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ROSENTHAL TYPE INEQUALITIES FOR RANDOM VARIABLES

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Abstract. It is shown that if the higher order upper Rosenthal inequality holds for the sum of random variables, then the lower order upper Rosenthal inequality also holds. The same result is also established for the maximum of partial sums of random variables. No additional assumptions are made on the random variables. As a corollary, we obtain that the upper Rosenthal inequality implies the Marcinkiewicz-Zygmund type inequality.

1. Introduction

One of the most interesting inequalities for independent random variables is the Marcinkiewicz-Zygmund inequality. It gives relations between moments of sums and moments of summands. For a sequence $\{X_i, 1 \leqslant i \leqslant n\}$ of independent random variables with mean 0 and $E|X_i|^p < \infty, 1 \leqslant i \leqslant n$, for some $p > 1$, there exist positive constants A_p and B_p depending only on p such that

$$A_p E\left(\sum_{i=1}^n X_i^2\right)^{p/2} \leqslant E\left|\sum_{i=1}^n X_i\right|^p \leqslant B_p E\left(\sum_{i=1}^n X_i^2\right)^{p/2}. \tag{1.1}$$

Using the Marcinkiewicz-Zygmund inequality, Rosenthal [10] obtained a moment inequality for independent random variables. For a sequence $\{X_i, 1 \leqslant i \leqslant n\}$ of independent random variables with mean 0 and $E|X_i|^p < \infty, 1 \leqslant i \leqslant n$, for some $p > 2$, there exist positive constants C_p and D_p depending only on p such that

$$C_p \max\left\{\sum_{i=1}^n E|X_i|^p, \left(\sum_{i=1}^n EX_i^2\right)^{p/2}\right\} \leqslant E\left|\sum_{i=1}^n X_i\right|^p \leqslant D_p \max\left\{\sum_{i=1}^n E|X_i|^p, \left(\sum_{i=1}^n EX_i^2\right)^{p/2}\right\}, \tag{1.2}$$

where $C_p = 2^{-p}$. Burkholder [3] proved that the Marcinkiewicz-Zygmund inequality holds for the maximum of partial sums of martingale differences. Using Burkholder [3]

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inequality, we can see that the Rosenthal [10] inequality also holds for the maximum of partial sums of independent random variables with mean zero and finite p-th moments for some $p > 2$. That is, there exist positive constants C_p^* and D_p^* depending only on p such that

$$C_p^* \max\left\{\sum_{i=1}^n E|X_i|^p, \left(\sum_{i=1}^n EX_i^2\right)^{p/2}\right\} \leqslant E \max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^k X_i\right|^p \leqslant D_p^* \max\left\{\sum_{i=1}^n E|X_i|^p, \left(\sum_{i=1}^n EX_i^2\right)^{p/2}\right\}. \quad (1.3)$$

Since $(|x|+|y|)/2 \leqslant \max\{|x|,|y|\} \leqslant |x|+|y|$ for real numbers x and y, the term $\max\left\{\sum_{i=1}^n E|X_i|^p, \left(\sum_{i=1}^n EX_i^2\right)^{p/2}\right\}$ in (1.2) and (1.3) can be replaced by $\sum_{i=1}^n E|X_i|^p + \left(\sum_{i=1}^n EX_i^2\right)^{p/2}$. Of course the constants in (1.2) and (1.3) may vary.

The right-hand side of (1.2) is called the p-th order upper Rosenthal inequality for the sum of random variables. The right-hand side of (1.3) is called the p-th order upper Rosenthal inequality for the maximum of partial sums of random variables.

The upper Rosenthal type inequalities have been established for dependent random variables. We refer to Shao [12] for ρ-mixing, Peligrad and Gut [9] and Utev and Peligrad [14] for ρ^*-mixing, Shao [13] for negatively associated random variables, Hu [6] and Wang et al. [17] for negatively superadditive dependent random variables, Asadian et al. [1] for negatively orthant dependent random variables, Wang and Lu [18] for asymptotically negatively associated random variables, and Yuan and An [19] for asymptotically almost negatively associated random variables.

When $1 < p < 2$, we can obtain from (1.1) that

$$E\left|\sum_{i=1}^n X_i\right|^p \leqslant B_p \sum_{i=1}^n E|X_i|^p,\ 1 < p < 2. \quad (1.4)$$

The inequality (1.4) is called the p-th order Marcinkiewicz-Zygmund type inequality. It is known that if the q-th order Marcinkiewicz-Zygmund type inequality holds for the truncated and centered random variables $\{X_i(x) - EX_i(x), 1 \leqslant i \leqslant n, x > 0\}$, where $X_i(x) = X_i I(|X_i| \leqslant x) + xI(X_i > x) - xI(X_i < -x)$, then the $p(1 < p < q)$-th order Marcinkiewicz-Zygmund type inequality also holds for the random variables $\{X_i - EX_i, 1 \leqslant i \leqslant n\}$ (see Chen and Sung [4] and Fazekas and Pecsora [5]). Note that no additional assumptions are made on the random variables $\{X_i\}$. Inspired by the above result, it is natural to consider that the above result holds for the Rosenthal inequality.

In this paper, we prove that if the q-th $(q > 2)$ order upper Rosenthal inequality holds for the sum of random variables, then the $p(2 \leqslant p < q)$-th order upper Rosenthal inequality also holds. We also prove the same result for the maximum of partial sums of random variables. As a corollary, we obtain that the upper Rosenthal inequality implies the Marcinkiewicz-Zygmund type inequality.

Throughout this paper, $I(A)$ denotes the indicator function of the event A.

2. The main results

The following theorem shows that if the higher order upper Rosenthal inequality holds for the sum of random variables, then the lower order upper Rosenthal inequality also holds.

THEOREM 2.1. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables with finite means. Assume that for some* $q > 2$, *there exists a positive function* $\alpha_q(x) \geqslant 1$ *such that*

$$E\left|\sum_{i=1}^{n}(X_i(x) - EX_i(x))\right|^q$$
$$\leqslant \alpha_q(n)\left\{\sum_{i=1}^{n} E\,|X_i(x)|^q + \left(\sum_{i=1}^{n} E\,|X_i(x)|^2\right)^{q/2}\right\},\ \forall n \geqslant 1, \forall x > 0, \tag{2.1}$$

where $X_i(x) = X_i I(|X_i| \leqslant x) + xI(X_i > x) - xI(X_i < -x)$. *Then for any* p *with* $2 \leqslant p < q$, *there exists a positive constant* C *depending only on* p *and* q *such that*

$$E\left|\sum_{i=1}^{n}(X_i - EX_i)\right|^p \leqslant C\alpha_q(n)\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right\}.$$

Proof. Without loss of generality, we may assume that $E|X_i|^p < \infty$ for $1 \leqslant i \leqslant n$. Let

$$A = \left\{x > 0:\ \sum_{i=1}^{n} E|X_i(x)|^q \geqslant \left(\sum_{i=1}^{n} E|X_i(x)|^2\right)^{q/2}\right\},$$
$$B = \left\{x > 0:\ \sum_{i=1}^{n} E|X_i(x)|^q < \left(\sum_{i=1}^{n} E|X_i(x)|^2\right)^{q/2}\right\}.$$

Set $Y_i(x) = X_i - X_i(x)$ for all $i \geqslant 1$ and $x > 0$. We proceed with two cases.

Case 1. B is an unbounded set.

By the Lebesgue convergence theorem, Hölder's inequality, and (2.1), we get that

$$E\left|\sum_{i=1}^{n}(X_i - EX_i)\right|^p = \lim_{x\to\infty, x\in B} E\left|\sum_{i=1}^{n}(X_i(x) - EX_i(x))\right|^p$$

$$\begin{aligned}
&\leqslant \limsup_{x\to\infty, x\in B}\left(E\left|\sum_{i=1}^{n}(X_i(x)-EX_i(x))\right|^q\right)^{p/q}\\
&\leqslant \alpha_q(n)^{p/q}\limsup_{x\to\infty, x\in B}\left(\sum_{i=1}^{n}E|X_i(x)|^q+\left(\sum_{i=1}^{n}E|X_i(x)|^2\right)^{q/2}\right)^{p/q}\\
&\leqslant (2\alpha_q(n))^{p/q}\limsup_{x\to\infty, x\in B}\left(\sum_{i=1}^{n}E|X_i(x)|^2\right)^{p/2}\\
&=(2\alpha_q(n))^{p/q}\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}\\
&\leqslant 2^{p/q}\alpha_q(n)\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2},
\end{aligned}$$

since $\alpha_q(n)\geqslant 1$.

Case 2. B is a bounded set.

Set $m=\sup_{x\in B}x$. Then $m\in B$ or $m\notin B$. We now proceed with two subcases.

Subcase 2.1. $m\in B$.

By the c_r-inequality, we have that

$$\begin{aligned}
E\left|\sum_{i=1}^{n}(X_i-EX_i)\right|^p &\leqslant 2^{p-1}\left\{E\left|\sum_{i=1}^{n}(X_i(m)-EX_i(m))\right|^p+E\left|\sum_{i=1}^{n}(Y_i(m)-EY_i(m))\right|^p\right\}\\
&:=2^{p-1}(I_1+I_2).
\end{aligned}$$

By Hölder's inequality, (2.1), and $m\in B$, we have that

$$\begin{aligned}
I_1 &\leqslant \left(E\left|\sum_{i=1}^{n}(X_i(m)-EX_i(m))\right|^q\right)^{p/q}\\
&\leqslant \alpha_q(n)^{p/q}\left(\sum_{i=1}^{n}E|X_i(m)|^q+\left(\sum_{i=1}^{n}E|X_i(m)|^2\right)^{q/2}\right)^{p/q}\\
&\leqslant (2\alpha_q(n))^{p/q}\left(\sum_{i=1}^{n}E|X_i(m)|^2\right)^{p/2}\\
&\leqslant (2\alpha_q(n))^{p/q}\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}.
\end{aligned}$$

For I_2, we first observe that $Y_i(m)I(|X_i|\leqslant x)=0$ if $0<x\leqslant m$ and $Y_i(m)I(|X_i|\leqslant x)=X_i(x)-X_i(m)+(m-x)I(X_i>x)+(x-m)I(X_i<-x)$ if $x>m$. We also observe that $Y_i(m)I(|X_i|>x)=(X_i-m)I(X_i>x\vee m)+(X_i+m)I(X_i<-(x\vee m))$, where $x\vee y=$

$\max\{x,y\}$. It follows that

$$
\begin{aligned}
I_2 =& p\int_0^\infty x^{p-1} P\left(\left|\sum_{i=1}^n (Y_i(m)-EY_i(m))\right| > x\right) dx \\
\leqslant& p\int_0^\infty x^{p-1} P\left(\left|\sum_{i=1}^n (Y_i(m)I(|X_i|\leqslant x)-EY_i(m)I(|X_i|\leqslant x))\right| > x/2\right) dx \\
&+ p\int_0^\infty x^{p-1} P\left(\left|\sum_{i=1}^n (Y_i(m)I(|X_i|> x)-EY_i(m)I(|X_i|> x))\right| > x/2\right) dx \\
=& p\int_m^\infty x^{p-1} P\left(\left|\sum_{i=1}^n (Y_i(m)I(|X_i|\leqslant x)-EY_i(m)I(|X_i|\leqslant x))\right| > x/2\right) dx \\
&+ p\int_0^\infty x^{p-1} P\left(\left|\sum_{i=1}^n (Y_i(m)I(|X_i|> x)-EY_i(m)I(|X_i|> x))\right| > x/2\right) dx \\
\leqslant& p\int_m^\infty x^{p-1} P\left(\left|\sum_{i=1}^n (X_i(x)-EX_i(x))\right| > x/8\right) dx \\
&+ p\int_m^\infty x^{p-1} P\left(\left|\sum_{i=1}^n (X_i(m)-EX_i(m))\right| > x/8\right) dx \\
&+ p\int_m^\infty x^{p-1} P\left(\left|\sum_{i=1}^n ((m-x)I(X_i>x)-E(m-x)I(X_i>x))\right| > x/8\right) dx \\
&+ p\int_m^\infty x^{p-1} P\left(\left|\sum_{i=1}^n ((x-m)I(X_i<-x)-E(x-m)I(X_i<-x))\right| > x/8\right) dx \\
&+ p\int_0^\infty x^{p-1} P\left(\left|\sum_{i=1}^n ((X_i-m)I(X_i>x\vee m)-E(X_i-m)I(X_i>x\vee m))\right| > x/4\right) dx \\
&+ p\int_0^\infty x^{p-1} P\left(\left|\sum_{i=1}^n ((X_i+m)I(X_i<-(x\vee m))-E(X_i+m)I(X_i<-(x\vee m)))\right| > x/4\right) dx \\
:=& I_{21}+I_{22}+I_{23}+I_{24}+I_{25}+I_{26}.
\end{aligned}
$$

For I_{21}, we note that $x\in A$ if $x>m$. It follows by Markov's inequality, (2.1), and the Fubini theorem that

$$
\begin{aligned}
I_{21} &\leqslant p8^q \int_m^\infty x^{p-q-1} E\left|\sum_{i=1}^n (X_i(x)-EX_i(x))\right|^q dx \\
&\leqslant p8^q \alpha_q(n) \int_m^\infty x^{p-q-1}\left\{\sum_{i=1}^n E|X_i(x)|^q + \left(\sum_{i=1}^n E|X_i(x)|^2\right)^{q/2}\right\} dx \\
&\leqslant 2p8^q\alpha_q(n)\int_m^\infty x^{p-q-1}\sum_{i=1}^n E|X_i(x)|^q\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= 2p8^q\alpha_q(n)\sum_{i=1}^{n}\int_m^\infty x^{p-q-1}\left\{E|X_i|^q I(|X_i|\leqslant x) + x^q P(|X_i|>x)\right\}\,dx \\
&= 2p8^q\alpha_q(n)\sum_{i=1}^{n}\left\{E\left[|X_i|^q\int_{max\{m,|X_i|\}}^\infty x^{p-q-1}\,dx\right] + E\left[I(|X_i|>m)\int_m^{|X_i|} x^{p-1}\,dx\right]\right\} \\
&= 2p8^q\alpha_q(n)\sum_{i=1}^{n}\Big[\frac{1}{q-p}E|X_i|^p I(|X_i|>m) + \frac{m^{p-q}}{q-p}E|X_i|^q I(|X_i|\leqslant m) \\
&\quad + \frac{1}{p}E|X_i|^p I(|X_i|>m) - \frac{m^p}{p}P(|X_i|>m)\Big] \\
&\leqslant 2p8^q\left(\frac{1}{q-p}+\frac{1}{p}\right)\alpha_q(n)\sum_{i=1}^{n}E|X_i|^p,
\end{aligned}
$$

since $m^{p-q}E|X_i|^q I(|X_i|\leqslant m)\leqslant E|X_i|^p I(|X_i|\leqslant m)$.

For I_{22}, we note that $2m\in A$. It follows by Markov's inequality and (2.1) that

$$
\begin{aligned}
I_{22} &\leqslant p8^q\int_m^\infty x^{p-q-1}E\left|\sum_{i=1}^{n}(X_i(m)-EX_i(m))\right|^q\,dx \\
&\leqslant p8^q\alpha_q(n)\int_m^\infty x^{p-q-1}\left\{\sum_{i=1}^{n}E|X_i(m)|^q + \left(\sum_{i=1}^{n}E|X_i(m)|^2\right)^{q/2}\right\}\,dx \\
&\leqslant p8^q\alpha_q(n)\int_m^\infty x^{p-q-1}\left\{\sum_{i=1}^{n}E|X_i(2m)|^q + \left(\sum_{i=1}^{n}E|X_i(2m)|^2\right)^{q/2}\right\}\,dx \\
&\leqslant 2p8^q\alpha_q(n)\int_m^\infty x^{p-q-1}\sum_{i=1}^{n}E|X_i(2m)|^q\,dx \\
&\leqslant 2p8^q(2m)^{q-p}\alpha_q(n)\sum_{i=1}^{n}E|X_i|^p\int_m^\infty x^{p-q-1}\,dx \\
&= 2p8^q2^{q-p}\frac{1}{q-p}\alpha_q(n)\sum_{i=1}^{n}E|X_i|^p.
\end{aligned}
$$

By Markov's inequality, we obtain

$$
\begin{aligned}
I_{23} &\leqslant 8p\int_m^\infty x^{p-2}E\left|\sum_{i=1}^{n}((m-x)I(X_i>x)-E(m-x)I(X_i>x))\right|\,dx \\
&\leqslant 16p\sum_{i=1}^{n}\int_m^\infty x^{p-2}(x-m)P(X_i>x)\,dx \\
&\leqslant 16p\sum_{i=1}^{n}\int_m^\infty x^{p-1}P(X_i>x)\,dx
\end{aligned}
$$

and

$$
I_{24}\leqslant 8p\int_m^\infty x^{p-2}E\left|\sum_{i=1}^{n}((x-m)I(X_i<-x)-E(x-m)I(X_i<-x))\right|\,dx
$$

$$\leqslant 16p\sum_{i=1}^{n}\int_{m}^{\infty}x^{p-2}(x-m)P(X_i<-x)\,dx$$
$$\leqslant 16p\sum_{i=1}^{n}\int_{m}^{\infty}x^{p-1}P(X_i<-x)\,dx,$$

which imply that

$$I_{23}+I_{24}\leqslant 16p\sum_{i=1}^{n}\int_{0}^{\infty}x^{p-1}P(|X_i|>x)\,dx=16\sum_{i=1}^{n}E|X_i|^p.$$

By Markov's inequality, we obtain

$$\begin{aligned}
I_{25}&\leqslant 4p\int_{0}^{\infty}x^{p-2}E\left|\sum_{i=1}^{n}((X_i-m)I(X_i>x\vee m)-E(X_i-m)I(X_i>x\vee m))\right|\,dx\\
&\leqslant 8p\sum_{i=1}^{n}\int_{0}^{\infty}x^{p-2}E\,|X_i-m|\,I(X_i>x\vee m)\,dx\\
&=8p\sum_{i=1}^{n}\int_{0}^{m}x^{p-2}E(X_i-m)I(X_i>m)\,dx+8p\sum_{i=1}^{n}\int_{m}^{\infty}x^{p-2}E(X_i-m)I(X_i>x)\,dx\\
&\leqslant 8p\sum_{i=1}^{n}\int_{0}^{m}x^{p-2}E|X_i|I(X_i>m)\,dx+8p\sum_{i=1}^{n}\int_{m}^{\infty}x^{p-2}E|X_i|I(X_i>x)\,dx
\end{aligned}$$

and

$$\begin{aligned}
I_{26}&\leqslant 4p\int_{0}^{\infty}x^{p-2}E\left|\sum_{i=1}^{n}((X_i+m)I(X_i<-(x\vee m))-E(X_i+m)I(X_i<-(x\vee m)))\right|\,dx\\
&\leqslant 8p\sum_{i=1}^{n}\int_{0}^{\infty}x^{p-2}E\,|X_i+m|\,I(X_i<-(x\vee m))\,dx\\
&=8p\sum_{i=1}^{n}\int_{0}^{m}x^{p-2}E(-X_i-m)I(X_i<-m)\,dx\\
&\quad+8p\sum_{i=1}^{n}\int_{m}^{\infty}x^{p-2}E(-X_i-m)I(X_i<-x)\,dx\\
&\leqslant 8p\sum_{i=1}^{n}\int_{0}^{m}x^{p-2}E|X_i|I(X_i<-m)\,dx+8p\sum_{i=1}^{n}\int_{m}^{\infty}x^{p-2}E|X_i|I(X_i<-x)\,dx,
\end{aligned}$$

which imply that

$$\begin{aligned}
I_{25}+I_{26}&\leqslant 8p\sum_{i=1}^{n}\int_{0}^{m}x^{p-2}E|X_i|I(|X_i|>m)\,dx+8p\sum_{i=1}^{n}\int_{m}^{\infty}x^{p-2}E|X_i|I(|X_i|>x)\,dx\\
&\leqslant 8p\sum_{i=1}^{n}\int_{0}^{\infty}x^{p-2}E|X_i|I(|X_i|>x)\,dx\\
&=\frac{8p}{p-1}\sum_{i=1}^{n}E|X_i|^p.
\end{aligned}$$

Thus, we get

$$
\begin{aligned}
&E\left|\sum_{i=1}^{n}(X_i-EX_i)\right|^p\\
&\leqslant 2^{p-1}\Bigg\{(2\alpha_q(n))^{p/q}\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}\\
&\qquad+\left(2p8^q\left(\frac{1}{q-p}+\frac{1}{p}\right)\alpha_q(n)+2p8^q2^{q-p}\frac{1}{q-p}\alpha_q(n)+16+\frac{8p}{p-1}\right)\sum_{i=1}^{n}E|X_i|^p\Bigg\}\\
&\leqslant C\alpha_q(n)\left\{\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}\right\},
\end{aligned}
$$

where $C=\left(2^{p+3q}+2^{4q}\right)\frac{p}{q-p}+\frac{2^{p+2}p}{p-1}+2^{p+3q}+2^{p+3}$.

Subcase 2.2. $m\notin B$.

In this case, m is a point of accumulation of the set B. Hence, there exists an increasing sequence $\{m_k\}$ in B such that $m/2\leqslant m_k\uparrow m$. Then $m_k+m/2\in A$ and $(m_k+m/2)/m_k\leqslant 2$.

The proof is similar to that of Subcase 2.1. By the c_r-inequality, we have that, for all $k\geqslant 1$,

$$
\begin{aligned}
E\left|\sum_{i=1}^{n}(X_i-EX_i)\right|^p &\leqslant 2^{p-1}\left\{E\left|\sum_{i=1}^{n}(X_i(m_k)-EX_i(m_k))\right|^p+E\left|\sum_{i=1}^{n}(Y_i(m_k)-EY_i(m_k))\right|^p\right\}\\
&:=2^{p-1}(J_1(k)+J_2(k)).
\end{aligned}
$$

By Hölder's inequality, (2.1), and $m_k\in B$, we have that

$$
\begin{aligned}
J_1(k)&\leqslant\left(E\left|\sum_{i=1}^{n}(X_i(m_k)-EX_i(m_k))\right|^q\right)^{p/q}\\
&\leqslant\alpha_q(n)^{p/q}\left(\sum_{i=1}^{n}E|X_i(m_k)|^q+\left(\sum_{i=1}^{n}E|X_i(m_k)|^2\right)^{q/2}\right)^{p/q}\\
&\leqslant(2\alpha_q(n))^{p/q}\left(\sum_{i=1}^{n}E|X_i(m_k)|^2\right)^{p/2}\\
&\leqslant(2\alpha_q(n))^{p/q}\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}.
\end{aligned}
$$

Since $J_2(k)$ is the same as I_2 except that m is replaced by m_k, we have that

$J_2(k)$

$$
\begin{aligned}
&\leqslant p\int_{m_k}^{\infty} x^{p-1}P\left(\left|\sum_{i=1}^{n}(X_i(x)-EX_i(x))\right|>x/8\right)dx\\
&\quad +p\int_{m_k}^{\infty} x^{p-1}P\left(\left|\sum_{i=1}^{n}(X_i(m_k)-EX_i(m_k))\right|>x/8\right)dx\\
&\quad +p\int_{m_k}^{\infty} x^{p-1}P\left(\left|\sum_{i=1}^{n}((m_k-x)I(X_i>x)-E(m_k-x)I(X_i>x))\right|>x/8\right)dx\\
&\quad +p\int_{m_k}^{\infty} x^{p-1}P\left(\left|\sum_{i=1}^{n}((x-m_k)I(X_i<-x)-E(x-m_k)I(X_i<-x))\right|>x/8\right)dx\\
&\quad +p\int_{0}^{\infty} x^{p-1}P\left(\left|\sum_{i=1}^{n}((X_i-m_k)I(X_i>x\vee m_k)-E(X_i-m_k)I(X_i>x\vee m_k))\right|>x/4\right)dx\\
&\quad +p\int_{0}^{\infty} x^{p-1}P\left(\left|\sum_{i=1}^{n}((X_i+m_k)I(X_i<-(x\vee m_k))\right.\right.\\
&\qquad\qquad\qquad \left.\left. -E(X_i+m_k)I(X_i<-(x\vee m_k)))\right|>x/4\right)dx\\
&:=J_{21}(k)+J_{22}(k)+J_{23}(k)+J_{24}(k)+J_{25}(k)+J_{26}(k).
\end{aligned}
$$

By the definition of $J_{21}(k)$, we obtain that

$$
\begin{aligned}
J_{21}(k)&=p\int_{m_k}^{m} x^{p-1}P\left(\left|\sum_{i=1}^{n}(X_i(x)-EX_i(x))\right|>x/8\right)\,dx+I_{21}\\
&\leqslant p\int_{m_k}^{m} x^{p-1}\,dx+I_{21}\\
&=m^p-m_k^p+I_{21}.
\end{aligned}
$$

It follows that

$$
\limsup_{k\to\infty} J_{21}(k)\leqslant I_{21}\leqslant 2p8^q\left(\frac{1}{q-p}+\frac{1}{p}\right)\alpha_q(n)\sum_{i=1}^{n}E|X_i|^p.
$$

Since $m_k+m/2\in A$ and $(m_k+m/2)/m_k\leqslant 2$, we have by Markov's inequality and (2.1) that

$$
\begin{aligned}
J_{22}(k)&\leqslant p8^q\int_{m_k}^{\infty} x^{p-q-1}E\left|\sum_{i=1}^{n}(X_i(m_k)-EX_i(m_k))\right|^q\,dx\\
&\leqslant p8^q\alpha_q(n)\int_{m_k}^{\infty} x^{p-q-1}\left\{\sum_{i=1}^{n}E|X_i(m_k)|^q+\left(\sum_{i=1}^{n}E|X_i(m_k)|^2\right)^{q/2}\right\}\,dx\\
&\leqslant p8^q\alpha_q(n)\int_{m_k}^{\infty} x^{p-q-1}\left\{\sum_{i=1}^{n}E|X_i(m_k+m/2)|^q+\left(\sum_{i=1}^{n}E|X_i(m_k+m/2)|^2\right)^{q/2}\right\}\,dx
\end{aligned}
$$

$$\leqslant 2p8^q\alpha_q(n)\int_{m_k}^{\infty} x^{p-q-1}\sum_{i=1}^{n} E|X_i(m_k+m/2)|^q\,dx$$
$$\leqslant 2p8^q\left(m_k+\frac{m}{2}\right)^{q-p}\alpha_q(n)\sum_{i=1}^{n}E|X_i|^p\int_{m_k}^{\infty}x^{p-q-1}\,dx$$
$$= 2p8^q\left(m_k+\frac{m}{2}\right)^{q-p}\alpha_q(n)\sum_{i=1}^{n}E|X_i|^p\frac{{m_k}^{p-q}}{q-p}$$
$$\leqslant 2p8^q2^{q-p}\frac{1}{q-p}\alpha_q(n)\sum_{i=1}^{n}E|X_i|^p.$$

As in Subcase 2.1,

$$J_{23}(k)+J_{24}(k)\leqslant 16\sum_{i=1}^{n}E|X_i|^p$$

and

$$J_{25}(k)+J_{26}(k)\leqslant \frac{8p}{p-1}\sum_{i=1}^{n}E|X_i|^p.$$

Thus, the upper bound of $E\left|\sum_{i=1}^{n}(X_i-EX_i)\right|^p$ is the same as that of Subcase 2.1. □

The following corollary shows that Theorem 2.1 still holds for $1<p<2$. But in this case, the term of the sum of second moments has disappeared. That is, the upper Rosenthal inequality implies the Marcinkiewicz-Zygmund type inequality.

COROLLARY 2.1. *Let $\{X_n, n\geqslant 1\}$ be a sequence of random variables with finite means. Assume that (2.1) holds for some $q>2$ and $\alpha_q(n)\geqslant 1$. Then for any p with $1<p<2$, there exists a positive constant C depending only on p and q such that*

$$E\left|\sum_{i=1}^{n}(X_i-EX_i)\right|^p\leqslant C\alpha_q(n)\sum_{i=1}^{n}E|X_i|^p.$$

Proof. Let $y>0$ be fixed and let $Y_i=X_i(y)$. Then $Y_i(x)=X_i(x)$ if $x\leqslant y$ and $Y_i(x)=X_i(y)$ if $x>y$. By the assumption, (2.1) also holds for $Y_i(x)$. By Theorem 2.1, there exists a positive constant D depending only on q such that

$$E\left|\sum_{i=1}^{n}(X_i(y)-EX_i(y))\right|^2\leqslant D\alpha_q(n)\sum_{i=1}^{n}E\,|X_i(y)|^2. \tag{2.2}$$

Since D is independent of y, (2.2) holds for all $y>0$. By Theorem 2.1 of Chen and Sung [4], we can see that

$$\left|\sum_{i=1}^{n}(X_i-EX_i)\right|^p\leqslant\left(4D\alpha_q(n)\frac{2}{2-p}+\frac{4}{p-1}\right)\sum_{i=1}^{n}E|X_i|^p$$

$$\leqslant \left(\frac{8D}{2-p}+\frac{4}{p-1}\right)\alpha_q(n)\sum_{i=1}^{n}E|X_i|^p.$$

Thus the proof is completed. □

A collection of random variables is k-wise independent if every k random variables of the collection are independent. When $k=2$, it is also called pairwise independent. The k-wise independent random variables are used in computer science for derandomizing algorithms because they can be constructed with less randomness than fully independent random variables (Berger and Rompel [2] and Motwani et al. [7]). They are also used in cryptography because k-wise independent permutations allow perfect secrecy if one allows k queries to the encryption oracle (Russell and Wang [11] and Vaudenay [15, 16]).

COROLLARY 2.2. *Let $1<p<4$ and let $\{X_n, n\geqslant 1\}$ be a sequence of 4-wise independent random variables with $E|X_n|^p<\infty$ for $n\geqslant 1$. Then there exists a positive constant C depending only on p such that*

$$E\left|\sum_{i=1}^{n}(X_i-EX_i)\right|^p \leqslant \begin{cases} C\sum_{i=1}^{n}E|X_i|^p, & 1<p<2,\\ C\left\{\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}\right\}, & 2\leqslant p<4.\end{cases}$$

Proof. Since $\{X_n, n\geqslant 1\}$ is a sequence of 4-wise independent random variables, we have

$$\begin{aligned}
&E\left|\sum_{i=1}^{n}(X_i(x)-EX_i(x))\right|^4\\
=&\sum_{i=1}^{n}E\left|X_i(x)-EX_i(x)\right|^4+\sum_{1\leqslant i\neq j\leqslant n}E\left|X_i(x)-EX_i(x)\right|^2E\left|X_j(x)-EX_j(x)\right|^2\\
\leqslant&\sum_{i=1}^{n}E\left|X_i(x)-EX_i(x)\right|^4+\left(\sum_{i=1}^{n}E\left|X_i(x)-EX_i(x)\right|^2\right)^2\\
\leqslant&2^4\sum_{i=1}^{n}E\left|X_i(x)\right|^4+\left(\sum_{i=1}^{n}E\left|X_i(x)\right|^2\right)^2.
\end{aligned}$$

Hence (2.1) holds with $q=4$ and $\alpha_q(n)=16$. If $1<p<2$, then the result follows from Corollary 2.1. If $2\leqslant p<4$, then the result follows from Theorem 2.1. □

REMARK 2.1. Corollary 2.2 does not hold for 2-wise (pairwise) independent random variables (see Theorem 3.1 in Pass and Spektor [8]).

The following theorem shows that if the higher order upper Rosenthal inequality holds for the maximum of partial sums of random variables, then the lower order upper Rosenthal inequality also holds.

THEOREM 2.2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables with finite means. Assume that for some* $q > 2$, *there exists a positive function* $\beta_q(x) \geqslant 1$ *such that*

$$
\begin{aligned}
&E \max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i(x)-EX_i(x))\right|^q \\
&\leqslant \beta_q(n)\left\{\sum_{i=1}^{n}E\,|X_i(x)|^q+\left(\sum_{i=1}^{n}E\,|X_i(x)|^2\right)^{q/2}\right\},\ \forall n\geqslant 1, \forall x>0, \qquad (2.3)
\end{aligned}
$$

where $X_i(x) = X_i I(|X_i| \leqslant x) + xI(X_i > x) - xI(X_i < -x)$. *Then for any* p *with* $2 \leqslant p < q$, *there exists a positive constant* C *depending only on* p *and* q *such that*

$$
E \max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i-EX_i)\right|^p \leqslant C\beta_q(n)\left\{\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}\right\}.
$$

Proof. The proof is similar to that of Theorem 2.1. Without loss of generality, we may assume that $E|X_i|^p < \infty$ for $1 \leqslant i \leqslant n$. Let

$$
A=\left\{x>0:\ \sum_{i=1}^{n}E|X_i(x)|^q\geqslant\left(\sum_{i=1}^{n}E|X_i(x)|^2\right)^{q/2}\right\},
$$

$$
B=\left\{x>0:\ \sum_{i=1}^{n}E|X_i(x)|^q<\left(\sum_{i=1}^{n}E|X_i(x)|^2\right)^{q/2}\right\}.
$$

Set $Y_i(x) = X_i - X_i(x)$ for all $i \geqslant 1$ and $x > 0$. We proceed with two cases.

Case 1. B is an unbounded set.

By the Lebesgue convergence theorem, Hölder's inequality, and (2.3), we get that

$$
\begin{aligned}
E \max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i-EX_i)\right|^p &= \lim_{x\to\infty, x\in B} E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{n}(X_i(x)-EX_i(x))\right|^p \\
&\leqslant \limsup_{x\to\infty, x\in B}\left(E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{n}(X_i(x)-EX_i(x))\right|^q\right)^{p/q} \\
&\leqslant \beta_q(n)^{p/q}\limsup_{x\to\infty, x\in B}\left(\sum_{i=1}^{n}E|X_i(x)|^q+\left(\sum_{i=1}^{n}E|X_i(x)|^2\right)^{q/2}\right)^{p/q} \\
&\leqslant (2\beta_q(n))^{p/q}\limsup_{x\to\infty, x\in B}\left(\sum_{i=1}^{n}E|X_i(x)|^2\right)^{p/2} \\
&= (2\beta_q(n))^{p/q}\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}
\end{aligned}
$$

$$\leqslant 2^{p/q}\beta_q(n)\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}.$$

Case 2. B is a bounded set.

The proof of this case is similar to that of Theorem 2.1 and is omitted. □

The following corollary shows that Theorem 2.2 still holds for $1<p<2$. But in this case, the term of the sum of second moments has disappeared.

COROLLARY 2.3. *Let* $\{X_n, n\geqslant 1\}$ *be a sequence of random variables with finite means. Assume that (2.3) holds for some* $q>2$ *and* $\beta_q(n)\geqslant 1$. *Then for any* p *with* $1<p<2$, *there exists a positive constant* C *depending only on* p *and* q *such that*

$$E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(X_i-EX_i)\right|^p\leqslant C\beta_q(n)\sum_{i=1}^{n}E|X_i|^p.$$

Proof. The proof is the same as that of Corollary 2.1 except that Theorem 2.1 of Chen and Sung [4] is replaced by Theorem 2.2 of Chen and Sung [4]. □

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