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GENERALIZATIONS OF CYCLIC REFINEMENTS OF JENSEN'S INEQUALITY BY LIDSTONE'S POLYNOMIAL WITH APPLICATIONS IN INFORMATION THEORY

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Abstract. Jensen's inequality plays pivotal role in attaining divergence between probability distributions. Shannon, Relative and Zipf-Mandelbrot entropies have ample applications in many applied sciences, especially in information theory, biology, economics, etc. In the present paper, we have obtained new generalizations of cyclic refinements of Jensen's inequality using different new Green functions by employing Lidstone's polynomial. As an application of our obtained results we have given new entropic bounds. Also, we have established the connections between Shannon and Relative entropy with Zipf-Mandelbrot entropy.

1. Introduction

Information theory is the branch of mathematics which illustrates how uncertainty should be quantified, manipulated and represented. Since the publication of groundbreaking paper of Claude Shanon in 1948 [25], the subject has had remarkable applications in almost every field of science and technology. It has also been shaping the theories of neural computation, statistics, economics, psychology etc. However, to work for such applications, Jensen's inequality is the key to success. Jensen's inequality for differentiable convex mappings has compelling applications in information theory.

To move on, we consider Lidstone series, a generalization of the Taylor series, approximating a given function in the neighborhood of two points instead of one by using the even derivatives. Such series have been studied by G. J. Lidstone (1929), H. Poritsky (1932), J. M. Wittaker (1934) and others (see [1, 2]). Widder proved the fundamental lemma:

LEMMA 1.1. [29] *If* $\phi \in C^{2n}[0,1]$, *then*

$$\phi(z) = \sum_{l=0}^{n-1} \left[\phi^{(2l)}(0)P_l(1-z) + \phi^{(2l)}(1)P_l(z)\right] + \int_0^1 G_n(z,r)\phi^{(2n)}(r)dr,$$

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where P_n *is a Lidstone's polynomial of degree* $(2n+1)$ *defined by the relations*

$$P_0(z) = z$$

$$P_n''(z) = P_{n-1}(z)$$

$$P_n(0) = P_n(1) = 0, \quad n \geqslant 1$$

and

$$G_1(z,r) = G(z,r) = \begin{cases} (z-1)r\,, & r \leqslant z, \\ (r-1)z, & z \leqslant r, \end{cases} \tag{1}$$

is homogeneous Green function of the differential operator $\frac{d^2}{dr^2}$ *on* $[0,1]$*, and with the successive iterates of* $G(z,r)$

$$G_n(z,r) = \int_0^1 G_1(z,s)G_{n-1}(s,r)ds, \quad n \geqslant 2. \tag{2}$$

The Lidstone's polynomial can be expressed in terms of $G_n(z,r)$ *as*

$$P_n(z) = \int_0^1 G_n(z,r)rdr. \tag{3}$$

For $j = 1,\ldots,5$, consider the well known Lagrange Green function along with new Green functions $G_{(j)} : [\alpha_1,\alpha_2]\times[\alpha_1,\alpha_2] \to \mathbb{R}$ defined as

$$G_{(1)}(z,r) = \begin{cases} \frac{(\alpha_2-z)(\alpha_1-r)}{\alpha_2-\alpha_1}\,, & \alpha_1 \leqslant r \leqslant z; \\ \frac{(\alpha_2-r)(\alpha_1-z)}{\alpha_2-\alpha_1}, & z \leqslant r \leqslant \alpha_2. \end{cases} \tag{4}$$

$$G_{(2)}(z,r) = \begin{cases} \alpha_1 - r, \ \alpha_1 \leqslant r \leqslant z, \\ \alpha_1 - z, \ z \leqslant r \leqslant \alpha_2. \end{cases} \tag{5}$$

$$G_{(3)}(z,r) = \begin{cases} z - \alpha_2, \ \alpha_1 \leqslant r \leqslant z, \\ r - \alpha_2, \ z \leqslant r \leqslant \alpha_2. \end{cases} \tag{6}$$

$$G_{(4)}(z,r) = \begin{cases} z - \alpha_1, \ \alpha_1 \leqslant r \leqslant z, \\ r - \alpha_1, \ z \leqslant r \leqslant \alpha_2. \end{cases} \tag{7}$$

$$G_{(5)}(z,r) = \begin{cases} \alpha_2 - r, \ \alpha_1 \leqslant r \leqslant z, \\ \alpha_2 - z, \ z \leqslant r \leqslant \alpha_2, \end{cases} \tag{8}$$

All these functions are convex and continuous. The following lemma holds:

LEMMA 1.2. [21] *Let* $\phi \in C^2[\alpha_1,\alpha_2]$ *, then the following identities hold:*

$$\phi(z) = \frac{\alpha_2 - z}{\alpha_2 - \alpha_1}\phi(\alpha_1) + \frac{z-\alpha_1}{\alpha_2-\alpha_1}\phi(\alpha_2) + \int_{\alpha_1}^{\alpha_2} G_{(1)}(z,r)\phi''(r)dr. \tag{9}$$

$$\phi(z) = \phi(\alpha_1) + (z-\alpha_1)\phi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} G_{(2)}(z,r)\phi''(r)dr, \tag{10}$$

$$\phi(z) = \phi(\alpha_2) + (\alpha_2 - z)\phi'(\alpha_1) + \int_{\alpha_1}^{\alpha_2} G_{(3)}(z,r)\phi''(r)dr, \tag{11}$$

$$\phi(z) = \phi(\alpha_2) - (\alpha_2-\alpha_1)\phi'(\alpha_2) + (z-\alpha_1)\phi'(\alpha_1) + \int_{\alpha_1}^{\alpha_2} G_{(4)}(z,r)\phi''(r)dr, \tag{12}$$

$$\phi(z) = \phi(\alpha_1) + (\alpha_2-\alpha_1)\phi'(\alpha_1) - (\alpha_2 - z)\phi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} G_{(5)}(z,r)\phi''(r)dr. \tag{13}$$

REMARK 1.3. The Green function $G_{(1)}(\cdot,\cdot)$ is called Lagrange Green function (see [29]). The new Green functions $G_{(j)}(\cdot,\cdot)$, $(j = 2,3,4,5)$, were introduced by Pečarić et al. in [21]. The result (10) given in the previous Lemma represents a special case of the representation of the function using the so-called 'two-point right focal' interpolating polynomial in case when $(n = 2 \ \& \ p = 0)$ (see [1]).

The most influential inequality dealing convex functions is the classical Jensen's inequality [12] having both discrete and continuous variants. Here, we present some recent work on the classical and discrete Jensen's inequalities (see [13]). To make statements of that work simple, we need the following hypothesis:

(M_1) Let $I \subset \mathbb{R}$ be an interval, $\mathbf{z} := (z_1,\cdots,z_m) \in I^m$ and let $p_1,\ldots,p_m$ and $\lambda_1,\ldots,\lambda_k$ represent positive probability distributions for integers $2 \leqslant k \leqslant m$.

(M$_2$) Let $(Z,\mathscr{B},\delta)$ be a probability space.

Let $l \geqslant 2$ be a fixed integer. For $j = 1,\cdots,l$, the σ-algebra in Z^l generated by the projection mappings $pr_j : Z^l \to Z$ defined by

$$pr_j(z_1,\ldots,z_l) := z_j$$

is denoted by $\mathscr{B}^l$. δ^l is the product measure on $\mathscr{B}^l$. This measure is uniquely (δ is σ-finite) specified by

$$\delta^l(B_1 \times \ldots \times B_l) := \delta(B_1)\ldots\delta(B_l), \quad B_j \in \mathscr{B}, \quad j = 1,\ldots,l.$$

(M$_3$) Let f be a δ-integrable function on Z taking values in an interval $I \subset \mathbb{R}$.
(M$_4$) Let ϕ be a convex function on I such that $\phi \circ f$ is δ-integrable on Z.
We state the following two main results proved in [13]:

THEOREM 1.4. *Assume* (M_1)*. If* $\phi : I \to \mathbb{R}$ *is a convex function with* $\mathbf{p} := (p_1, ..., p_m)$ *and* $\lambda := (\lambda_1, ..., \lambda_k)$ *then*

$$\phi\left(\sum_{u=1}^{m} p_u z_u\right) \leqslant C_{dis}(\phi, \mathbf{z}, \mathbf{p}, \lambda) \tag{14}$$

$$:= \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right)\phi\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}z_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right) \leqslant \sum_{u=1}^{m} p_u\phi(z_u)$$

where $u+v$ *means* $u+v-m$ *in case of* $u+v>m$*.*

THEOREM 1.5. *Assume* (M_1) *and* (M_2*-*M_4)*. Then for* $\mathbf{p} := (p_1, ..., p_m)$ *and* $\lambda := (\lambda_1, ..., \lambda_k)$

$$\phi\left(\int_Z f d\delta\right) \leqslant C_{par}(t) \leqslant C_{int} \leqslant \int_Z \phi \circ f d\delta, \quad t \in [0,1],$$

where

$$C_{int} = C_{int}(\phi, f, \delta, \mathbf{p}, \lambda)$$

$$:= \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right)\int_{Z^m}\phi\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}f(z_{u+v})}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right)d\delta^m(z_1,\ldots,z_m), \tag{15}$$

and for $t \in [0,1]$

$$C_{par}(t) = C_{par}(t, \phi, f, \delta, \mathbf{p}, \lambda) := \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right)$$

$$\times \int_{Z^m}\phi\left(t\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}f(z_{u+v})}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}} + (1-t)\int_Z f d\delta\right)d\delta^m(z_1,\ldots,z_m), \tag{16}$$

where $u+v$ *means* $u+v-m$ *in case of* $u+v>m$*.*

REMARK 1.6. Theorem 1.4 can be considered as the weighted form of Theorem 2.1 in [3]. Lemma 2.1 (b) in [10] assures that the integrals in (15) and (16) exist and are finite.

REMARK 1.7. Under the conditions (M_1), with $\mathbf{p} := (p_1,...,p_m)$ and $\lambda := (\lambda_1,...,\lambda_k)$ we define

$$J_1(\phi) = J_1(\mathbf{z},\mathbf{p},\lambda;\phi) := \sum_{u=1}^{m} p_u\phi(z_u) - C_{dis}(\phi,\mathbf{z},\mathbf{p},\lambda)$$

$$J_2(\phi) = J_1(\mathbf{z},\mathbf{p},\lambda;\phi) := C_{dis}(\phi,\mathbf{z},\mathbf{p},\lambda) - \phi\left(\sum_{u=1}^{m} p_u z_u\right),$$

where $\phi : I \to \mathbb{R}$ is a function. The functionals $\phi \to J_i(\phi)$ are linear and Theorem 1.4 implies that

$$J_i(\phi) \geqslant 0, \quad i = 1,2$$

provided that ϕ is a convex function.

Assume (M_1-M_4). Then we have the following additional linear functionals

$$J_3(\phi) = J_3(\phi,f,\delta,\mathbf{p},\lambda) := \int_Z \phi\circ f d\delta - C_{int}(\phi,f,\delta,\mathbf{p},\lambda) \geqslant 0,$$

$$J_4(\phi) = J_4(t,\phi,f,\delta,\mathbf{p},\lambda) := \int_Z \phi\circ f d\delta - C_{par}(t,\phi,f,\delta,\mathbf{p},\lambda) \geqslant 0;\ \ t\in[0,1],$$

$$J_5(\phi) = J_5(t,\phi,f,\delta,\mathbf{p},\lambda) := C_{int}(\phi,f,\delta,\mathbf{p},\lambda) - C_{par}(t,\phi,f,\delta,\mathbf{p},\lambda) \geqslant 0; \quad t\in[0,1],$$

$$J_6(\phi) = J_6(t,\phi,f,\delta,\mathbf{p},\lambda) := C_{par}(t,\phi,f,\delta,\mathbf{p},\lambda) - \phi\left(\int_Z f d\delta\right) \geqslant 0; \quad t\in[0,1].$$

2. Extensions of cyclic refinements of Jensen's inequality by Lidstone's interpolating polynomial

To start for real weights, we need the following assumptions:

(A_1) For the linear functionals $J_i(\cdot)$ $(i = 1,2)$, assume further that $\dfrac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}z_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}} \in [\alpha_1,\alpha_2]$ for $u = 1,\ldots m$.

(A_2) For the linear functionals $J_i(\cdot)$ $(i = 3,\ldots,6)$, assume further that $\dfrac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}f(z_{u+v})}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}} \in [\alpha_1,\alpha_2]$ for $u = 1,\ldots m$.

We propose the following Lemma in which we construct the generalized identities having real weights utilizing Lidstone's interpolating polynomial and Green functions.

LEMMA 2.1. *Let $m,k \in \mathbb{N}$, $\mathbf{p} := (p_1,...,p_m)$ and $\lambda := (\lambda_1,...,\lambda_k)$ be real tuples for $2 \leqslant k \leqslant m$, such that $\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \neq 0$ for $u = 1,\ldots m$ with $\sum_{u=1}^{m} p_u = 1$ and $\sum_{v=1}^{k} \lambda_v = 1$. Also let $z \in [\alpha_1,\alpha_2] \subset \mathbb{R}$ and $\mathbf{z} \in [\alpha_1,\alpha_2]^m$. Consider the function $\phi \in C^{2n}[\alpha_1,\alpha_2]$ such that $\phi(\alpha_1) = \phi(\alpha_2)$, $\phi'(\alpha_1) = 0 = \phi'(\alpha_2)$, G_n and $G_{(j)}$, $(j = 1,\ldots,5)$ be the same as defined in (2) and (4)–(8), respectively. Then for $(i = 1,\ldots,6)$ along with the assumptions (A_1) and (A_2), we have the following generalized identities:*

(a) For $n \geqslant 1$

$$J_i(\phi) = \sum_{l=0}^{n-1} (\alpha_2-\alpha_1)^{2l} \left[\phi^{(2l)}(\alpha_1) J_i\left(P_l\left(\frac{\alpha_2 - z}{\alpha_2-\alpha_1}\right)\right) + \phi^{(2l)}(\alpha_2) J_i\left(P_l\left(\frac{z-\alpha_1}{\alpha_2-\alpha_1}\right)\right)\right] + (\alpha_2-\alpha_1)^{2n-1} \int_{\alpha_1}^{\alpha_2} J_i\left(G_n\left(\frac{z-\alpha_1}{\alpha_2-\alpha_1},\frac{r-\alpha_1}{\alpha_2-\alpha_1}\right)\right)\phi^{(2n)}(r)dr. \tag{17}$$

(b) For $n \geqslant 2$

$$J_i(\phi) = \int_{\alpha_1}^{\alpha_2} J_i\Big(G_{(j)}(z,r)\Big) \times \left(\sum_{l=0}^{n-2} (\alpha_2-\alpha_1)^{2l}\left[\phi^{(2l+2)}(\alpha_1) P_l\left(\frac{\alpha_2-r}{\alpha_2-\alpha_1}\right) + \phi^{(2l+2)}(\alpha_2) P_l\left(\frac{r-\alpha_1}{\alpha_2-\alpha_1}\right)\right]\right) dr + (\alpha_2-\alpha_1)^{2n-3} \int_{\alpha_1}^{\alpha_2} \phi^{(2n)}(s) \left(\int_{\alpha_1}^{\alpha_2} J_i\Big(G_{(j)}(z,r)\Big) G_{n-1}\left(\frac{r-\alpha_1}{\alpha_2-\alpha_1},\frac{s-\alpha_1}{\alpha_2-\alpha_1}\right) dr\right) ds. \tag{18}$$

Proof. Fix $(i = 1,\ldots,6)$.

(a) As $\phi \in C^{2n}([\alpha_1,\alpha_2])$, by Widder's lemma we have

$$\phi(z) = \sum_{l=0}^{n-1} (\alpha_2-\alpha_1)^{2l}\left[\phi^{(2l)}(\alpha_1) P_l\left(\frac{\alpha_2 - z}{\alpha_2-\alpha_1}\right) + \phi^{(2l)}(\alpha_2) P_l\left(\frac{z-\alpha_1}{\alpha_2-\alpha_1}\right)\right] + (\alpha_2-\alpha_1)^{2n-1} \int_{\alpha_1}^{\alpha_2} G_n\left(\frac{z-\alpha_1}{\alpha_2-\alpha_1},\frac{r-\alpha_1}{\alpha_2-\alpha_1}\right)\phi^{(2n)}(r)dr. \tag{19}$$

Now employing our respective cyclic Jensen's functional $J_i(\cdot)$ on (19) and practicing its linearity, we get (17) for $(i = 1,\ldots,6)$.

(b) For fix $j=2$, testing identity (10) in cyclic Jensen's functional $J_i(\cdot)$ and employing its properties along with the assumed condition, we have

$$J_i(\phi) = J_i(\phi(\alpha_1)) + J_i\left((z-\alpha_1)\phi^{'}(\alpha_2)\right) + \int_{\alpha_1}^{\alpha_2} J_i\left(G_{(2)}(z,r)\right)\phi^{''}(r)dr$$
$$= \phi^{'}(\alpha_2)J_i(z) + \int_{\alpha_1}^{\alpha_2} J_i\left(G_{(2)}(z,r)\right)\phi^{''}(r)dr$$
$$= \int_{\alpha_1}^{\alpha_2} J_i(G_{(2)}(z,r))\phi^{''}(r)dr. \tag{20}$$

Using representation (19) for ϕ'', we get

$$\phi''(r) = \sum_{l=0}^{n-2}(\alpha_2-\alpha_1)^{2l}\left[\phi^{(2l+2)}(\alpha_1)P_l\left(\frac{\alpha_2-r}{\alpha_2-\alpha_1}\right)+\phi^{(2l+2)}(\alpha_2)P_l\left(\frac{r-\alpha_1}{\alpha_2-\alpha_1}\right)\right]$$
$$+(\alpha_2-\alpha_1)^{2n-3}\int_{\alpha_1}^{\alpha_2} G_{n-1}\left(\frac{r-\alpha_1}{\alpha_2-\alpha_1},\frac{s-\alpha_1}{\alpha_2-\alpha_1}\right)\phi^{(2n)}(s)ds. \tag{21}$$

Now, using (21) in (20) and applying Fubini's theorm, we get (18) for $j=2$ and $(i=1,\ldots,6)$. The cases for $(j=1,3,4,5)$ can be treated analogously.

Now we obtain generalizations of discrete and integral cyclic Jensen's type linear functionals, with real weights for $2n-$ convex functions.

THEOREM 2.2. *Consider $\phi \in C^{2n}[\alpha_1,\alpha_2]$ be such that ϕ is $2n-$convex function along with the suppositions of Lemma 2.1. Then we conclude the following results:*

(a) If for all $(i=1,\ldots,6)$,

$$J_i\left(G_n\left(\frac{z-\alpha_1}{\alpha_2-\alpha_1},\frac{r-\alpha_1}{\alpha_2-\alpha_1}\right)\right) \geqslant 0, \quad r\in[\alpha_1,\alpha_2] \tag{22}$$

holds, then we have

$$J_i(\phi) \geqslant \sum_{l=0}^{n-1}(\alpha_2-\alpha_1)^{2l}\left[\phi^{(2l)}(\alpha_1)J_i\left(P_l\left(\frac{\alpha_2-z}{\alpha_2-\alpha_1}\right)\right)\right.$$
$$\left.+\phi^{(2l)}(\alpha_2)J_i\left(P_l\left(\frac{z-\alpha_1}{\alpha_2-\alpha_1}\right)\right)\right]. \tag{23}$$

(b) If for all $(i=1,\ldots,6)$ and $(j=1,\ldots,5)$

$$\int_{\alpha_1}^{\alpha_2} J_i\left(G_{(j)}(z,r)\right)G_{n-1}\left(\frac{r-\alpha_1}{\alpha_2-\alpha_1},\frac{s-\alpha_1}{\alpha_2-\alpha_1}\right)dr \geqslant 0,\ r\in[\alpha_1,\alpha_2] \tag{24}$$

holds, then we have

$$J_i(\phi) \geqslant \int_{\alpha_1}^{\alpha_2} J_i\Big(G_{(j)}(z,r)\Big)$$
$$\times \left(\sum_{l=0}^{n-2}(\alpha_2-\alpha_1)^{2l}\left[\phi^{(2l+2)}(\alpha_1)P_l\left(\frac{\alpha_2-r}{\alpha_2-\alpha_1}\right)+\phi^{(2l+2)}(\alpha_2)P_l\left(\frac{r-\alpha_1}{\alpha_2-\alpha_1}\right)\right]\right)dr. \tag{25}$$

Proof. We start with the proof of (a) and its assumed conditions. Fix $(i=1,\ldots,6)$. By our assumption $\phi \in C^{2n}[\alpha_1,\alpha_2]$ and is $2n-$convex function, we have $\phi^{(2n)}(\cdot) \geqslant 0$ (see [22], p. 16). Therefore applying Lemma 2.1 (a) by taking into account assumption (22) and $\phi^{(2n)} \geqslant 0$, we get (23).
In the similar passion, we can give the proof of (25).
We will finish the present section by the following results:

THEOREM 2.3. *If the assumptions of Lemma 2.1 be fulfilled with additional conditions that $\mathbf{p} := (p_1,...,p_m)$ and $\lambda := (\lambda_1,...,\lambda_k)$ be non negative tuples for $2 \leqslant k \leqslant m$, such that $\sum_{u=1}^{m} p_u = 1$ and $\sum_{v=1}^{k} \lambda_v = 1$. Then for $\phi : [\alpha_1,\alpha_2] \to \mathbb{R}$ being $2n-$convex function, we conclude the following results:*

(a) (23) is valid for odd $n \geqslant 1$. Besides, for function

$$H(z) := \sum_{l=0}^{n-1}(\alpha_2-\alpha_1)^{2l}\left[\phi^{(2l)}(\alpha_1)P_l\left(\frac{\alpha_2-z}{\alpha_2-\alpha_1}\right)+\phi^{(2l)}(\alpha_2)P_l\left(\frac{z-\alpha_1}{\alpha_2-\alpha_1}\right)\right] \tag{26}$$

to be convex, the right side of (23) is non negative, means

$$J_i(\phi) \geqslant 0, \qquad i=1,\ldots,6. \tag{27}$$

(b) For odd $n \geqslant 3$, (25) holds. Moreover, let (25) is valid and

$$\sum_{l=0}^{n-2}(\alpha_2-\alpha_1)^{2l}\left[\phi^{(2l+2)}(\alpha_1)P_l\left(\frac{\alpha_2-r}{\alpha_2-\alpha_1}\right)+\phi^{(2l+2)}(\alpha_2)P_l\left(\frac{r-\alpha_1}{\alpha_2-\alpha_1}\right)\right] \geqslant 0, \tag{28}$$

then, we get (27) for all $(i=1,\ldots,6)$ and $(j=1,\ldots,5)$.

Proof.

(a) Fix $(i=1,\ldots,6)$.
From (2), we get $G_n(z,r) \leqslant 0$ for odd n and $G_n(z,r) \geqslant 0$ for even n. Moreover G_1 in (1) is convex and G_{n-1} is positive for odd n. Thus taking into account (2), G_n is convex in first variable if n is odd. Therefore (22) holds by virtue of

Remark 1.7 on account of given weights to be positive. Hence (23) is established by taking into account Theorem 2.2 (a). Moreover, the R.H.S. of (23) can be written in the functional form $J_i(H)$ for all $(i = 1,\dots,6)$ after reorganizing this side. Employing Remark 1.7 the nonnegativity of R.H.S. of (23) is secure, especially (27) is established.

(b) Fix $(i = 1,\dots,6)$.
For odd $n \geqslant 3$, G_{n-1} is positive. Also we have assumed positive weights and for all $(j = 1,\dots,5)$, $G_{(j)}(z,r)$ is convex. Thus by practicing Remark 1.7, $J_i\left(G_{(j)}(z,r)\right) \geqslant 0$ which together with positivity of G_{n-1} yields (24). As ϕ is $2n-$convex, hence by following Theorem 2.2 (b), we obtain (25). Finally, taking into account the positivity of $J_i\left(G_{(j)}(z,r)\right)$ and (28), we get (27).

3. Applications to entropic bounds

Let $\phi : (0,\infty) \to (0,\infty)$ be a convex function with $\mathbf{p} := (p_1,\dots,p_m)$ and $\mathbf{q} := (q_1,\dots,q_m)$ be positive probability distributions. Then ϕ-divergence functional is defined (in [14]) as follows

$$I_\phi(\mathbf{p},\mathbf{q}) = \sum_{u=1}^{m} q_u \phi\left(\frac{p_u}{q_u}\right).$$

Surveying the classical Csiszár divergence functional, we propose a new functional:

DEFINITION 1. Let $\phi : I \to \mathbb{R}$ be a function with I an interval in $\mathbb{R}$. Let $\mathbf{p} := (p_1,\dots,p_m) \in \mathbb{R}^m$, and $\mathbf{q} := (q_1,\dots,q_m) \in (0,\infty)^m$ such that

$$\frac{p_u}{q_u} \in I, \quad u = 1,\dots,m.$$

Then let

$$\widetilde{I}_\phi(\mathbf{p},\mathbf{q}) = \sum_{u=1}^{m} q_u \phi\left(\frac{p_u}{q_u}\right). \tag{29}$$

REMARK 3.1. Under the assumptions of Theorem 2.2 (a), we consider the discrete extensions of cyclic refinements of Jensen's inequalities for $(i = 1)$, from (23) with respect to $2n-$convex function ϕ in the explicit form:

$$\sum_{u=1}^{m} p_u \phi(z_u) - \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}\right) \phi\left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} z_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}}\right)$$
$$\geqslant \sum_{l=0}^{n-1} (\alpha_2 - \alpha_1)^{2l} \phi^{(2l)}(\alpha_1)$$

$$\times\left(\sum_{u=1}^{m} p_u\cdot P_l\left(\frac{\alpha_2-z_u}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right)P_l\left(\frac{\alpha_2-\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}z_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}}{\alpha_2-\alpha_1}\right)\right)$$

$$+\sum_{l=0}^{n-1}(\alpha_2-\alpha_1)^{2l}\phi^{(2l)}(\alpha_2)$$

$$\times\left(\sum_{u=1}^{m} p_u\cdot P_l\left(\frac{z_u-\alpha_1}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right)P_l\left(\frac{\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}z_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}-\alpha_1}{\alpha_2-\alpha_1}\right)\right), \tag{30}$$

where P_n is a Lidstone's polynomial defined in Lemma 1.1.

THEOREM 3.2. *Let $m,k\in\mathbb{N}$ $(2\leqslant k\leqslant m)$, $\lambda_1,\dots,\lambda_k$ be positive probability distributions. Let $\mathbf{p}:=(p_1,\dots,p_m)\in\mathbb{R}^m$ and $\mathbf{q}:=(q_1,\dots,q_m)\in(0,\infty)^m$ be such that*

$$\frac{p_u}{q_u}\in[\alpha_1,\alpha_2],\quad u=1,\dots,m.$$

Also let $\phi\in C^{2n}[\alpha_1,\alpha_2]$ such that ϕ is $2n-$convex function. Then the following inequalities hold:

$$\widetilde{I}_\phi(\mathbf{p},\mathbf{q})$$

$$\geqslant\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\phi\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}\right)+\sum_{l=0}^{n-1}(\alpha_2-\alpha_1)^{2l}\phi^{(2l)}(\alpha_1)$$

$$\times\left(\sum_{u=1}^{m} q_u\cdot P_l\left(\frac{\alpha_2-\frac{p_u}{q_u}}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\alpha_2-\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}}{\alpha_2-\alpha_1}\right)\right)$$

$$+\sum_{l=0}^{n-1}(\alpha_2-\alpha_1)^{2l}\phi^{(2l)}(\alpha_2)$$

$$\times\left(\sum_{u=1}^{m} q_u \cdot P_l\left(\frac{\frac{p_u}{q_u}-\alpha_1}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}-\alpha_1}{\alpha_2-\alpha_1}\right)\right). \tag{31}$$

Proof. Replacing p_u with q_u and z_u with $\frac{p_u}{q_u}$ for $(u=1,\ldots,m)$ in (30) , we get (31).

REMARK 3.3. Under the assumptions of Theorem 2.3 (a) for $(i=1)$, (31) becomes

$$\widetilde{I}_\phi(\mathbf{p},\mathbf{q}) \geqslant \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\phi\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}\right). \tag{32}$$

We now explore two exceptional cases of the previous result.
One corresponds to the entropy of a discrete probability distribution.
Shannon entropy and related measures are increasingly used in molecular ecology and population genetics, information theory, dynamical systems and statistical physics (see [7, 20]). For positive m-tuple $\mathbf{q}=(q_1,\ldots,q_m)$ such that $\sum_{u=1}^{m} q_u = 1$, the **Shannon entropy** is defined by

$$S(\mathbf{q}) = -\sum_{u=1}^{m} q_u \ln q_u. \tag{33}$$

Some recent bounds for Shannon entropy can be seen in [17, 11]. We propose the following results:

COROLLARY 3.4. *Let $m,k\in\mathbb{N}$ $(2\leqslant k\leqslant m)$, $\lambda_1,\ldots,\lambda_k$ be positive probability distributions.*

(a) If $\mathbf{q}:=(q_1,\ldots,q_m)\in(0,\infty)^m$, then

$$\begin{aligned}
&\sum_{u=1}^{m} q_u \ln q_u \\
&\geqslant \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\ln\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)+\sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_1)^{2l}} \\
&\quad\times\left(\sum_{u=1}^{m} q_u\cdot P_l\left(\frac{\alpha_2-\frac{1}{q_u}}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\alpha_2-\frac{1}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}}{\alpha_2-\alpha_1}\right)\right) \\
&\quad+\sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_2)^{2l}}
\end{aligned}$$

$$\times\left(\sum_{u=1}^{m} q_u \cdot P_l\left(\frac{\frac{1}{q_u}-\alpha_1}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\frac{1}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}-\alpha_1}{\alpha_2-\alpha_1}\right)\right). \tag{34}$$

(b) If $\mathbf{q} := (q_1,\ldots,q_m)$ *is a positive probability distribution, then we get the bounds for the Shannon entropy of* $\mathbf{q}$.

$$\begin{aligned} &S(\mathbf{q})\\ &\leqslant -\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\ln\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)-\sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_1)^{2l}}\\ &\quad\times\left(\sum_{u=1}^{m} q_u \cdot P_l\left(\frac{\alpha_2-\frac{1}{q_u}}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\alpha_2-\frac{1}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}}{\alpha_2-\alpha_1}\right)\right)\\ &\quad-\sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_2)^{2l}}\\ &\quad\times\left(\sum_{u=1}^{m} q_u \cdot P_l\left(\frac{\frac{1}{q_u}-\alpha_1}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\frac{1}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}-\alpha_1}{\alpha_2-\alpha_1}\right)\right). \end{aligned} \tag{35}$$

Proof.

(a) Using $\phi(z) := -\ln z$ and $\mathbf{p} := (1,1,\ldots,1)$ in Theorem 3.2, we get the required results.

(b) It is a special case of (a).

REMARK 3.5. Using Remark 3.3, (35) becomes

$$S(\mathbf{q}) \leqslant -\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\ln\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right). \tag{36}$$

The second case corresponds to the Relative entropy or Kullback-Leibler divergence between two probability distributions. One of the best known distance function used in mathematical statistics, information theory and signal processing is Kullback-Leibler distance (see [28]). The **Kullback-Leibler** distance [19] between the positive probability distributions $\mathbf{p} = (p_1,\ldots,p_m)$ and $\mathbf{q} = (q_1,\ldots,q_m)$ is defined by

$$D(\mathbf{q}\,\|\,\mathbf{p}) = \sum_{u=1}^{m} q_u \ln\left(\frac{q_u}{p_u}\right). \tag{37}$$

Some recent bounds for Relative entropy can be seen in [17, 11] (see also [9]). We propose the following results:

COROLLARY 3.6. *Let $m,k \in \mathbb{N}$ $(2 \leqslant k \leqslant m)$, $\lambda_1,\ldots,\lambda_k$ be positive probability distributions.*

(a) If $\mathbf{q} := (q_1,\ldots,q_m), \mathbf{p} := (p_1,\ldots,p_m) \in (0,\infty)^m$, then

$$\begin{aligned}
&\sum_{u=1}^{m} q_u \ln\left(\frac{q_u}{p_u}\right) \\
&\geqslant \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\ln\left(\frac{\sum\limits_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}{\sum\limits_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right) + \sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_1)^{2l}} \\
&\times\left(\sum_{u=1}^{m} q_u \cdot P_l\left(\frac{\alpha_2-\frac{p_u}{q_u}}{\alpha_2-\alpha_1}\right) - \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\alpha_2-\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}}{\alpha_2-\alpha_1}\right)\right) \\
&+\sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_2)^{2l}} \\
&\times\left(\sum_{u=1}^{m} q_u \cdot P_l\left(\frac{\frac{p_u}{q_u}-\alpha_1}{\alpha_2-\alpha_1}\right) - \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}-\alpha_1}{\alpha_2-\alpha_1}\right)\right).
\end{aligned} \tag{38}$$

(b) If If $\mathbf{q} := (q_1,\ldots,q_m), \mathbf{p} := (p_1,\ldots,p_m)$ are positive probability distributions, then we have

$$\begin{aligned}
&D(\mathbf{q}\parallel\mathbf{p}) \\
&\geqslant \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\ln\left(\frac{\sum\limits_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}{\sum\limits_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right) + \sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_1)^{2l}}
\end{aligned}$$

$$\times\left(\sum_{u=1}^{m} q_u \cdot P_l\left(\frac{\alpha_2-\frac{p_u}{q_u}}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\alpha_2-\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}}{\alpha_2-\alpha_1}\right)\right)$$

$$+\sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_2)^{2l}}$$

$$\times\left(\sum_{u=1}^{m} q_u \cdot P_l\left(\frac{\frac{p_u}{q_u}-\alpha_1}{\alpha_2-\alpha_1}\right)-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)P_l\left(\frac{\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}-\alpha_1}{\alpha_2-\alpha_1}\right)\right). \tag{39}$$

Proof.

(a) Using $\phi(z) := -\ln z$ in Theorem 3.2, we get the desired results.

(b) It is special case of (a).

REMARK 3.7. Using Remark 3.3, (39) becomes

$$D(\mathbf{q} \,\|\, \mathbf{p}) \geqslant \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\ln\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right). \tag{40}$$

Zipf's law [26, 27] is one of the basic laws in information science and is extensively applied in linguistics. Let $m \in \{1,2,\ldots\}$, $c \geqslant 0$, $d > 0$, then **Zipf-Mandelbrot entropy** can be given as:

$$Z(H,c,d) = \frac{d}{H_{c,d}^{m}}\sum_{u=1}^{m}\frac{\ln(u+c)}{(u+c)^d} + \ln(H_{c,d}^{m}) \tag{41}$$

where

$$H_{c,d}^{m} = \sum_{\sigma=1}^{m}\frac{1}{(\sigma+c)^d}.$$

Consider

$$q_u = \phi(u;m,c,d) = \frac{1}{(u+c)^d H_{c,d}^{m}} \tag{42}$$

where $\phi(u;m,c,d)$ is discrete probability distribution known as **Zipf-Mandelbrot** law. Application of Zipf-Mandelbrot law can be found in linguistics, information sciences and also is often applicable in ecological field studies. Some of the recent study regarding Zipf-Mandelbrot law can be seen in the listed references (see [15, 17, 18, 11]). Now we state our results involving entropy introduced by Mandelbrot Law by establishing the relationship with Shannon and relative entropies:

THEOREM 3.8. *Let* $m,k \in \mathbb{N}$ $(2 \leqslant k \leqslant m)$, $\lambda := (\lambda_1,...,\lambda_k)$ *be a probability distribution and* $\mathbf{q}$ *be as defined in* (42) *by Zipf-Mandelbrot law with parameters* $m \in \{1,2,\ldots\}$, $c \geqslant 0$, $d > 0$. *Then, the following holds*

$$\begin{aligned}
&S(\mathbf{q}) = Z(H,c,d)\\
&\leqslant -\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^d H_{c,d}^m)}\right)\ln\left(\frac{1}{H_{c,d}^m}\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^d)}\right)\\
&\quad-\sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_1)^{2l}}\left(\sum_{u=1}^{m}\frac{1}{((u+c)^d H_{c,d}^m)}\cdot P_l\left(\frac{\alpha_2-((u+c)^d H_{c,d}^m)}{\alpha_2-\alpha_1}\right)\right.\\
&\quad\left.-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^d H_{c,d}^m)}\right)P_l\left(\frac{\alpha_2-\sum_{v=0}^{k-1}\frac{((u+v+c)^d H_{c,d}^m)}{\lambda_{v+1}}}{\alpha_2-\alpha_1}\right)\right)\\
&\quad-\sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_2)^{2l}}\left(\sum_{u=1}^{m}\frac{1}{((u+c)^d H_{c,d}^m)}\cdot P_l\left(\frac{((u+c)^d H_{c,d}^m)-\alpha_1}{\alpha_2-\alpha_1}\right)\right.\\
&\quad\left.-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^d H_{c,d}^m)}\right)P_l\left(\frac{\sum_{v=0}^{k-1}\frac{((u+v+c)^d H_{c,d}^m)}{\lambda_{v+1}}-\alpha_1}{\alpha_2-\alpha_1}\right)\right).
\end{aligned} \tag{43}$$

Proof. It is interesting to see that for q_u be as defined in (42), $\sum_{u=1}^{m} q_u = 1$. Therefore, using above q_u in Shannon entropy (33), we get Mandelbrot entropy(41)

$$\begin{aligned}
S(q) = -q_u \ln q_u &= -\sum_{u=1}^{m}\frac{1}{((u+c)^d H_{c,d}^m)}\ln\frac{1}{((u+c)^d H_{c,d}^m)}\\
&= \frac{-1}{(H_{c,d}^m)}\sum_{u=1}^{m}\frac{1}{(u+c)^d}\ln\frac{1}{(u+c)^d H_{c,d}^m}\\
&= \frac{-1}{(H_{c,d}^m)}\sum_{u=1}^{m}\frac{1}{(u+c)^d}\left(\ln(1)-d\ln(u+c)-\ln(H_{c,d}^m)\right)\\
&= \frac{1}{(H_{c,d}^m)}\sum_{u=1}^{m}\frac{1}{(u+c)^d}\left(d\ln(u+c)+\ln(H_{c,d}^m)\right)
\end{aligned}$$

$$= \frac{d}{(H_{c,d}^m)} \sum_{u=1}^{m} \frac{\ln(u+c)}{(u+c)^d} + \ln(H_{c,d}^m). \tag{44}$$

Finally, substituting this $q_u = \frac{1}{((u+c)^d H_{c,d}^m)}$ in Corollary 3.4(b), we get the desired result.

The next result establish the relationship of Relative entropy with Mandelbrot entropy:

COROLLARY 3.9. *Let $m,k \in \mathbb{N}$ $(2 \leqslant k \leqslant m)$, $\lambda := (\lambda_1, ..., \lambda_k)$ be a probability distribution and for $c_1, c_2 \in [0, \infty)$, $d_1, d_2 > 0$, let $H_{c_1,d_1}^m = \sum_{\sigma=1}^{m} \frac{1}{(\sigma+c_1)^{d_1}}$ and $H_{c_2,d_2}^m = \sum_{\sigma=1}^{m} \frac{1}{(\sigma+c_2)^{d_2}}$. Now using $q_u = \dfrac{1}{(u+c_1)^{d_1} H_{c_1,d_1}^m}$ and $p_u = \dfrac{1}{(u+c_2)^{d_2} H_{c_2,d_2}^m}$ in Corollary 3.6(b), then the following holds*

$$\begin{aligned}
& D(\mathbf{q} \parallel \mathbf{p}) \\
&= \sum_{u=1}^{m} \frac{1}{(u+c_1)^{d_1} H_{c_1,d_1}^m} \ln\left(\frac{(u+c_2)^{d_2} H_{c_2,d_2}^m}{(u+c_1)^{d_1} H_{c_1,d_1}^m}\right) \\
&= -Z(H, c_1, d_1) + \frac{d_2}{H_{c_1,d_1}^m} \sum_{u=1}^{m} \frac{\ln(u+c_2)}{(u+c_1)^{d_1}} + \ln\left(H_{c_2,d_2}^m\right) \\
&\geqslant \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_1)^{d_1} H_{c_1,d_1}^m}\right) \ln\left(\frac{\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_1)^{d_1} H_{c_1,d_1}^m}}{\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_2)^{d_2} H_{c_2,d_2}^m}}\right) \\
&\quad + \sum_{l=0}^{n-1} \frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_1)^{2l}} \left(\sum_{u=1}^{m} \frac{1}{(u+c_1)^{d_1} H_{c_1,d_1}^m} \cdot P_l\left(\frac{\alpha_2 - \left(\frac{(u+c_1)^{d_1} H_{c_1,d_1}^m}{(u+c_2)^{d_2} H_{c_2,d_2}^m}\right)}{\alpha_2-\alpha_1}\right) \right. \\
&\quad \left. - \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_1)^{d_1} H_{c_1,d_1}^m}\right) P_l\left(\frac{\alpha_2 - \frac{\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_2)^{d_2} H_{c_2,d_2}^m}}{\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_1)^{d_1} H_{c_1,d_1}^m}}}{\alpha_2-\alpha_1}\right)\right)
\end{aligned}$$

$$+\sum_{l=0}^{n-1}\frac{(\alpha_2-\alpha_1)^{2l}(2l-1)!}{(\alpha_2)^{2l}}\left(\sum_{u=1}^{m}\frac{1}{(u+c_1)^{d_1}H^m_{c_1,d_1}}\cdot P_l\left(\frac{\left(\frac{(u+c_1)^{d_1}H^m_{c_1,d_1}}{(u+c_2)^{d_2}H^m_{c_2,d_2}}\right)-\alpha_1}{\alpha_2-\alpha_1}\right)\right.$$

$$\left.-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{(u+v+c_1)^{d_1}H^m_{c_1,d_1}}\right)P_l\left(\frac{\frac{\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{(u+v+c_2)^{d_2}H^m_{c_2,d_2}}}{\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{(u+v+c_1)^{d_1}H^m_{c_1,d_1}}}-\alpha_1}{\alpha_2-\alpha_1}\right)\right). \tag{45}$$

REMARK 3.10. Using Remark 3.3, (43) and (47) becomes

$$S(\mathbf{q})=Z(H,c,d)\leqslant-\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^dH^m_{c,d})}\right)\ln\left(\frac{1}{H^m_{c,d}}\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^d)}\right). \tag{46}$$

$$\begin{aligned}D(\mathbf{q}\parallel\mathbf{p})&=\sum_{u=1}^{m}\frac{1}{(u+c_1)^{d_1}H^m_{c_1,d_1}}\ln\left(\frac{(u+c_2)^{d_2}H^m_{c_2,d_2}}{(u+c_1)^{d_1}H^m_{c_1,d_1}}\right)\\&=-Z(H,c_1,d_1)+\frac{d_2}{H^m_{c_1,d_1}}\sum_{u=1}^{m}\frac{\ln(u+c_2)}{(u+c_1)^{d_1}}+\ln\left(H^m_{c_2,d_2}\right)\\&\geqslant\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{(u+v+c_1)^{d_1}H^m_{c_1,d_1}}\right)\ln\left(\frac{\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{(u+v+c_1)^{d_1}H^m_{c_1,d_1}}}{\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{(u+v+c_2)^{d_2}H^m_{c_2,d_2}}}\right).\end{aligned} \tag{47}$$

REMARK 3.11. It is interesting to note that, in the similar passion we are able to construct different estimations of ϕ-divergences along with their applications to Shannon, Relative and Mandelbrot entropies using the other inequalities for $2n-$convex functions constructed in Theorem 2.2 for discrete case of cyclic refinements of Jensen's inequality.

4. Monotonicity of functionals for $2n-$convex functions at a point

In the present section, we shall give related inequalities for $n-$convex functions at a point, a generalization of the class of $n-$convex functions introduced by Pečarić et al. in [24].

DEFINITION 2. Let $\phi : I \to \mathbb{R}$ be a function and ξ be any point belonging to the interior of I. ϕ called $(n+1)-$convex at ξ if there exists a constant C such that

$$\Phi(z) = \phi(z) - \frac{C}{n!}z^n \tag{48}$$

is $n-$concave on $I \cap (-\infty, \xi]$ and $n-$convex on $I \cap [\xi, \infty)$. A function ϕ is called $(n+1)-$concave at ξ if $-\phi$ is $(n+1)-$convex at ξ.

Witkowski et al in [24] deduced the conditions which are necessary and sufficient on two linear functionals

$$\Gamma : C([\alpha_1, \xi]) \to \mathbb{R}$$

and

$$\Upsilon : C([\xi, \alpha_2]) \to \mathbb{R}$$

such that the inequality $\Gamma(\phi) \leqslant \Upsilon(\phi)$ is valid for every function ϕ which is n-convex at the point ξ. In the present section we shall obtain the monotonicity of particular linear functionals which were obtained from the inequalities in the previous section. Suppose ζ^i represents the monomials $\zeta^i(x) = x^i$, $i \in \mathbb{N}_0$. For the remaining part of this section, $\Gamma_i(\phi_{[\alpha_1,\xi]})$ and $\Upsilon_i(\phi_{[\xi,\alpha_2]})$ $(i = 1, \dots, 6)$, will represent the linear cyclic Jensen's functionals obtained as the difference of the L. H. S. and R. H. S. of inequality (23) applied to the intervals $[\alpha_1, \xi]$ and $[\xi, \alpha_2]$ respectively i. e., for $\mathbf{z} \in [\alpha_1, \xi]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^k$, $\mathbf{y} \in [\xi, \alpha_2]^{\tilde{m}}$, $\mathbf{q} \in \mathbb{R}^{\tilde{m}}$ and $\tilde{\lambda} \in \mathbb{R}^{\tilde{k}}$ let

$$\Gamma_i(\phi_{[\alpha_1,\xi]}) := J_i(\phi) - \sum_{l=0}^{n-1} (\xi - \alpha_1)^{2l} \left[\phi^{(2l)}(\alpha_1) J_i\left(P_l\left(\frac{\xi - z}{\xi - \alpha_1} \right) \right) + \phi^{(2l)}(\xi) J_i\left(P_l\left(\frac{z - \alpha_1}{\xi - \alpha_1} \right) \right) \right]. \tag{49}$$

$$\Upsilon_i(\phi_{[\xi,\alpha_2]}) := J_i(\phi) - \sum_{l=0}^{n-1} (\alpha_2 - \xi)^{2l} \left[\phi^{(2l)}(\xi) J_i\left(P_l\left(\frac{\alpha_2 - y}{\alpha_2 - \xi} \right) \right) + \phi^{(2l)}(\alpha_2) J_i\left(P_l\left(\frac{y - \xi}{\alpha_2 - \xi} \right) \right) \right]. \tag{50}$$

In the similar passion, by using the inequality (25), for $(i = 1, \dots, 6)$ and $(j = 1, \dots, 5)$, we define linear functionals as:

$$\hat{\Gamma}_i(\phi_{[\alpha_1,\xi]}) := J_i(\phi) - \int_{\alpha_1}^{\xi} J_i\Big(G_{(j)}(z, r) \Big) \left(\sum_{l=0}^{n-2} (\xi - \alpha_1)^{2l} \left[\phi^{(2l+2)}(\alpha_1) P_l\left(\frac{\xi - r}{\xi - \alpha_1} \right) + \phi^{(2l+2)}(\xi) P_l\left(\frac{r - \alpha_1}{\xi - \alpha_1} \right) \right] \right) dr. \tag{51}$$

$$\hat{\Upsilon}_i(\phi_{[\xi,\alpha_2]}) := J_i(\phi) - \int_{\xi}^{\alpha_2} J_i\Big(G_{(j)}(y,r)\Big) \left(\sum_{l=0}^{n-2} (\alpha_2-\xi)^{2l} \left[\phi^{(2l+2)}(\xi) P_l\left(\frac{\alpha_2-r}{\alpha_2-\xi}\right) + \phi^{(2l+2)}(\alpha_2) P_l\left(\frac{r-\xi}{\alpha_2-\xi}\right) \right] \right) dr. \tag{52}$$

By constructing new linear functionals $\Gamma_i(\phi_{[\alpha_1,\xi]})$ and $\Upsilon_i(\phi_{[\xi,\alpha_2]})$, identity (17) for $(i=1,\ldots,6)$ enforced to the corresponding intervals $[\alpha_1,\xi]$ and $[\xi,\alpha_2]$ becomes:

$$\Gamma_i(\phi_{[\alpha_1,\xi]}) = (\xi-\alpha_1)^{2n-1} \int_{\alpha_1}^{\xi} J_i\left(G_n\left(\frac{z-\alpha_1}{\xi-\alpha_1}, \frac{r-\alpha_1}{\xi-\alpha_1}\right)\right) \phi^{(2n)}(r)dr. \tag{53}$$

$$\Upsilon_i(\phi_{[\xi,\alpha_2]}) = (\alpha_2-\xi)^{2n-1} \int_{\xi}^{\alpha_2} J_i\left(G_n\left(\frac{y-\xi}{\alpha_2-\xi}, \frac{r-\xi}{\alpha_2-\xi}\right)\right) \phi^{(2n)}(r)dr. \tag{54}$$

Further, by applying identity (18) for $(i=1,\ldots,6)$ and $(j=1,\ldots,5)$ on the intervals $[\alpha_1,\xi]$ and $[\xi,\alpha_2]$ we get:

$$\hat{\Gamma}_i(\phi_{[\alpha_1,\xi]}) = (\xi-\alpha_1)^{2n-3} \int_{\alpha_1}^{\xi} \phi^{(2n)}(s) \left(\int_{\alpha_1}^{\xi} J_i\Big(G_{(j)}(z,r)\Big) G_{n-1}\left(\frac{r-\alpha_1}{\xi-\alpha_1}, \frac{s-\alpha_1}{\xi-\alpha_1}\right) dr \right) ds. \tag{55}$$

$$\hat{\Upsilon}_i(\phi_{[\xi,\alpha_2]}) = (\alpha_2-\xi)^{2n-3} \int_{\xi}^{\alpha_2} \phi^{(2n)}(s) \left(\int_{\xi}^{\alpha_2} J_i\Big(G_{(j)}(y,r)\Big) G_{n-1}\left(\frac{r-\xi}{\alpha_2-\xi}, \frac{s-\xi}{\alpha_2-\xi}\right) dr \right) ds. \tag{56}$$

Promptly, we are in position to state our main theorem of the present section for inequalities involving $2n-$convex function at a point:

THEOREM 4.1. *Consider* $\mathbf{z} \in [\alpha_1,\xi]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^k$, $\mathbf{y} \in [\xi,\alpha_2]^{\tilde{m}}$, $\mathbf{q} \in \mathbb{R}^{\tilde{m}}$ *and* $\tilde{\lambda} \in \mathbb{R}^{\tilde{k}}$ *in such a way that*

(a) For $(i=1,\ldots,6)$ *consider*

$$J_i\left(G_n\left(\frac{z-\alpha_1}{\xi-\alpha_1}, \frac{r-\alpha_1}{\xi-\alpha_1}\right)\right) \geqslant 0, \quad r \in [\alpha_1,\xi], \tag{57}$$

$$J_i\left(G_n\left(\frac{y-\xi}{\alpha_2-\xi}, \frac{r-\xi}{\alpha_2-\xi}\right)\right) \geqslant 0, \quad r \in [\xi,\alpha_2], \tag{58}$$

$$\int_{\alpha_1}^{\xi} J_i\left(G_n\left(\frac{z-\alpha_1}{\xi-\alpha_1}, \frac{r-\alpha_1}{\xi-\alpha_1}\right)\right) dr = \left(\frac{\alpha_2-\xi}{\xi-\alpha_1}\right)^{2n-1} \int_{\xi}^{\alpha_2} J_i\left(G_n\left(\frac{y-\xi}{\alpha_2-\xi}, \frac{r-\xi}{\alpha_2-\xi}\right)\right) dr \tag{59}$$

and let $\Gamma_i(\phi_{[\alpha_1,\xi]})$ *and* $\Upsilon_i(\phi_{[\xi,\alpha_2]})$ *be the linear functionals introduced in* (49) *and* (50). *If* $\phi : [\alpha_1,\alpha_2] \to \mathbb{R}$ *is* $(2n+1)-$*convex at the point* ξ, *then*

$$\Gamma_i(\phi_{[\alpha_1,\xi]}) \leqslant \Upsilon_i(\phi_{[\xi,\alpha_2]}) \text{ for } (i=1,\ldots,6). \tag{60}$$

(b) Similarly, for $(i=1,\ldots,6)$ *and* $(j=1,\ldots,5)$ *suppose*

$$\int_{\alpha_1}^{\xi} J_i\Big(G_{(j)}(z,r)\Big) G_{n-1}\left(\frac{r-\alpha_1}{\xi-\alpha_1}, \frac{s-\alpha_1}{\xi-\alpha_1}\right) dr \geqslant 0,\ r \in [\alpha_1,\xi], \tag{61}$$

$$\int_{\xi}^{\alpha_2} J_i\Big(G_{(j)}(y,r)\Big) G_{n-1}\left(\frac{r-\xi}{\alpha_2-\xi}, \frac{s-\xi}{\alpha_2-\xi}\right) dr \geqslant 0, \quad r \in [\xi,\alpha_2], \tag{62}$$

$$\begin{aligned}&\int_{\alpha_1}^{\xi}\left(\int_{\alpha_1}^{\xi} J_i\Big(G_{(j)}(z,r)\Big) G_{n-1}\left(\frac{r-\alpha_1}{\xi-\alpha_1}, \frac{s-\alpha_1}{\xi-\alpha_1}\right) dr\right) ds\\ &= \left(\frac{\alpha_2-\xi}{\xi-\alpha_1}\right)^{2n-3}\int_{\xi}^{\alpha_2}\left(\int_{\xi}^{\alpha_2} J_i\Big(G_{(j)}(y,r)\Big) G_{n-1}\left(\frac{r-\xi}{\alpha_2-\xi}, \frac{s-\xi}{\alpha_2-\xi}\right) dr\right) ds\end{aligned} \tag{63}$$

and let $\hat{\Gamma}_i(\phi_{[\alpha_1,\xi]})$ *and* $\hat{\Upsilon}_i(\phi_{[\xi,\alpha_2]})$ *be the linear functionals which are given by* (51) *and* (52). *If* $\phi : [\alpha_1,\alpha_2] \to \mathbb{R}$ *is* $(2n+1)-$*convex at the point* ξ, *then*

$$\hat{\Gamma}_i(\phi_{[\alpha_1,\xi]}) \leqslant \hat{\Upsilon}_i(\phi_{[\xi,\alpha_2]}) \text{ for } (i=1,\ldots,6) \text{ and } (j=1,\ldots,5). \tag{64}$$

Proof.

(a) Fix $i=1,\ldots,6$. Employing Definition 2, set up function

$$\Phi = \phi - \frac{C}{(2n)!}\zeta^{2n}$$

such that the function Φ is $2n-$concave on $[\alpha_1,\xi]$ and is $2n-$convex on $[\xi,\alpha_2]$. Executing Theorem 2.2 on Φ over interval $[\alpha_1,\xi]$, we get

$$0 \geqslant \Gamma_i(\Phi) = \Gamma_i(\phi_{[\alpha_1,\xi]}) - \frac{C}{(2n)!}\Gamma_i(\zeta^{2n}_{[\alpha_1,\xi]}). \tag{65}$$

Identically practicing Theorem 2.2 on Φ over interval $[\xi,\alpha_2]$, we have

$$0 \leqslant \Upsilon_i(\Phi) = \Upsilon_i(\phi_{[\xi,\alpha_2]}) - \frac{C}{(2n)!}\Upsilon_i(\zeta^{2n}_{[\xi,\alpha_2]}). \tag{66}$$

Furthermore, using monomials $\zeta^{2n}(\cdot)$ in identities (53) and (54) gives

$$\Gamma_i(\zeta^{2n}_{[\alpha_1,\xi]}) = (2n)!\,(\xi-\alpha_1)^{2n-1}\int_{\alpha_1}^{\xi} J_i\left(G_n\left(\frac{z-\alpha_1}{\xi-\alpha_1},\frac{r-\alpha_1}{\xi-\alpha_1}\right)\right)dr, \tag{67}$$

$$\Upsilon_i(\zeta^{2n}_{[\xi,\alpha_2]}) = (2n)!\,(\alpha_2-\xi)^{2n-1}\int_{\xi}^{\alpha_2} J_i\left(G_n\left(\frac{y-\xi}{\alpha_2-\xi},\frac{r-\xi}{\alpha_2-\xi}\right)\right)dr. \tag{68}$$

Therefore assumption (59) is equivalent to

$$\Gamma_i(\zeta^{2n}_{[\alpha_1,\xi]}) = \Upsilon_i(\zeta^{2n}_{[\xi,\alpha_2]}).$$

So form (65) and (66), one can get

$$\Gamma_i(\phi_{[\alpha_1,\xi]}) \leqslant \frac{C}{(2n)!}\Gamma_i(\zeta^{2n}_{[\alpha_1,\xi]}) = \frac{C}{(2n)!}\Delta_i(\zeta^{2n}_{[\xi,\alpha_2]}) \leqslant \Upsilon_i(\phi_{[\xi,\alpha_2]}). \tag{69}$$

So (60) is obtained for $(i=1,\ldots,6)$.

(b) Similar method as above can be employed by using the identities (55) and (56). Finally by deducing supposition (63), we have (64) for $(i=1,\ldots,6)$ and $(j=1,\ldots,5)$.

We conclude with the following significant remarks:

REMARK 4.2. Note that inequality (60) and (64) are also valid on replacing assumptions (59) and (63) with the weaker assumptions that $C\left(\Upsilon_i(\zeta^{2n}_{[\xi,\alpha_2]})-\Gamma_i(\zeta^{2n}_{[\alpha_1,\xi]})\right) \geqslant 0$ and $C\left(\hat{\Upsilon}_i(\zeta^{2n}_{[\xi,\alpha_2]})-\hat{\Gamma}_i(\zeta^{2n}_{[\alpha_1,\xi]})\right) \geqslant 0$ for $(i=1,\ldots,6)$ respectively.

5. Concluding remarks

It is refreshing to note that obtained inequalities for $2n-$convex functions in Section 2 are worth more as they enable us to give variety of new and sharp upper bounds for Grüss and Ostrowski type inequalities (see [6]) as an application of the results obtained by Dragomir et al in [8]. Furthermore, we can construct variety of functionals from the inequalities introduced in the Theorem 2.2 and present Cauchy and Lagrange type mean value theorems for $2n-$convex functions. More than that, taking into account $n-$exponentially convex approach in [16] and [23](see also [5] and [4]), a new collection of non trivial examples of $n-$exponentially and exponentially convex functions can be established. Finally, we are also able to construct monotonic Cauchy means.

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