

## A MAPPING ASSOCIATED TO $h$ -CONVEX VERSION OF THE HERMITE–HADAMARD INEQUALITY WITH APPLICATIONS

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*Abstract.* This paper deals with a real mapping  $L(t)$  related to the  $h$ -convex version of the Hermite-Hadamard inequality. In special cases some generalized form of the Hermite-Hadamard inequality for convex functions are obtained. Also, as an application some inequalities for special means are given.

### 1. Introduction

The following integral inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, \quad (1)$$

where  $f : [a, b] \rightarrow \mathbb{R}$  is convex, are known in the literature as Hermite-Hadamard inequality. Without exaggeration this inequality is the first fundamental result related to convex functions and so usually for any newly introduced generalized convex function in the literature obtaining the corresponding version of this inequality is one of the most important aims.

On the other hand, one of the most known generalization of convex functions is the concept of  $h$ -convex functions related to the nonnegative real functions which in 2006 has been introduced in [11] by S. Varošanec. The class of  $h$ -convex functions ( $SX(h, I)$ ) is including a large class of nonnegative functions defined on interval  $I$  in  $\mathbb{R}$  such as nonnegative convex functions, Godunova-Levin functions [5],  $s$ -convex functions in the second sense [1] and  $P$ -functions [4].

The following definition is a modified version of the concept of  $h$ -convex functions introduced in [11], which is used in this paper:

**DEFINITION 1.1.** Let  $h : [0, 1] \rightarrow \mathbb{R}^+$  be a function such that  $h \not\equiv 0$ . We say that  $f : I \rightarrow \mathbb{R}$  is a  $h$ -convex function, if for all  $x, y \in I$ ,  $\lambda \in [0, 1]$  we have

$$f(\lambda x + (1 - \lambda)y) \leq h(\lambda)f(x) + h(1 - \lambda)f(y).$$

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The Hermite-Hadamard inequality related to  $h$ -convex functions has been introduced in [10] by M. Z. Sarikaya et. al. as the following:

**THEOREM 1.1.** *Let  $f \in SX(h, I)$ ,  $a, b \in I$ , with  $a < b$  and  $f \in L^1([a, b])$ . Then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq [f(a) + f(b)] \int_0^1 h(t)dt, \quad (2)$$

For other inequalities in connection to Hermite-Hadamard inequality we invite interested readers to study [2]-[4], [6]-[8] and references therein.

In this paper we investigate some generalized Hermite-Hadamard type inequalities by the use of a real mapping  $L(t)$  related to the  $h$ -convex functions. In special cases we obtain some generalized form of Hermite-Hadamard inequality for convex functions. As an application we give some inequalities for special means.

## 2. Main results

### The mapping $L$ .

For a function  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  with  $a, b \in I$ ,  $a < b$  and  $f \in L^1[a, b]$ , the mapping  $L : [0, 1] \rightarrow \mathbb{R}$  defined by

$$L(t) := \frac{1}{2(b-a)} \int_a^b [f(ta + (1-t)x) + f((1-t)x + tb)] dx, \quad (3)$$

has been introduced in [2] and the following result is obtained for it.

**THEOREM 2.1.** *Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a convex mapping on  $I$  and  $a, b$  are as above. Then:*

- (i)  $L$  is convex on  $[0, 1]$ .
- (ii) We have the inequalities:

$$L(t) \leq \frac{1-t}{b-a} \int_a^b f(x)dx + t \cdot \frac{f(a) + f(b)}{2} \leq \frac{f(a) + f(b)}{2},$$

for all  $t \in [0, 1]$  and

$$\sup_{t \in [0, 1]} L(t) = \frac{f(a) + f(b)}{2}.$$

In the following result, we give a generalization for assertions (i) and (ii) of Theorem 2.1. Also we obtain new Hermite-Hadamard type inequalities in connection with  $L(t)$  which extend (1) and (2) to a generalized form.

**THEOREM 2.2.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is a  $h$ -convex function with  $h(\frac{1}{2}) > 0$ , then:*

(i)  $L$  is  $h$ -convex on  $[0, 1]$ .

(ii) The following inequalities hold:

$$\begin{aligned} L(t) &\leq h(t) \cdot \frac{f(a) + f(b)}{2} + \frac{h(1-t)}{b-a} \int_a^b f(x) dx \\ &\leq \left[ h(t) + 2h(1-t) \int_0^1 h(u) du \right] \frac{f(a) + f(b)}{2}. \end{aligned} \tag{4}$$

(iii) For  $0 \leq t < \frac{1}{2}$ , we have

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) &\leq \frac{1-t}{\frac{1}{2}-t} L(t) - \frac{1}{(b-a)(1-2t)} \int_a^b f(x) dx \\ &\leq [(f(ta + (1-t)b) + f(tb + (1-t)a))] \int_0^1 h(u) du, \end{aligned} \tag{5}$$

and for  $\frac{1}{2} < t < 1$ ,

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) &\leq \frac{1-t}{t-\frac{1}{2}} L(t) - \frac{1}{(b-a)(2t-1)} \int_a^b f(x) dx \\ &\leq [f(a) + f(b)] \int_0^1 h(u) du. \end{aligned} \tag{6}$$

Furthermore for  $t = \frac{1}{2}$  we have

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq L\left(\frac{1}{2}\right) \leq [f(a) + f(b)] \int_0^1 h(t) dt.$$

*Proof.*

(i) Consider  $t_1, t_2, \alpha, \beta \in [0, 1]$  with  $t_1 + t_2 = 1$ . It follows from  $h$ -convexity of  $f$  that

$$\begin{aligned} &L(t_1\alpha + t_2\beta) \\ &= \frac{1}{2(b-a)} \int_a^b \left[ f((t_1\alpha + t_2\beta)a + (1-t_1\alpha - t_2\beta)x) \right. \\ &\quad \left. + f((1-t_1\alpha - t_2\beta)x + (t_1\alpha + t_2\beta)b) \right] dx \\ &= \frac{1}{2(b-a)} \int_a^b \left[ f(t_1(\alpha a + x - \alpha x) + t_2(\beta a + x - \beta x)) \right. \\ &\quad \left. + f(t_1(x - \alpha x + \alpha b) + t_2(x - \beta x + \beta b)) \right] dx \\ &\leq \frac{h(t_1)}{2(b-a)} \int_a^b \left[ f(\alpha a + (1-\alpha)x) + f((1-\alpha)x + \alpha b) \right] dx \\ &\quad + \frac{h(t_2)}{2(b-a)} \int_a^b \left[ f(\beta a + (1-\beta)x) + f((1-\beta)x + \beta b) \right] dx \\ &= h(t_1)L(\alpha) + h(t_2)L(\beta). \end{aligned}$$

- (ii) The left part of (4) follows from  $h$ -convexity of  $f$ . For the right part of (4), by the use of (2) we have:

$$\begin{aligned} & h(t) \cdot \frac{f(a) + f(b)}{2} + \frac{h(1-t)}{b-a} \int_a^b f(x) dx \\ & \leq h(t) \cdot \frac{f(a) + f(b)}{2} + h(1-t)[f(a) + f(b)] \int_0^1 h(u) du \\ & \leq [h(t) + 2h(1-t) \int_0^1 h(u) du] \frac{f(a) + f(b)}{2}. \end{aligned}$$

- (iii) If we consider two changes of variable  $u = ta + (1-t)x$  and  $u = (1-t)x + tb$  in (3) respectively, then we have

$$L(t) = \frac{1}{2(1-t)(b-a)} \left[ \int_a^{ta+(1-t)b} f(u) du + \int_{(1-t)a+tb}^b f(u) du \right]. \quad (7)$$

With some calculations we obtain from (7) that

$$2(1-t)(b-a)L(t) = \begin{cases} \int_a^b f(u) du + \int_{tb+(1-t)a}^{ta+(1-t)b} f(u) du, & 0 \leq t \leq \frac{1}{2}; \\ \int_a^b f(u) du + \int_{ta+(1-t)b}^{tb+(1-t)a} f(u) du, & \frac{1}{2} \leq t < 1, \end{cases}$$

which implies that

$$\frac{1-t}{\frac{1}{2}-t} L(t) - \frac{1}{(b-a)(1-2t)} \int_a^b f(x) dx = \frac{1}{(b-a)(1-2t)} \int_{tb+(1-t)a}^{ta+(1-t)b} f(u) du, \quad (8)$$

for  $0 \leq t < \frac{1}{2}$  and

$$\frac{1-t}{t-\frac{1}{2}} L(t) - \frac{1}{(b-a)(2t-1)} \int_a^b f(x) dx = \frac{1}{(b-a)(2t-1)} \int_{ta+(1-t)b}^{tb+(1-t)a} f(u) du, \quad (9)$$

for  $\frac{1}{2} < t < 1$ .

On the other hand from  $h$ -convexity of  $f$ , inequality (2) and the fact that

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{ta + (1-t)b + tb + (1-t)a}{2}\right)$$

for all  $t \in [0, 1]$ , we get

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) & \leq \frac{1}{(b-a)(1-2t)} \int_{tb+(1-t)a}^{ta+(1-t)b} f(u) du \\ & \leq [f(tb + (1-t)a) + f(ta + (1-t)b)] \int_0^1 h(u) du, \end{aligned} \quad (10)$$

for  $0 \leq t < \frac{1}{2}$  and

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{(b-a)(2t-1)} \int_{ta+(1-t)b}^{tb+(1-t)a} f(u)du \\ &\leq [f(ta+(1-t)b) + f(tb+(1-t)a)] \int_0^1 h(u)du, \end{aligned} \tag{11}$$

for  $\frac{1}{2} < t < 1$ . By putting (10), (11) in (8), (9) respectively, we obtain the desired results.

Finally for the case that  $t = \frac{1}{2}$  we only need to the following identity:

$$\begin{aligned} L\left(\frac{1}{2}\right) &= \frac{1}{2(b-a)} \int_a^b \left[ f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] dx \\ &= \frac{1}{(b-a)} \left[ \int_a^{\frac{a+b}{2}} f(t)dt + \int_{\frac{a+b}{2}}^b f(t)dt \right] = \frac{1}{(b-a)} \int_a^b f(t)dt. \end{aligned}$$

REMARK 1. Assertions (i) and (ii) of Theorem 2.2 are respectively generalization of assertions (i) and (ii) of Theorem 3 in [2], from convex to  $h$ -convex version.

COROLLARY 2.1. [9] *If in assertion (iii) of Theorem 2.2 we consider that  $f$  is convex, then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1-t}{\frac{1}{2}-t}L(t) - \frac{1}{(b-a)(1-2t)} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

for  $0 \leq t < \frac{1}{2}$  and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1-t}{t-\frac{1}{2}}L(t) - \frac{1}{(b-a)(2t-1)} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

for  $\frac{1}{2} < t < 1$ . In special case if  $t = 0$ , then we recapture (1).

EXAMPLE 1. The following means for real numbers  $a, b \in \mathbb{R}$  are known:

$$\begin{aligned} A(a, b) &= \frac{a+b}{2} && \text{arithmetic mean,} \\ L_n(a, b) &= \left[ \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right]^{\frac{1}{n}} && \text{generalized log-mean, } n \in \mathbb{R}, a < b. \end{aligned} \tag{12}$$

In Theorem 2.2, for  $0 \leq a \leq b$  consider

$$\begin{cases} f(x) = x^r, & r \in (-\infty, -1) \cup (-1, 0] \cup [1, \infty); \\ h(t) = t^s, & s \leq 1. \end{cases}$$

From Example 7 in [11],  $f$  is  $h$ -convex and also with some calculations we have

$$L(t) = \frac{1}{2(b-a)(1-t)(r+1)} \times [(ta + (1-t)b)^{r+1} - (tb + (1-t)a)^{r+1} + b^{r+1} - a^{r+1}],$$

which implies that

$$\begin{aligned} \frac{1-t}{\frac{1}{2}-t}L(t) &= \frac{1}{(b-a)(1-2t)(r+1)} \\ &\times [(ta + (1-t)b)^{r+1} - (tb + (1-t)a)^{r+1} + b^{r+1} - a^{r+1}]. \end{aligned} \tag{13}$$

On the other hand it is clear that

$$\frac{1}{(b-a)(1-2t)} \int_a^b f(x)dx = \frac{b^{r+1} - a^{r+1}}{(b-a)(1-2t)(r+1)}. \tag{14}$$

So from (13) and (14) we obtain that

$$\begin{aligned} \frac{1-t}{\frac{1}{2}-t}L(t) - \frac{1}{(b-a)(1-2t)} \int_a^b f(x)dx \\ = \frac{1}{(b-a)(1-2t)(r+1)} \times [(ta + (1-t)b)^{r+1} - (tb + (1-t)a)^{r+1}]. \end{aligned} \tag{15}$$

Finally, from (5), (12) and (15) for  $0 \leq t < \frac{1}{2}$ , we get the following special means type inequality

$$\begin{aligned} 2^{s-1}A^r(a, b) &\leq L_r^r\left((tb + (1-t)a), (ta + (1-t)b)\right) \\ &\leq \frac{2[t^s + (1-t)^s]}{s+1}A\left((ta + (1-t)b)^r, (tb + (1-t)a)^r\right). \end{aligned} \tag{16}$$

Also with the same argument as above for  $\frac{1}{2} < t < 1$ , from (6) we have

$$\begin{aligned} 2^{s-1}A^r(a, b) &\leq L_r^r\left((ta + (1-t)b), (tb + (1-t)a)\right) \\ &\leq \frac{2[t^s + (1-t)^s]}{s+1}A\left((ta + (1-t)b)^r, (tb + (1-t)a)^r\right). \end{aligned} \tag{17}$$

If in inequalities (16) and (17) we consider  $s = 1$ , then for  $r \in (-\infty, -1) \cup (-1, 0] \cup [1, \infty)$  we obtain that

$$\begin{cases} A^r(a, b) \leq L_r^r\left((tb + (1-t)a), (ta + (1-t)b)\right) \leq A(a^r, b^r), 0 \leq t < \frac{1}{2}; \\ A^r(a, b) \leq L_r^r\left((ta + (1-t)b), (tb + (1-t)a)\right) \leq A(a^r, b^r), \frac{1}{2} < t < 1. \end{cases}$$

In a more special case for  $t = 0$  from above, we recapture the following well known special means inequality

$$A^r(a, b) \leq L_r^r(a, b) \leq A(a^r, b^r),$$

where  $0 \leq a < b$  and  $r \in [1, \infty)$ .

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