{j\neq i_0}L_j\left|a_{ij}\right|-e^{\beta\tau}\sum_{i=1}^{m}\sum_{j\neq i_0}K_j\left|b_{ij}\right|\right]W(t)\\ &+\left(\sum_{i=1}^{m}\left|a_{ii_0}\right|\right)W(t)\ \tilde{f}_{i_0}(W(t))+\left(\sum_{i=1}^{m}\left|b_{ii_0}\right|\right)W(t-\tau)\ \tilde{g}_{i_0}(W(t-\tau)),t\geqslant 0\end{aligned}$$

or

$$D^{+}W(t) \leqslant -aW(t) + \sup_{t-\tau \leqslant s \leqslant t} [W(s)\eta(W(s))], \ t \geqslant 0, \tag{13}$$

where

$$a := d^{*} - \sum_{i=1}^{m} \sum_{j \neq i_0} L_j |a_{ij}| - e^{\beta\tau} \sum_{i=1}^{m} \sum_{j \neq i_0} K_j |b_{ij}| > 0 \tag{14}$$

and

$$\eta(W(t)) := 2\left[\left(\sum_{i=1}^{m} |a_{ii_0}|\right) \tilde{f}_{i_0}(W(s)) + \left(\sum_{i=1}^{m} |b_{ii_0}|\right) \tilde{g}_{i_0}(W(s))\right].$$

By our assumption

$$\eta(0) = 2\left[\left(\sum_{i=1}^{m} |a_{ii_0}|\right) \tilde{f}_{i_0}(0) + \left(\sum_{i=1}^{m} |b_{ii_0}|\right) \tilde{g}_{i_0}(0)\right] < a \tag{15}$$

and Lemma 1 is applicable. Therefore, there exist an $M > 0$ and $\alpha > 0$ such that

$$W(t) \leqslant Me^{-\alpha t}, \ t \geqslant 0$$

whenever

$$\sup_{-\tau \leqslant s \leqslant 0} |y_0(s)e^{\alpha s}| + \int_{-\tau}^{0} e^{\beta(\sigma+\tau)} \sum_{i=1}^{m} \left[\sum_{j \neq i_0} K_j |b_{ij}| |y_{0j}(\sigma)|\right] d\sigma \leqslant M. \tag{16}$$

This completes the proof. □

REMARK 5. Regarding the stability of the equilibrium state, we first have existence of a unique equilibrium in case f_i and g_i are Lipschitz continuous when fixing component $i_0 \in \{1,...,n\}$ and f_{i_0} and g_{i_0} are of Osgood type (that is $\Phi_{f_{i_0}}$ and $\Phi_{g_{i_0}}$ satisfy Osgood criterion (3)) or in the even more general situation when all f_i and g_i satisfy Osgood condition. In case the modulus of continuity is of the form $|x|\tilde{h}(|x|)$ with $\tilde{h}(|x|)$ defined at 0, then the equilibrium is locally exponentially stable under the previous conditions.

4. Example

In this section we consider the following example

$$\begin{cases} x_1'(t) = -d_1x_1(t) + a_{11}f_1(x_1(t)) + a_{12}f_2(x_2(t)) + b_{11}g_1(x_1(t-\tau)) \\ \qquad\qquad +b_{12}g_2(x_2(t-\tau)) + c_1, \quad t > 0, \\ x_2'(t) = -d_2x_2(t) + a_{21}f_1(x_1(t)) + a_{22}f_2(x_2(t)) + b_{21}g_1(x_1(t-\tau)) \\ \qquad\qquad +b_{22}g_2(x_2(t-\tau)) + c_2, \quad t > 0, \\ x_i(t) = x_{0i}(t), \quad t \in [-\tau, 0], \ i = 1, 2. \end{cases} \tag{17}$$

In both cases: existence of a unique equilibrium (Theorem 3) or not (Theorem 2), we may take $f_1(x)$ and $g_1(x)$ any two Lipschitz continuous functions, say $f_1(x) = x$ and $g_1(x) = \tanh x$ (with $f_2(0) = 0$) and $f_2(x)$ and $g_2(x)$ any two non-Lipschitz continuous functions with modulus of continuity, say x^2 and x^3, resp. The inputs are irrelevant as we will compare our system (17) (after subtraction) with the system

$$\begin{cases} D^+y_1(t) = -d_1\,|y_1(t)| + |a_{11}|\,L_1\,|y_1(t)| + |b_{11}|\,K_1\,|y_1(t-\tau)| \\ \qquad\qquad + |a_{12}|\,|y_2(t)|\,\tilde{f}_2\,(|y_2(t)|) + |b_{12}|\,|y_2(t-\tau)|\,\tilde{g}_2\,(|y_2(t-\tau)|), \quad t>0 \\ D^+y_2(t) = -d_2\,|y_2(t)| + |a_{21}|\,L_1\,|y_1(t)| + |b_{21}|\,K_1\,|y_1(t-\tau)| \\ \qquad\qquad + |a_{22}|\,|y_2(t)|\,\tilde{f}_2\,(|y_2(t)|) + |b_{22}|\,|y_2(t-\tau)|\,\tilde{g}_2\,(|y_2(t-\tau)|), \quad t>0 \\ y_i(t) = y_{0i}(t), \quad t\in[-\tau,0],\ i=1,2. \end{cases} \tag{18}$$

We select the different parameters so as to satisfy the conditions. Let $a_{11}=0.5,\ a_{12}=2,\ a_{21}=0.7,\ a_{22}=1,\ b_{11}=0.3,\ b_{12}=3,\ b_{21}=0.4,\ b_{22}=4,\ \beta=4,\ \tau=0.2,\ d_1=5,\ d_2=6$. Clearly $L_1=K_1=1,\ d^*=\min\{d_1,d_2,\beta\}=1$ and the condition (15) is satisfied. The system (18) reads

$$\begin{cases} D^+y_1(t) = -\,|y_1(t)| + 0.3\,|y_1(t)| + 0.5\,|y_1(t-\tau)| + 2\,|y_2(t)|^2 + 3\,|y_2(t-\tau)|^3, \\ D^+y_2(t) = -2\,|y_2(t)| + 0.4\,|y_1(t)| + 0.2\,|y_1(t-\tau)| + |y_2(t)|^2 + 2\,|y_2(t-\tau)|^3, \\ y_i(t) = y_{0i}(t), \quad t\in[-\tau,0],\ i=1,2. \end{cases}$$

The assumption (14) holds as

$$\begin{aligned} & d^* - \textstyle\sum_{i=1}^m \sum_{j\neq i_0} L_j\,|a_{ij}| - e^{\beta\tau} \sum_{i=1}^m \sum_{j\neq i_0} K_j\,|b_{ij}| \\ =& d^* - L_1\,(|a_{11}|+|a_{21}|) - K_1 e^{\beta\tau}\,(|b_{11}|+|b_{21}|) = 4-(1.2)-(0.7)\,e^{0.8} \backsimeq 1.24. \end{aligned}$$

Therefore, Lemma 1 applies and gives the local exponential stability provided that the initial data satisfy

$$\sup_{-\tau\leqslant s\leqslant 0} |y_0(s)e^{\alpha s}| + (0.7)\int_{-0.2}^{0} e^{4(\sigma+0.2)}\,|y_{01}(\sigma)|\,d\sigma \leqslant M$$

where $y_0(s)=|y_{01}(s)|+|y_{02}(s)|,\ \alpha=\alpha_b$ and $M=\bar{y}_\delta$ in the proof of Lemma 1.

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