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HERMITE–HADAMARD TYPE INEQUALITIES FOR THE s–HH CONVEX FUNCTIONS VIA k–FRACTIONAL INTEGRALS AND APPLICATIONS

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Abstract. In this paper, we first extend the concept of the HH convex function (harmonic harmonically function (see[18,19])) to s-HH convex functions and establish some fractional integral inequalities of Hermite-Hadamard type for s-HH convex functions via fractional integrals and k-fractional integrals.

1. Introduction

In this article, we set $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_{++} = (0,\infty)$ $\mathbb{R}_+ = [0,\infty)$.
First, let us recall some definitions of various convex functions.

DEFINITION 1.1 ([3,11]). A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is convex on I, if

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y), \qquad \forall x,y \in I, t \in [0,1]. \tag{1.1}$$

f is concave function if $-f$ is convex function.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function. The following inequality is the well-known Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \quad a,b \in I \text{ with } a<b. \tag{1.2}$$

In [16], the following Hermite-Hadamard type integral inequality of convex function is proved.

THEOREM A [16]. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leqslant a < b$ and $f \in L[a,b]$. If f is convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(x)+J^\alpha_{b-}f(x)] \leqslant \frac{f(a)+f(b)}{2}, \quad \alpha \geqslant 0. \tag{1.3}$$

Hudzik and Maligranda in [6] define an s-convex function in the second sense, as follows.

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DEFINITION 1.2 ([6]). A function $f: I \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be s-convex in the second sense if for some fixed $s \in (0,1]$

$$f(tx+(1-t)y) \leqslant t^s f(x) + (1-t)^s f(y), \qquad \forall x,y \in I, t \in [0,1]. \tag{1.4}$$

f is be s-concave function if $-f$ is s-convex function.

Dragomir and Fitzpatrick in [4] prove a kind of Hermite-Hadamard type inequality which holds for s-convex functions in the second sense.

THEOREM B ([4]). *Assume that a function $f: \mathbb{R}_+ \to \mathbb{R}_+$ is an s-convex in the second sense, and $a,b \in \mathbb{R}_+$ with $a < b$. If $f \in L[0,1]$, then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{s+1}. \tag{1.5}$$

DEFINITION 1.3 ([7,15]). A function $f: I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonic convex function on I, if

$$f\left(\frac{1}{tx^{-1}+(1-t)y^{-1}}\right) \leqslant tf(x) + (1-t)f(y), \qquad \forall x,y \in I, t \in [0,1]. \tag{1.6}$$

f is a harmonic concave function if $-f$ is a harmonic convex function.

THEOREM C ([7,15]). *Let $f: I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonic convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then*

$$f\left(\frac{2ab}{a+b}\right) \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.7}$$

DEFINITION 1.4 ([21]). A function $f: I \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ is said to be m-AH convex on I, if $m \in (0,1]$ and

$$f(tx+m(1-t)y) \leqslant \frac{1}{t[f(x)]^{-1}+m(1-t)[f(y)]^{-1}}, \qquad \forall x,y \in I, t \in [0,1]. \tag{1.8}$$

f is said to be m-AH concave function if $-f$ is m-AH convex function.

The concept of the harmonic harmonically convex and m-harmonic harmonically convex function may be introduced as follows.

DEFINITION 1.5 ([18]). A function $f: I \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ is harmonic harmonically (briefly HH convex) convex on I, if

$$f\left(\frac{1}{tx^{-1}+(1-t)y^{-1}}\right) \leqslant \frac{1}{t[f(x)]^{-1}+(1-t)[f(y)]^{-1}}, \qquad \forall x,y \in I, t \in [0,1]. \tag{1.9}$$

f is a harmonic harmonically concave if $-f$ is a harmonic harmonically convex.

DEFINITION 1.6 ([19]). A function $f: I \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ is m-harmonic harmonically convex (briefly m-HH convex) on I, if $\forall x,y \in I, t \in [0,1]$

$$f\left(\frac{1}{tx^{-1}+m(1-t)y^{-1}}\right) \leqslant \frac{1}{t[f(x)]^{-1}+m(1-t)[f(y)]^{-1}}, \tag{1.10}$$

where $m \in (0,1]$. f is said to be m-HH concave if $-f$ is m-HH convex.

We next give some sample examples of functions satisfying s-convex, harmonic convex, m-AH convex, HH-convex and $m-$ HH convex.

EXAMPLES.

(1) The function $f:(0,+\infty)\to(0,+\infty)$, $f(x)=x^s+C$ is s-convex for $0<s<1$ and $C\geqslant 0$.

(2) Let $g:(0,+\infty)\to(-\infty,+\infty)$, $g(x)=x$, and $g:(-\infty,0)\to(-\infty,+\infty)$, $h(x)=x$, then g is harmonically convex and h is harmonically concave.

(3) The function $j(x)=\frac{1}{x}$ is m-AH convex fumction for $x>0$.

(4) Let $t:(0,+\infty)\to(0,+\infty)$, $t(x)=x^r$. If $r\leqslant 0$ or $r\geqslant 1$, then $t(x)=x^r$ is a harmonoically concave function; If $0<r<1$, then $t(x)=x^r$ is a harmonically convex function.

(5) The function $u(x)=x^p$ is m-HH convex for $0<p\leqslant 1$ and $x>0$. The function $u(x)=x^p$ is m-HH concave for $p>1$ and $x>0$. The function $v(x)=-x$ is m-HH concave for $x>0$. The function $j(x)=\frac{1}{x}$ is m-HH concave for $x>0$.

REMARK. The concept of the HH convex function and m-HH convex function are defined by the second author of this article and co-authors. So, the proofs of examples (4) and (5) can be found in references [18] and [19].

DEFINITION 1.7 ([5,9,12]). Let $f\in L[a,b]$. The Riemann-Liouville integrals J^{α}_{a+} and J^{α}_{b+} of order $\alpha>0$ with $a\geqslant 0$ are defined by

$$J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\quad x<b,$$

respectively, where

$$\Gamma(x)=\int_0^{\infty} e^{-x}t^{x-1}dt,\quad x>0.$$

is the Gamma function and $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.

For many recent results related to Hermite-Hadamard type inequalities for some special functions, see [1,4,7,8,14,15,18,19,20,21].

Recently, Diaz and Pariguan in [2] define two new functions called k-gamma and k-beta functions and the Pochhammer k-symbol that is respectively generalization of the classical gamma and beta functions and the classical Pochhammer symbol:

$$\Gamma_k(\alpha)=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},(k>0),$$

where

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k), \quad x \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N}^+$$

is the Pochhammer k-symbol for factorial function. It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, explicitly given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{x^k}{k}} dt, x > 0.$$

Obviously, $\Gamma(x) = \lim_{k \to 1} \Gamma_k(x)$, $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma_k(\frac{x}{k})$ and $\Gamma_k(x+k) = x\Gamma_k(x)$.

The k-Beta function $B_k(x,y)$ is given by the formula

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} = \int_0^\infty t^{x-1}(1+t^k)^{-\frac{x+y}{k}} dt, \mathbf{R}e(x) > 0, \mathbf{R}e(y) > 0.$$

Later, under the above definitions, Muben and Habibullah in [10] also introduce the k-fractional integral of the Reimann type as follows:

$${}_kJ_a^\alpha[f(t)] = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau) d\tau, \alpha > 0, t > a, k > 0.$$

For many recent results related to Hermite-Hadamard type inequalities via k-fractional integrals, see [13,17].

The aim of this paper is first to introduce the concept of the s-harmonic harmonically convex functions. Afterwards, we establish some Hermite-Hadamard type integral inequalities for s-harmonic harmonically convex functions, including fractional integral inequalities and k-fractional integral inequalities.

2. Definition and lemmas

The concept of the s-harmonic harmonically convex (briefly s-HH convex) function may be introduced as follows.

DEFINITION 2.1. A function $f : I \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ is said to be s-harmonic harmonically convex (briefly s-HH convex) on I, if for fixed $s \in (0,1]$

$$f\left(\frac{1}{tx^{-1}+(1-t)y^{-1}}\right) \leqslant \frac{1}{t^s[f(x)]^{-1}+(1-t)^s[f(y)]^{-1}}, \qquad \forall x,y \in I, t \in [0,1]. \tag{2.1}$$

f is said to be s-HH concave if $-f$ is s-HH convex.

Take $s = 1$ in Definition 2.1, we deduce the Definition 1.5.

To illustrate the Definition 2.1, we give a sample example.

EXAMPLE. The function $f : [0,1] \to [0,1]$, $f(x) = x^s$ is s-HH convex for $0 < s \leqslant 1$.

Proof. By Definition 1.2, we get that the function x^s is s-convex for $0 < s \leqslant 1$. Then for $0 < a < b$

$$[ta+(1-t)b]^s \leqslant t^s a^s + (1-t)^s b^s.$$

Let $a = \frac{1}{x}$ and $b = \frac{1}{y}$ in above inequality, we have

$$\left[\frac{t}{x} + \frac{m(1-t)}{y}\right]^s \leqslant \frac{t^s}{x^s} + \frac{(1-t)^s}{y^s}.$$

From this we have

$$\frac{1}{\left[\frac{t}{x} + \frac{(1-t)}{y}\right]^s} \geqslant \frac{1}{\frac{t^s}{x^s} + \frac{(1-t)^s}{y^s}},$$

So, the conclusion valid. □

LEMMA 2.1. [1] *The Beta function:*

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x,y > 0. \tag{2.2}$$

LEMMA 2.2. *Let A and B are two positive real numbers. Then for $p > 0$ and $0 < s < 1$, the following inequalities are valid.*

$$\int_0^1 \frac{t^{p-1}}{At^s + B(1-t)^2} \leqslant \begin{cases} \frac{1}{A+B} \cdot \beta(\alpha, 1-s), & 0 < s < 1, p \geqslant 1, \\ \frac{1}{A+B} \cdot \frac{1}{p-s}, & 0 < s < 1, 0 < s < p < 1. \end{cases} \tag{2.3}$$

Proof. The following inequalities is obviously true for $r > 0$ and $0 \leqslant t \leqslant 1$.

$$\begin{cases} t^r \geqslant (1-t)^r, & t \geqslant \frac{1}{2}, \\ t^r \leqslant (1-t)^r, & t < \frac{1}{2}. \end{cases} \tag{2.4}$$

We first prove for the case $0 < s < 1$ and $p \geqslant 1$.
By using (2.4), we have

$$\frac{t^{p-1}}{t^s A + (1-t)^s B} \leqslant \begin{cases} \frac{t^{p-1}}{(A+B)(1-t)^s}, & t \geqslant \frac{1}{2}, \\ \frac{t^{p-1}}{(A+B)t^s}, & t < \frac{1}{2}. \end{cases} \tag{2.5}$$

Integrating for $t \in [0,1]$, we get for $0 < s < 1$ and $p \geqslant 1$

$$\int_0^1 \frac{t^{p-1}}{t^s A + (1-t)^s B} dt \leqslant \begin{cases} \frac{1}{A+B} \cdot \int_0^1 \frac{t^{p-1}}{(1-t)^s} dt, & t \geqslant \frac{1}{2}, \\ \frac{1}{A+B} \cdot \int_0^1 \frac{(1-t)^{p-1}}{t^s} dt, & t < \frac{1}{2} \end{cases}$$

$$\leqslant \begin{cases} \dfrac{1}{A+B}\cdot\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{1-s-1}dt, & t\geqslant \dfrac{1}{2},\\ \dfrac{1}{A+B}\cdot\displaystyle\int_0^1 t^{1-s-1}(1-t)^{\alpha-1}dt, & t<\dfrac{1}{2}\end{cases}$$

$$\leqslant \begin{cases} \dfrac{1}{A+B}\cdot\beta(\alpha,1-s), & t\geqslant \dfrac{1}{2},\\ \dfrac{1}{A+B}\cdot\beta(1-s,\alpha), & t<\dfrac{1}{2}.\end{cases} \tag{2.6}$$

Since the function $\beta(x,y)$ is symmetric, then for $0<s<1$ and $p\geqslant 1$

$$\int_0^1 \frac{t^{p-1}}{t^sA+(1-t)^sB}dt \leqslant \frac{1}{A+B}\cdot\beta(\alpha,1-s). \tag{2.7}$$

In the following, we calculate for the case $0<s<p<1$.
By (2.5) and integrating for $t\in[0,1]$, we get for $0<s<p<1$

$$\int_0^1 \frac{t^{p-1}}{t^sA+(1-t)^sB}dt \leqslant \begin{cases} \dfrac{1}{A+B}\cdot\displaystyle\int_0^1 \frac{(1-t)^{p-1}}{(1-t)^s}dt, & t\geqslant \dfrac{1}{2},\\ \dfrac{1}{A+B}\cdot\displaystyle\int_0^1 \frac{t^{p-1}}{t^s}dt, & t<\dfrac{1}{2}\end{cases}$$

$$\leqslant \begin{cases} \dfrac{1}{A+B}\cdot\dfrac{1}{p-s}, & t\geqslant \dfrac{1}{2},\\ \dfrac{1}{A+B}\cdot\dfrac{1}{p-s}, & t<\dfrac{1}{2}.\end{cases} \tag{2.8}$$

From (2.7) and (2.8), we obtain the desired result. □

3. Hermite-Hadamard type inequalities

THEOREM 3.1. *Let $f: I\subset \mathbf{R}\backslash\{0\}\to\mathbf{R}_{++}$ be a s-HH convex function with $0\leqslant s\leqslant 1$. If $a,b\in I$ with $a<b$ and $f\in L[a,b]$, then*

$$\frac{1}{2^{s-1}}f\left(\frac{2ab}{a+b}\right)\leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leqslant \begin{cases} \dfrac{1}{1-s}\cdot\dfrac{f(a)f(b)}{f(a)+f(b)}, & 0<s<1,\\ f(a)f(b)\cdot\dfrac{\ln f(a)-\ln f(b)}{f(a)-f(b)}, & s=1.\end{cases} \tag{3.1}$$

Proof. Since $f(x)$ is s-HH convex, for all $x,y\in I$ (with $t=\frac{1}{2}$), we have

$$f\left(\frac{2xy}{x+y}\right)\leqslant \frac{2^sf(x)f(y)}{f(x)+f(y)}\leqslant 2^{s-1}\cdot\frac{f(x)+f(y)}{2}.$$

Let $x=\frac{ab}{ta+(1-t)b}$, $y=\frac{ab}{tb+(1-t)a}$, we have

$$f\left(\frac{2ab}{a+b}\right)\leqslant 2^{s-1}\cdot\frac{f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)}{2}. \tag{3.2}$$

Further, by integrating for $t \in [0,1]$, we obtain

$$f\left(\frac{2ab}{a+b}\right) \leqslant 2^{s-1} \cdot \frac{1}{2}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt + \int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt\right]$$
$$= 2^{s-1} \cdot \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx. \tag{3.3}$$

On the other hand, take $x = b$ and $y = a$ in (2.1), we get

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leqslant \frac{f(a)f(b)}{t^s f(a)+(1-t)^s f(b)}. \tag{3.4}$$

Further, by integrating for $t \in [0,1]$, we obtain

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt \leqslant \int_0^1 \frac{f(a)f(b)}{t^s f(a)+(1-t)^s f(b)}dt. \tag{3.5}$$

By computation we get

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx, \tag{3.6}$$

and for $0 < s < 1$

$$\int_0^1 \frac{f(a)f(b)}{t^s f(a)+(1-t)^s f(b)}dt \leqslant \begin{cases} \displaystyle\int_0^1 \frac{f(a)f(b)}{[f(a)+f(b)](1-t)^s}dt & t \geqslant \frac{1}{2}, \\ \displaystyle\int_0^1 \frac{f(a)f(b)}{[f(a)+f(b)]t^s}dt & t < \frac{1}{2} \end{cases}$$
$$= \begin{cases} \dfrac{1}{1-s} \cdot \dfrac{f(a)f(b)}{f(a)+f(b)} & t \geqslant \dfrac{1}{2}, \\ \dfrac{1}{1-s} \cdot \dfrac{f(a)f(b)}{f(a)+f(b)} & t < \dfrac{1}{2}. \end{cases} \tag{3.7}$$

For $s = 1$ we have

$$\int_0^1 \frac{f(a)f(b)}{tf(a)+(1-t)f(b)}dt = \int_0^1 \frac{f(a)f(b)}{f(b)+[f(a)-f(b)]t}dt$$
$$\leqslant f(a)f(b) \cdot \frac{\ln f(a) - \ln f(b)}{f(a)-f(b)}. \tag{3.8}$$

From (3.4) and (3.6-3.8), we get (3.1). The proof is complete. □

4. Fractional integral inequalities

THEOREM 4.1. *Let $f : I \subset \mathbf{R}\backslash\{0\} \to \mathbf{R}_{++}$ be a s-HH convex function with $0 < s \leqslant 1$. If $a, b \in I$ with $a < b$ and $f \in L[a,b]$. Set*

$$I_f(\alpha, a, b) = \frac{\Gamma(\alpha+1)}{2\alpha}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}+}(f \circ g)(\frac{1}{a}) + J^{\alpha}_{\frac{1}{a}-}(f \circ g)(\frac{1}{b})\right].$$

Then

$$\frac{1}{\alpha\cdot 2^{s-1}}f\left(\frac{2ab}{a+b}\right)\leqslant I_f(\alpha,a,b)$$
$$\leqslant\begin{cases}\dfrac{f(a)f(b)}{f(a)+f(b)}\cdot\begin{cases}\beta(\alpha,1-s), & 0<s<1,\alpha\geqslant 1,\\ \dfrac{1}{\alpha-s}, & 0<s<\alpha<1,\end{cases}\\ \qquad\qquad\qquad\qquad \dfrac{f(a)f(b)}{m\alpha}, \quad s=1,\alpha>0,\end{cases} \tag{4.1}$$

where $g(x)=\frac{1}{x}$ *and* $m=\min\{f(a),f(b)\}$.

Proof. A sample computation yields

$$\begin{aligned}&\int_0^1 t^{\alpha-1}f\left(\frac{ab}{ta+(1-t)b}\right)dt+\int_0^1 t^{\alpha-1}f\left(\frac{ab}{tb+(1-t)a}\right)dt\\ =&\left[\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\alpha-1}f(\frac{1}{x})dx+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}f(\frac{1}{x})dx\right]\\ =&\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b})\right]\\ =&\frac{\Gamma(\alpha+1)}{\alpha}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b})\right].\end{aligned} \tag{4.2}$$

We first prove that the left-hand inequality of (4.1) is valid. Multiplying by $t^{\alpha-1}$ in (3.2) integrating for $t\in[0,1]$, we get

$$\int_0^1 t^{\alpha-1}f\left(\frac{2ab}{a+b}\right)dt\leqslant 2^{s-1}\cdot\int_0^1 t^{\alpha-1}\frac{f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)}{2}dt. \tag{4.3}$$

From (4.2) and (4.3) we have

$$\begin{aligned}\frac{1}{\alpha}f\left(\frac{2ab}{a+b}\right)&\leqslant 2^{s-1}\frac{\Gamma(\alpha)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b})\right]\\ &=2^{s-1}\frac{\Gamma(\alpha+1)}{2\alpha}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b})\right].\end{aligned} \tag{4.4}$$

In the following, we prove the right-hand of (4.1). By (2.1) one has

$$f\left(\frac{ab}{tb+(1-t)a}\right)\leqslant\frac{f(a)f(b)}{t^sf(b)+(1-t)^sf(a)}. \tag{4.5}$$

Integrating for $t\in[0,1]$ and applying Lemma 2.2, we get for

$$\int_0^1 t^{\alpha-1}f\left(\frac{ab}{tb+(1-t)a}\right)dt$$

$$\leqslant \frac{f(a)f(b)}{f(a)+f(b)} \cdot \begin{cases} \beta(\alpha,1-s), & 0<s<1,\alpha \geqslant 1, \\ \dfrac{1}{\alpha-s}, & 0<s<\alpha<1. \end{cases} \tag{4.6}$$

Similarly, we have

$$\int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt$$
$$\leqslant \frac{f(a)f(b)}{f(a)+f(b)} \cdot \begin{cases} \beta(\alpha,1-s), & 0<s<1,\alpha \geqslant 1, \\ \dfrac{1}{\alpha-s}, & 0<s<\alpha<1. \end{cases} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.2), we obtain

$$\frac{\Gamma(\alpha+1)}{\alpha}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b})\right]$$
$$=\int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt+\int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt.$$
$$\leqslant \frac{2f(a)f(b)}{f(a)+f(b)} \cdot \begin{cases} \beta(\alpha,1-s), & 0<s<1,\alpha \geqslant 1, \\ \dfrac{1}{\alpha-s}, & 0<s<\alpha<1. \end{cases} \tag{4.8}$$

From (1.9) and by integrating for $t \in [0,1]$, we get for $s=1$

$$\int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt \leqslant \int_0^1 \frac{t^{\alpha-1}f(a)f(b)}{tf(b)+(1-t)f(a)}dt \leqslant \frac{f(a)f(b)}{\alpha m}, \tag{4.9}$$
$$\int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt \leqslant \int_0^1 \frac{t^{\alpha-1}f(a)f(b)}{tf(a)+(1-t)f(b)}dt \leqslant \frac{f(a)f(b)}{\alpha m}. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.2), we obtain for $s=1$

$$\frac{\Gamma(\alpha+1)}{\alpha}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b})\right] \leqslant \frac{2f(a)f(b)}{\alpha m}. \tag{4.11}$$

By (4.4), (4.8) and (4.11), we can derive the conclusion. So the proof is complete. □

5. k-Fractional integral inequalities

THEOREM 5.1. *Let $f: I \subset \mathbf{R}\backslash\{0\} \to \mathbf{R}_{++}$ be a s-HH convex function with $0 \leqslant s \leqslant 1$. If $a,b \in I$ with $a<b$ and $f \in L[a,b]$. Set*

$$I_f(\alpha,k,a,b)=\frac{k\Gamma_k(\alpha)}{2}\frac{ab}{b-a}\left[\left(\frac{b}{b-a}\right)^{\frac{\alpha}{k}-1}{}_kJ^{\alpha}_a g(b)+\left(\frac{a}{b-a}\right)^{\frac{\alpha}{k}-1}{}_kJ^{\alpha}_a h(b)\right].$$

Then for $\alpha > 0$ *and* $k > 0$

$$\frac{1}{2^{s-1}}\frac{k}{\alpha}f\left(\frac{2ab}{a+b}\right) \leqslant I_f(\alpha,k,a,b) \leqslant \begin{cases} \frac{f(a)f(b)}{f(a)+f(b)} \cdot \begin{cases} \beta(\frac{\alpha}{k},1-s), & 0<s<1, \frac{\alpha}{k} \geqslant 1, \\ \frac{1}{\frac{\alpha}{k}-s}, & 0<s<\frac{\alpha}{k}<1, \end{cases} \\ \frac{kf(a)f(b)}{m\alpha}, \quad s=1, \alpha>0, \end{cases} \tag{5.1}$$

where $g(x) = f(a+b-x)\left(\frac{1}{a+b-x}\right)^{\frac{\alpha}{k}+1}$, $h(x) = f(x)\left(\frac{1}{x}\right)^{\frac{\alpha}{k}+1}$.

Proof. We first prove the left-hand inequality of (5.1). Multiplying by $t^{\frac{\alpha}{k}-1}$ in (3.2) and integrating for $t \in [0,1]$, we get

$$\int_0^1 t^{\frac{\alpha}{k}-1} f\left(\frac{2ab}{a+b}\right)dt \leqslant 2^{s-1} \cdot \int_0^1 t^{\frac{\alpha}{k}-1}\frac{f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)}{2}dt. \tag{5.2}$$

Computation we get

$$\frac{k}{\alpha}f\left(\frac{2ab}{a+b}\right) \leqslant 2^{s-2}\cdot\left[\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{ab}{ta+(1-t)b}\right)dt + \int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{ab}{tb+(1-t)a}\right)dt\right]. \tag{5.3}$$

Calculate the right-hand integrals of (5.3), we have

$$\begin{aligned} &\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{ab}{ta+(1-t)b}\right)dt \\ =&\frac{ab}{b-a}\left(\frac{b}{b-a}\right)^{\frac{\alpha}{k}-1}\int_a^b (x-a)^{\frac{\alpha}{k}-1}f(x)\left(\frac{1}{x}\right)^{\frac{\alpha}{k}+1}dx \\ =&\frac{ab}{b-a}\left(\frac{b}{b-a}\right)^{\frac{\alpha}{k}-1}\int_a^b (b-u)^{\frac{\alpha}{k}-1}f(b+a-u)\left(\frac{1}{b+a-u}\right)^{\frac{\alpha}{k}+1}du \\ =&\frac{ab}{b-a}\left(\frac{b}{b-a}\right)^{\frac{\alpha}{k}-1}k\Gamma_k(\alpha)_kJ_a^\alpha g(b). \end{aligned} \tag{5.4}$$

$$\begin{aligned} \int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{ab}{tb+(1-t)a}\right)dt &= \frac{ab}{b-a}\left(\frac{a}{b-a}\right)^{\frac{\alpha}{k}-1}\int_a^b (b-x)^{\frac{\alpha}{k}-1}f(x)\left(\frac{1}{x}\right)^{\frac{\alpha}{k}+1}dx \\ &= \frac{ab}{b-a}\left(\frac{a}{b-a}\right)^{\frac{\alpha}{k}-1}k\Gamma_k(\alpha)_kJ_a^\alpha h(b). \end{aligned} \tag{5.5}$$

Substituting (5.4) and (5.5) into (5.3), we have

$$\frac{k}{\alpha} f\left(\frac{2ab}{a+b}\right) \leqslant 2^{s-1} \frac{k\Gamma_k(\alpha)}{2} \frac{ab}{b-a} \left[\left(\frac{b}{b-a}\right)^{\frac{\alpha}{k}-1} {}_kJ_a^\alpha g(b) + \left(\frac{a}{b-a}\right)^{\frac{\alpha}{k}-1} {}_kJ_a^\alpha h(b).\right]. \tag{5.6}$$

In the following, we check the right-hand inequality of (5.1). Multiplying by $t^{\frac{\alpha}{k}-1}$ in (4.5) and integrating for $t \in [0,1]$, furthermore, using Lemma 2.2, we have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt \\
&\leqslant \int_0^1 \frac{t^{\frac{\alpha}{k}-1} f(a)f(b)}{f(b)t^s + f(a)(1-t)^s} dt \\
&\leqslant \frac{f(a)f(b)}{f(a)+f(b)} \cdot \begin{cases} \beta(\frac{\alpha}{k}, 1-s), & 0<s<1, \frac{\alpha}{k} \geqslant 1, \\ \dfrac{1}{\frac{\alpha}{k}-s}, & 0<s<\dfrac{\alpha}{k}<1. \end{cases}
\end{aligned} \tag{5.7}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt \\
&\leqslant \frac{f(a)f(b)}{f(a)+f(b)} \cdot \begin{cases} \beta(\frac{\alpha}{k}, 1-s), & 0<s<1, \frac{\alpha}{k} \geqslant 1, \\ \dfrac{1}{\frac{\alpha}{k}-s}, & 0<s<\dfrac{\alpha}{k}<1. \end{cases}
\end{aligned} \tag{5.8}$$

By (5.4), (5.5), (5.7) and (5.8), we obtain

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{2} \frac{ab}{b-a} \left[\left(\frac{b}{b-a}\right)^{\frac{\alpha}{k}-1} {}_kJ_a^\alpha g(b) + \left(\frac{a}{b-a}\right)^{\frac{\alpha}{k}-1} {}_kJ_a^\alpha h(b).\right] \\
&= \int_0^1 t^{\frac{\alpha}{k}-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt + \int_0^1 t^{\frac{\alpha}{k}-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt. \\
&\leqslant \frac{2f(a)f(b)}{f(a)+f(b)} \cdot \begin{cases} \beta(\frac{\alpha}{k}, 1-s), & 0<s<1, \frac{\alpha}{k} \geqslant 1, \\ \dfrac{1}{\frac{\alpha}{k}-s}, & 0<s<\dfrac{\alpha}{k}<1. \end{cases}
\end{aligned} \tag{5.9}$$

Lastly, from (1.9) and by integrating for $t \in [0,1]$, we get for $s=1$

$$\int_0^1 t^{\frac{\alpha}{k}-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt \leqslant \int_0^1 \frac{t^{\frac{\alpha}{k}-1} f(a)f(b)}{tf(b)+(1-t)f(a)} dt \leqslant \frac{kf(a)f(b)}{\alpha m}, \tag{5.10}$$

$$\int_0^1 t^{\frac{\alpha}{k}-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt \leqslant \int_0^1 \frac{t^{\frac{\alpha}{k}-1} f(a)f(b)}{tf(a)+(1-t)f(b)} dt \leqslant \frac{kf(a)f(b)}{\alpha m}. \tag{5.11}$$

Substituting (5.10) and (5.11) into (5.4) and (5.5), we obtain for $s=1$

$$k\Gamma_k(\alpha)\frac{ab}{b-a}\left[\left(\frac{b}{b-a}\right)^{\frac{\alpha}{k}-1}{}_kJ_a^\alpha g(b)+\left(\frac{a}{b-a}\right)^{\frac{\alpha}{k}-1}{}_kJ_a^\alpha h(b)\right]\leqslant\frac{2kf(a)f(b)}{\alpha m}. \tag{5.12}$$

By (5.6), (5.9) and (5.12), we can derive the conclusion. So the proof is complete. □

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