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HARDY INEQUALITIES AND CAFFARELLI–KOHN–NIRENBERG INEQUALITIES WITH RADIAL DERIVATIVE

TUAN DUY NGUYEN, LE-LONG PHI* AND WEIJIA YIN

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Abstract. In this paper, we study several inequalities of Hardy and Caffarelli-Kohn-Nirenberg type. We set up some optimal versions of these inequalities using the radial derivatives or the convex combinations of the full gradient and its radial part. We also exhibit their optimizers in some certain cases.

1. Introduction

Let $N \geqslant 3$ and Ω be a domain in $\mathbb{R}^N$ containing 0. The classical Hardy inequality that plays an important role in many areas such as analysis, probability and partial differential equations says that for all $u \in C_0^\infty(\Omega)$:

$$\left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx \leqslant \int_\Omega |\nabla u|^2 \, dx. \tag{1.1}$$

It is well-known that the constant $\left(\frac{N-2}{2}\right)^2$ is sharp but the equality in (1.1) is never happened by nontrivial functions. See, for instance, [27, 38] for historical backgrounds and some standard references on Hardy inequalities [3, 26, 28, 42].

It was showed in [23] that the operator $-\Delta - \left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2}$ is critical on the whole space $\mathbb{R}^N$ in the sense that there is no strictly positive $W \in V^1((0,\infty))$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \geqslant \int_{\mathbb{R}^N} W(|x|) \, |u|^2 \, dx$$

holds for all $u \in C_0^\infty\left(\mathbb{R}^N\right)$. However, the situation on bounded domain is different. Indeed, to study the stability of singular solutions of certain nonlinear elliptic equations,

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* Corresponding author.

Brezis and Vázquez established in [4] the following Hardy type inequality: for all $u \in W_0^{1,2}(\Omega)$:

$$\int_\Omega |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx \geqslant z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_\Omega |u|^2 dx. \tag{1.2}$$

Here ω_N is the volume of the unit ball and $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$. The constant $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is optimal when Ω is a ball. However, $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is not achieved in $W_0^{1,2}(\Omega)$. Actually $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_\Omega |u|^2 dx$ is just a first term of an infinite series of extra terms that can be added to the RHS of (1.2). See, for instance, [1, 11, 13, 18, 19, 20, 24, 35, 36].

In [21], Ghoussoub and Moradifam introduced the notion of Bessel pair and established the following general Hardy inequality with radial weights:

THEOREM A. *Let* $0 < R \leqslant \infty$, V *and* W *be positive* C^1*−functions on* $(0,R)$ *such that* $\int_0^R \frac{1}{r^{N-1}V(r)} dr = \infty$ *and* $\int_0^R r^{N-1} V(r)\, dr < \infty$. *Then the following are equivalent:*

(1) (V,W) *is a* N*−dimensional Bessel pair on* $(0,R)$.

(2) There exists $c > 0$ *such that* $\int_{B_R} V(x) |\nabla u|^2 dx \geqslant c \int_{B_R} W(x) |u|^2 dx$ *for all* $u \in C_0^\infty(B_R)$.

Moreover, the largest c *for which the inequality holds is equal to* $\beta(V,W;R)$.

Here a couple of C^1-functions (V,W) is a N-dimensional Bessel pair on $(0,R)$ if there exists $c > 0$ such that the ordinary differential equation

$$y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)}\right) y'(r) + \frac{cW(r)}{V(r)} y(r) = 0$$

has a positive solution on the interval $(0,R)$. Also, $\beta(V,W;R)$ is defined as the supremum of such c. See the book [22] for various examples of the N-dimensional Bessel pair.

Motivated by the general Hardy inequalities with Bessel pair in [21], and the recent results in [29, 30], our first aim of this note is to study an improved version of Theorem A using the radial derivative. We note that the following improved version of (1.1): for all $u \in C_0^\infty(\mathbb{R}^N)$

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leqslant \int_{\mathbb{R}^N} \left|\frac{x}{|x|} \cdot \nabla u\right|^2 dx \tag{1.3}$$

has been set up in, for instance, [37]. The constant $\left(\frac{N-2}{2}\right)^2$ is still optimal. In the polar coordinate, $\left|\frac{x}{|x|} \cdot \nabla u\right| = |\partial_r u(r\omega)|$. Hence $\frac{x}{|x|} \cdot \nabla u$ is actually the radial derivative of u.

Obviously, (1.3) is (1.1) when u is radial. Actually, the Hardy type inequalities with radial derivative have been studied extensively. See [25, 34, 37, 39, 40, 41, 44, 45], for example. It is also worthy to mention that radial derivative plays an important role in the study of several functional inequalities on homogeneous groups. We refer the interested reader to the monograph [43] for this subject.

Our first result can be read as follows:

THEOREM 1.1. *Let $p>1$, $0<R\leqslant\infty$, V and W be positive functions on $(0,R)$. Then the following are equivalent:*

(A) $\int_{B_R} V(|x|)|\nabla u|^p\,dx \geqslant \int_{B_R} W(|x|)|u|^p\,dx$ *for all* $u\in C_0^\infty(B_R)$.

(B) $\int_{B_R} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^p dx \geqslant \int_{B_R} W(|x|)|u|^p\,dx$ *for all* $u\in C_0^\infty(B_R)$.

(C) $\int_{B_R} V(|x|)|\nabla u|^p\,dx \geqslant \int_{B_R} W(|x|)|u|^p\,dx$ *for all radial functions* $u\in C_0^\infty(B_R)$.

Combining Theorem 1.1 and Theorem A, we get

COROLLARY 1.1. *Let $0<R\leqslant\infty$, V and W be positive C^1-functions on $(0,R)$ such that* $\int_0^R \frac{1}{r^{N-1}V(r)}dr=\infty$ *and* $\int_0^R r^{N-1}V(r)\,dr<\infty$. *Then the following are equivalent:*

(1) (V,W) *is a $N-$dimensional Bessel pair on* $(0,R)$.

(2) There exists $c>0$ such that $\int_{B_R} V(x)|\nabla u|^2\,dx \geqslant c\int_{B_R} W(x)|u|^2\,dx$ *for all* $u\in C_0^\infty(B_R)$.

(3) There exists $c>0$ such that $\int_{B_R} V(x)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx \geqslant c\int_{B_R} W(x)|u|^2\,dx$ *for all* $u\in C_0^\infty(B_R)$.

(4) There exists $c>0$ such that $\int_{B_R} V(x)|\nabla u|^2\,dx \geqslant c\int_{B_R} W(x)|u|^2\,dx$ *for all radial functions* $u\in C_0^\infty(B_R)$.

Moreover, the largest c for which the inequality holds is equal to $\beta(V,W;R)$.

We have the following optimal Hardy type inequalities as some examples of our results: Let $a\leqslant\frac{N-2}{2}$, then for all $u\in C_0^\infty(B_R)$:

$$\int_{B_R}\frac{\left|\frac{x}{|x|}\cdot\nabla u\right|^2}{|x|^{2a}}dx \geqslant \left(\frac{N-2a-2}{2}\right)^2\int_{B_R}\frac{|u|^2}{|x|^{2a+2}}dx + z_0^2\omega_N^{\frac{2}{N}}|B_R|^{-\frac{2}{N}}\int_{B_R}\frac{|u|^2}{|x|^{2a}}dx,$$

$$\int_{B_R} \frac{\left|\frac{x}{|x|}\cdot\nabla u\right|^2}{|x|^{2a}}dx \geqslant \left(\frac{N-2a-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^{2a+2}}dx + \frac{1}{4}\sum_{j=1}^{k}\int_{B_R} \frac{|u|^2}{|x|^{2a+2}} \left(\prod_{i=1}^{j} \log^{(i)} \frac{\rho}{|x|}\right)^{-2} dx$$

for every integer k, where $\rho > R\left(e^{e^{e^{\cdot^{\cdot^{e(k-\text{times})}}}}}\right)$,

$$\int_{B_R} \frac{\left|\frac{x}{|x|}\cdot\nabla u\right|^2}{|x|^{2a}}dx \geqslant \left(\frac{N-2a-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^{2a+2}}dx + \frac{1}{4}\sum_{j=1}^{\infty}\int_{B_R} \frac{|u|^2}{|x|^{2a+2}} X_1^2\left(\frac{|x|}{R}\right)\ldots X_j^2\left(\frac{|x|}{R}\right)dx.$$

Here $X_1(t) = (1-\log t)^{-1}$, $X_k(t) = X_1(X_{k-1}(t))$.

In the same line of thought, we will also investigate in this paper the optimal Caffarelli-Kohn-Nirenberg (CKN) inequalities with the radial derivative. Here, we will use the following form of the CKN inequality (see [46]):

$$\left(\int_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s}\right)^{1/r} \leqslant C\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\right)^{a/p} \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\right)^{(1-a)/q} \tag{CKN}$$

where

$$a = \frac{[(N-\theta)r-(N-s)q]\,p}{[(N-\theta)p-(N-\mu-p)q]\,r}.$$

The CKN inequalities were first introduced in 1984 by Caffarelli, Kohn and Nirenberg in their celebrated work [5]. It is worth noting that many well-known and important inequalities such as Gagliardo-Nirenberg inequalities, Sobolev inequalities, Hardy-Sobolev inequalities, Nash's inequalities, etc are just the special cases of the CKN inequalities.

Assume that we can find $d > 0$ such that $d(p+\mu-N)+N-p=0$. Denote $D^{p,q}_{\mu,\theta}(\mathbb{R}^N)$ the completion of the space of smooth compactly supported functions with the norm

$$\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\right)^{1/p} + \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\right)^{1/q}$$

and

$$GN(N,p,q,r,\mu,\theta,s) = \sup_{u\in D^{p,q}_{0,N+\theta d-Nd}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}}dx\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}}$$

where

$$a = \frac{[(N-\theta)r-(N-s)q]p}{[(N-\theta)p-(N-\mu-p)q]r}.$$

For $\alpha \geqslant 0$, we also denote

$$\mathscr{R}_\alpha(u) = \left(\alpha|\nabla u|^2 + (1-\alpha)\left|\frac{x}{|x|}\cdot\nabla u\right|^2\right)^{\frac{1}{2}}.$$

We note that $\mathscr{R}_0(u)$ is the radial derivative of u while $\mathscr{R}_1(u)$ is the standard gradient of u. Also, $\mathscr{R}_0(u) \leqslant \mathscr{R}_\alpha(u) \leqslant \mathscr{R}_1(u)$ for $0 \leqslant \alpha \leqslant 1$ and $\mathscr{R}_\alpha(u) \leqslant \alpha^{\frac{1}{2}}\mathscr{R}_1(u)$ for $\alpha \geqslant 1$. When u is radial, then $\mathscr{R}_\alpha(u) = |\nabla u|$.

Our next purpose is to show that in some situations, we can get sharp versions of the Caffarelli-Kohn-Nirenberg inequalities with radial derivative $\mathscr{R}_0(u)$. For instance, we have

THEOREM 1.2. *For all* $(a,b) \in \mathscr{A} = \left\{a<b+1,\ b\leqslant\frac{N-2}{2}\right\} \cup \left\{a>b+1,\ b\geqslant\frac{N-2}{2}\right\}$ *and* $u \in C_0^\infty\left(\mathbb{R}^N\setminus\{0\}\right)\setminus\{0\}$ *one has*

$$\left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx\right)\left(\int_{\mathbb{R}^N}\frac{|\mathscr{R}_0(u)|^2}{|x|^{2b}}dx\right) - \left(\frac{|N-a-b-1|}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx\right)^2$$
$$= \frac{\int_{\mathbb{R}^N}\left|\left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx\right)\frac{(\mathscr{R}_0(u))x}{|x|^{b+1}} - \left(\frac{N-a-b-1}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx\right)\frac{x}{|x|^{a+1}}u\right|^2 dx}{\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx}.$$

As a consequence,

$$\left(\frac{|N-a-b-1|}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx\right)^2 \leqslant \left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx\right)\left(\int_{\mathbb{R}^N}\frac{|\mathscr{R}_0(u)|^2}{|x|^{2b}}dx\right). \tag{1.4}$$

The constant $\frac{|N-a-b-1|}{2}$ *is optimal.*

As a by-product, we can obtain the following sharp CKN inequalities in [12]:

$$\left(\frac{|N-a-b-1|}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx\right)^2 \leqslant \left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx\right)\left(\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}dx\right). \tag{1.5}$$

The constant $\frac{|N-a-b-1|}{2}$ is optimal. Moreover, optimizers for (1.5), if exist, must be radial since in this situation $|\mathscr{R}_0(u)| = |\nabla u|$. Actually, it was proved in [7] that the extremizers for (1.5) is of the form $D\exp\left(\frac{t}{b+1-a}|x|^{b+1-a}\right)$. Now since we are just using the radial derivative $\mathscr{R}_0$ in (1.4), it is an interesting and challenging problem to investigate the symmetry and symmetry breaking phenomena for (1.4). This is indeed our next goal. More precisely, using spherical harmonic decomposition, we will show that indeed all the optimizers for (1.4) have to be radially symmetric.

THEOREM 1.3. *Let u be an optimizer of (1.4). Then u is radially symmetric and hence is of the form* $D\exp\left(\frac{t}{b+1-a}|x|^{b+1-a}\right)$.

However, there are many situations where we could not replace the standard gradient by the radial derivative in the CKN inequality. For instance, if $GN(N,p,q,r,\mu,\theta,s)$ can be attained by nonradial optimizers (such as in [14, 15]), then we could not expect that

$$\left(\int_{\mathbb{R}^N}\frac{|u|^r}{|x|^{N+sd-Nd}}dx\right)^{1/r} \tag{1.6}$$
$$\leqslant GN(N,p,q,r,\mu,\theta,s)\left(\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla u\right|^p dx\right)^{\frac{a}{p}}\left(\int_{\mathbb{R}^N}\frac{|u|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}.$$

Indeed, let U be the nonradial maximizer for $GN(N,p,q,r,\mu,\theta,s)$. If (1.6) holds true, then

$$\begin{aligned}
&GN(N,p,q,r,\mu,\theta,s)\left(\int_{\mathbb{R}^N}|\nabla U|^p dx\right)^{\frac{a}{p}}\left(\int_{\mathbb{R}^N}\frac{|U|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}\\
&=\left(\int_{\mathbb{R}^N}\frac{|U|^r}{|x|^{N+sd-Nd}}dx\right)^{1/r}\\
&\leqslant GN(N,p,q,r,\mu,\theta,s)\left(\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla U\right|^p dx\right)^{\frac{a}{p}}\left(\int_{\mathbb{R}^N}\frac{|U|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}\\
&\leqslant GN(N,p,q,r,\mu,\theta,s)\left(\int_{\mathbb{R}^N}|\nabla U|^p dx\right)^{\frac{a}{p}}\left(\int_{\mathbb{R}^N}\frac{|U|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}.
\end{aligned}$$

Hence, we must have $\int_{\mathbb{R}^N}|\nabla U|^p dx = \int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla U\right|^p dx$ which is impossible since U is nonradial.

Nevertheless, we will set up improved versions of the CKN inequalities in some special cases using the convex combination of the full gradient and its radial part

$\mathscr{R}_\alpha(u)$. Our motivation is the results in [9, 17, 31, 32, 33] where the authors studied the CKN inequalities in some special cases using suitable quasi-conformal mappings. Moreover, for some certain 1-parameter families of inequalities, the best constants and the optimizers for the CKN inequalities were calculated explicitly there. Their method is that under convenient vector fields and for some particular families of parameters, CKN inequalities can be transformed to simpler versions such as the Hardy–Sobolev inequalities and the Gagliardo–Nirenberg inequalities. Since the sharp constants and optimizers of those inequalities are easier to study, and are known in some particular classes (see, for instance, [2, 6, 8, 10, 12, 16]), they could deduce the best constants and extremizers for CKN inequalities in the corresponding regions.

Using the approach as in [33], our next aim is to show that

THEOREM 1.4. *Assume that $GN(N,p,q,r,\mu,\theta,s)$ is finite. Then for any smooth function u, we have*

$$\left(\int_{\mathbb{R}^N}|u|^r\frac{dx}{|x|^s}\right)^{1/r}\leqslant d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}GN(N,p,q,r,\mu,\theta,s)$$
$$\times\left(\int_{\mathbb{R}^N}\left(\mathscr{R}_{\frac{1}{d^2}}(u)\right)^p\frac{dx}{|x|^\mu}\right)^{\frac{a}{p}}\left(\int_{\mathbb{R}^N}|u|^q\frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}.$$

Moreover, the equality happens at V if and only if $\left(\frac{1}{d}\right)^{\frac{p-1}{p}}V(|x|^{d-1}x)$ is an optimizer for $GN(N,p,q,r,\mu,\theta,s)$.

As a byproduct, we obtain the following results that studied in [9, 33]

COROLLARY 1.2. *Assume that $GN(N,p,q,r,\mu,\theta,s)$ is finite.*

(1) If $d>1$, then for any smooth function u, we have

$$\left(\int_{\mathbb{R}^N}|u|^r\frac{dx}{|x|^s}\right)^{1/r}\leqslant d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}GN(N,p,q,r,\mu,\theta,s)$$
$$\times\left(\int_{\mathbb{R}^N}|\nabla u|^p\frac{dx}{|x|^\mu}\right)^{\frac{a}{p}}\left(\int_{\mathbb{R}^N}|u|^q\frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}.$$

(2) If $d<1$, then for any smooth function u, we have

$$\left(\int_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s}\right)^{1/r} \leqslant d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)-a} GN(N,p,q,r,\mu,\theta,s) \times \left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}.$$

Another interesting consequence of Theorem 1.4 is the symmetry and symmetry-breaking phenomena of the Cafferelli-Kohn-Nirenberg inequalities that will be studied in Section 3. More precisely, if we denote

$$CKN(N,\mu,\theta,s,p,q,r) = \sup_{u\in D^{p,q}_{\mu,\theta}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}},$$

then one has

THEOREM 1.5. *Assume that $GN(N,p,q,r,\mu,\theta,s)$ is achieved by both radial and nonradial optimizers. Then:*

(1) If $d>1$, $CKN(N,\mu,\theta,s,p,q,r)$ is attained and its maximizers are radial.

(2) If $d=1$, then $CKN(N,\mu,\theta,s,p,q,r)$ is achieved by both radial and nonradial optimizers.

(3) If $0<d<1$, then extremal functions for $CKN(N,\mu,\theta,s,p,q,r)$, if exist, are nonradial.

See [9, 33] for related results.

2. Hardy inequalities-proof of Theorem 1.1

Proof of Theorem 1.1. It is clear that we just need to show (C) $\Rightarrow$ (B). Now suppose that we have (C). Let $u\in C_0^\infty(B_R)$, we set

$$U(r) = \left(\frac{1}{|\mathbb{S}^{N-1}|}\int_{\mathbb{S}^{N-1}} |u(r\omega)|^p\, d\omega\right)^{\frac{1}{p}},$$

we get

$$p\,|U(r)|^{p-2}U(r)U'(r)=\frac{1}{|\mathbb{S}^{N-1}|}\int_{\mathbb{S}^{N-1}}p\,|u(r\omega)|^{p-2}u(r\omega)\frac{\partial u}{\partial r}(r\omega)\,d\omega.$$

Hence,

$$p\,|U(r)|^{p-1}\left|U'(r)\right|\leqslant p\,|U(r)|^{p-1}\left(\frac{1}{|\mathbb{S}^{N-1}|}\int_{\mathbb{S}^{N-1}}\left|\frac{\partial u}{\partial r}(r\omega)\right|^{p}d\omega\right)^{\frac{1}{p}}$$

and

$$\left|U'(r)\right|\leqslant\left(\frac{1}{|\mathbb{S}^{N-1}|}\int_{\mathbb{S}^{N-1}}\left|\frac{\partial u}{\partial r}(r\omega)\right|^{p}d\omega\right)^{\frac{1}{p}}.$$

So

$$\begin{aligned}\int_{B_R}V(|x|)\,|\nabla U(x)|^{p}\,dx&=\left|\mathbb{S}^{N-1}\right|\int_0^R V(r)\left|U'(r)\right|^{p}r^{N-1}dr\\&\leqslant\left|\mathbb{S}^{N-1}\right|\int_0^R V(r)\frac{1}{|\mathbb{S}^{N-1}|}\int_{\mathbb{S}^{N-1}}\left|\frac{\partial u}{\partial r}(r\omega)\right|^{p}r^{N-1}d\omega dr\\&=\int_0^R\int_{\mathbb{S}^{N-1}}V(r)\left|\frac{\partial u}{\partial r}(r\omega)\right|^{2}r^{N-1}d\omega dr=\int_{B_R}V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^{p}dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{B_R}W(|x|)\,|U(x)|^{p}\,dx&=\left|\mathbb{S}^{N-1}\right|\int_0^R W(r)\,|U(r)|^{p}\,r^{N-1}dr\\&=\left|\mathbb{S}^{N-1}\right|\int_0^R W(r)\frac{1}{|\mathbb{S}^{N-1}|}\int_{\mathbb{S}^{N-1}}|u(r\omega)|^{p}\,r^{N-1}d\omega dr\\&=\int_0^R\int_{\mathbb{S}^{N-1}}W(r)\,|u(r\omega)|^{p}\,r^{N-1}d\omega dr=\int_{B_R}W(|x|)\,|u|^{p}\,dx.\end{aligned}$$

Applying (C) for the radial function U, we obtain

$$\int_{B_R}V(|x|)\,|\nabla U(x)|^{p}\,dx\geqslant\int_{B_R}W(|x|)\,|U(x)|^{p}\,dx.$$

That is

$$\int_{B_R}V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^{p}dx\geqslant\int_{B_R}W(|x|)\,|u|^{p}\,dx.$$

3. Caffarelli-Kohn-Nirenberg inequalities with radial derivative

Let $d>0$. As in [33], we define the vector-valued function $L_{N,d}:\mathbb{R}^N\setminus\{0\}\to\mathbb{R}^N\setminus\{0\}$:

$$L_{N,d}(x):=|x|^{d-1}x$$

and the mapping $D_{N,d,p}$, $p>1$:

$$D_{N,d,p}u(x):=\left(\frac{1}{d}\right)^{\frac{p-1}{p}}u(L_{N,d}(x)). \tag{3.1}$$

It was showed that the Jacobian matrix of this function $L_{N,d}$ is

$$\mathbf{J_{L_{N,d}}}=|x|^{d-1}\mathbb{I}_N+(d-1)|x|^{d-3}\begin{pmatrix} x_1^2 & x_1x_2 & \dots & x_1x_N\\ x_2x_1 & x_2^2 & \dots & x_2x_N\\ \vdots & \vdots & \ddots & \vdots\\ x_Nx_1 & x_Nx_2 & \dots & x_N^2\end{pmatrix}.$$

and

$$\det(\mathbf{J}_{L_{N,d}})=d|x|^{N(d-1)}. \tag{3.2}$$

Moreover, under the transform $D_{N,d,p}$, it was established in [33] that: for continuous function f, we have

$$\int_{\mathbb{R}^N}\frac{f\left(\left(\frac{1}{d}\right)^{\frac{p-1}{p}}u(x)\right)}{|x|^t}dx=d\int_{\mathbb{R}^N}\frac{f\left(D_{N,d,p}u(x)\right)}{|x|^{N+td-Nd}}dx.$$

In particular, we obtain that $u\in L^s\left(\frac{dx}{|x|^t}\right)$ if and only if $D_{N,d,p}u\in L^s\left(\frac{dx}{|x|^{N+td-Nd}}\right)$. It was also proved that if $\nabla u\in L^p\left(\frac{dx}{|x|^\mu}\right)$, then $\nabla D_{N,d,p}u\in L^p\left(\frac{dx}{|x|^{d(p+\mu-N)+N-p}}\right)$ and

$$\int_{\mathbb{R}^N}\frac{|\nabla D_{N,d,p}u(x)|^p}{|x|^{d(p+\mu-N)+N-p}}dx\leqslant\int_{\mathbb{R}^N}|\nabla u|^p\frac{dx}{|x|^\mu}. \tag{3.3}$$

In order to achieve our result, we will first improve (3.3) as follows:

LEMMA 3.1. *If* $\mathscr{R}_{\frac{1}{d^2}}u\in L^p\left(\frac{dx}{|x|^\mu}\right)$, *then* $\nabla D_{N,d,p}u\in L^p\left(\frac{dx}{|x|^{d(p+\mu-N)+N-p}}\right)$. *Moreover,*

$$\int_{\mathbb{R}^N}\frac{|\nabla D_{N,d,p}u(x)|^p}{|x|^{d(p+\mu-N)+N-p}}dx=\int_{\mathbb{R}^N}\left(\mathscr{R}_{\frac{1}{d^2}}u\right)^p\frac{dx}{|x|^\mu}.$$

Proof. By direct calculations, we have

$$\frac{\partial D_{N,d,p}u}{\partial x_i}(x) = \left(\frac{1}{d}\right)^{\frac{p-1}{p}} \left(|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) + A_i\right),$$

for $i = 1, 2, ...N$, where

$$A_i := \sum_{j=1}^{N} (d-1)\,|x|^{d-3} x_i x_j \frac{\partial u}{\partial x_j}(|x|^{d-1}x).$$

Hence, we obtain

$$\begin{aligned}
&|\nabla D_{N,d,p}u(x)|^2 \\
=&\sum_{i=1}^{N}\left(\frac{\partial D_{N,d,p}u}{\partial x_i}(x)\right)^2 \\
=&d^{-2\frac{p-1}{p}}\left[\sum_{i=1}^{N}|x|^{2(d-1)}\left(\frac{\partial u}{\partial x_i}(|x|^{d-1}x)\right)^2 + \sum_{i=1}^{N}2A_i|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) + \sum_{i=1}^{N}A_i^2\right].
\end{aligned}$$

Direct computations show

$$\sum_{i=1}^{N}|x|^{2(d-1)}\left(\frac{\partial u}{\partial x_i}(|x|^{d-1}x)\right)^2 = |x|^{2(d-1)}\left|\nabla u(|x|^{d-1}x)\right|^2,$$

$$\sum_{i=1}^{N}2A_i|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) = 2\,(d-1)\,|x|^{2d-2}\left|\frac{x}{|x|}\cdot\nabla u(|x|^{d-1}x)\right|^2,$$

$$\sum_{i=1}^{N}A_i^2 = (d-1)^2\,|x|^{2d-2}\left|\frac{x}{|x|}\cdot\nabla u(|x|^{d-1}x)\right|^2.$$

Combining them together, we obtain

$$|\nabla D_{N,d,p}u(x)|^2 = d^{-2\frac{p-1}{p}}\,|x|^{2(d-1)}\left(\left|\nabla u(|x|^{d-1}x)\right|^2 + \left(d^2-1\right)\left|\frac{x}{|x|}\cdot\nabla u(|x|^{d-1}x)\right|^2\right).$$

Using the change of variables again, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\frac{|\nabla D_{N,d,p}u(x)|^p}{|x|^{d(p+\mu-N)+N-p}}dx \\
=&\frac{1}{d}\int_{\mathbb{R}^N}\frac{|\nabla D_{N,d,p}u(x)|^p}{|x|^{p(d-1)}\,||x|^{d-1}x|^{\mu}}d|x|^{N(d-1)}dx \\
=&\frac{1}{d}\int_{\mathbb{R}^N}\frac{d^{1-p}\left(\left|\nabla u(|x|^{d-1}x)\right|^2 + \left(d^2-1\right)\left|\frac{x}{|x|}\cdot\nabla u(|x|^{d-1}x)\right|^2\right)^{\frac{p}{2}}}{||x|^{d-1}x|^{\mu}}d|x|^{N(d-1)}dx
\end{aligned}$$

$$= \int_{\mathbb{R}^N} \left(\frac{1}{d^2} |\nabla u(y)|^2 + \left(1 - \frac{1}{d^2}\right) \left| \frac{y}{|y|} \cdot \nabla u(y) \right|^2 \right)^{\frac{p}{2}} \frac{dy}{|y|^\mu}$$
$$= \int_{\mathbb{R}^N} \left(\mathscr{R}_{\frac{1}{d^2}} u \right)^p \frac{dx}{|x|^\mu}.$$

We are now ready to provide a proof for Theorem 1.4.

Proof of Theorem 1.4. Denote

$$ICKN(N,\mu,\theta,s,p,q,r) = \sup_{u \in D^{p,q}_{\mu,\theta}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s} \right)^{1/r}}{\left(\int_{\mathbb{R}^N} \left(\mathscr{R}_{\frac{1}{d^2}} u \right)^p \frac{dx}{|x|^\mu} \right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta} \right)^{\frac{1-a}{q}}}.$$

By Lemma 3.1, it is easy to see that

$$\frac{\left(\int_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s} \right)^{1/r}}{\left(\int_{\mathbb{R}^N} \left(\mathscr{R}_{\frac{1}{d^2}} v \right)^p \frac{dx}{|x|^\mu} \right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta} \right)^{\frac{1-a}{q}}}$$
$$= \frac{\left(d^{1+\frac{p-1}{p}r} \int_{\mathbb{R}^N} \frac{\left|D_{N,d,p} v\right|^r}{|x|^{N+sd-Nd}} dx \right)^{1/r}}{\left(\int_{\mathbb{R}^N} \left|\nabla D_{N,d,p} v\right|^p dx \right)^{\frac{a}{p}} \left(d^{1+\frac{p-1}{p}q} \int_{\mathbb{R}^N} \frac{\left|D_{N,d,p} v\right|^q}{|x|^{N+\theta d-Nd}} dx \right)^{\frac{1-a}{q}}}.$$

As a consequence

$$ICKN(N,\mu,\theta,s,p,q,r) = d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} GN(N,p,q,r,\mu,\theta,s).$$

Also, V is a maximizer for $ICKN(N,\mu,\theta,s,p,q,r)$ if and only if $U = D_{N,d,p}V$ is an optimizer for $GN(N,p,q,r,\mu,\theta,s)$.

3.1. The case $d > 1$

We will also assume that $GN(N,p,q,r,\mu,\theta,s)$ can be achieved by radial maximizers. In this case, we will show that $CKN(N,\mu,\theta,s,p,q,r)$ can be attained, its extremal functions must be radially symmetric and

$$CKN(N,\mu,\theta,s,p,q,r) = d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} GN(N,p,q,r,\mu,\theta,s).$$

THEOREM 3.1. *Assume that $d>1$ and $GN(N,p,q,r,\mu,\theta,s)$ can be attained by radial maximizers. Then $CKN(N,\mu,\theta,s,p,q,r)$ can be attained. Moreover its extremal functions must be radially symmetric and*

$$CKN(N,\mu,\theta,s,p,q,r) = d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} GN(N,p,q,r,\mu,\theta,s).$$

Proof. Indeed, let U_0 be the radial optimizer for $GN(N,p,q,r,\mu,\theta,s)$ and let $V_0 = D_{N,d,p}^{-1} U_0$, that is $U_0 = D_{N,d,p} V_0$. Since V_0 is also radial, we obtain:

$$\frac{\left(\int |V_0|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla V_0|^p \frac{dx}{|x|^{\mu}}\right)^{\frac{a}{p}} \left(\int |V_0|^q \frac{dx}{|x|^{\theta}}\right)^{\frac{1-a}{q}}}$$

$$= \frac{\left(\int |V_0|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} \left(\mathscr{R}_{\frac{1}{d^2}}(V_0)\right)^p \frac{dx}{|x|^{\mu}}\right)^{\frac{a}{p}} \left(\int |V_0|^q \frac{dx}{|x|^{\theta}}\right)^{\frac{1-a}{q}}}$$

$$= d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\left(\int\limits_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla U_0|^p dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}}.$$

Also, for any v, we get

$$d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\left(\int\limits_{\mathbb{R}^N} \frac{\left|D_{N,d,p}v\right|^r}{|x|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} \left|\nabla D_{N,d,p}v\right|^p dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} \frac{\left|D_{N,d,p}v\right|^q}{|x|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}}$$

$$= \frac{\left(\int\limits_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} \left(\mathscr{R}_{\frac{1}{d^2}}(v)\right)^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}} \geqslant \frac{\left(\int\limits_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla v|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}.$$

Hence V_0 is an optimizer for $CKN(N,\mu,\theta,s,p,q,r)$ and $ICKN(N,\mu,\theta,s,p,q,r)$. As a consequence

$$\begin{aligned} CKN(N,\mu,\theta,s,p,q,r) &= ICKN(N,\mu,\theta,s,p,q,r) \\ &= d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} GN(N,p,q,r,\mu,\theta,s). \end{aligned}$$

Now, let V be a maximizer for $CKN(N,\mu,\theta,s,p,q,r)$:

$$\begin{aligned} CKN(N,\mu,\theta,s,p,q,r) &= \frac{\left(\int\limits_{\mathbb{R}^N} |V|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla V|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |V|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}} \\ &\leqslant \frac{\left(\int\limits_{\mathbb{R}^N} |V|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} \left(\mathscr{R}_{\frac{1}{d^2}}(V)\right)^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |V|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}} \\ &\leqslant ICKN(N,\mu,\theta,s,p,q,r). \end{aligned}$$

Hence V must be radially symmetric.

3.2. The case $0 < d < 1$

In the case $0 < d < 1$, we have

$$\int\limits_{\mathbb{R}^N} \left(\mathscr{R}_{\frac{1}{d^2}} v\right)^p \frac{dx}{|x|^\mu} \leqslant \frac{1}{d^p} \int\limits_{\mathbb{R}^N} |\nabla v|^p \frac{dx}{|x|^\mu}.$$

Hence

$$\begin{aligned} CKN(N,\mu,\theta,s,p,q,r) &\leqslant \frac{1}{d^a} ICKN(N,\mu,\theta,s,p,q,r) \\ &= d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)-a} GN(N,p,q,r,\mu,\theta,s). \end{aligned}$$

Now, if we also assume that $GN(N,p,q,r,\mu,\theta,s)$ can be attained by both radial and nonradial maximizers, then we will show that optimizers for $CKN(N,\mu,\theta,s,p,q,r)$, if exist, are not radial. In other words, the symmetry breaking phenomenon happens.

THEOREM 3.2. *Assume $0<d<1$ and $GN(N,p,q,r,\mu,\theta,s)$ can be attained by both radial and nonradial maximizers. Then optimizers for $CKN(N,\mu,\theta,s,p,q,r)$, if exist, are not radial.*

Proof. Indeed, let us assume that $CKN(N,\mu,\theta,s,p,q,r)$ can be attained by a radial maximizer V. Then we will show that $U=D_{N,d,p}V$ is a maximizer for $GN(N,p,q,r,\mu,\theta,s)$. Indeed, noting that since $GN(N,p,q,r,\mu,\theta,s)$ can be attained by radial maximizers, we have

$$\begin{aligned}&GN(N,p,q,r,\mu,\theta,s)\\&=\sup_{u\in D^{p;q}_{0,N+\theta d-Nd}(\mathbb{R}^N):u\text{ is radial}}\frac{\left(\int\limits_{\mathbb{R}^N}\frac{|u|^r}{|x|^{N+sd-Nd}}dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{\frac{a}{p}}\left(\int\limits_{\mathbb{R}^N}\frac{|u|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}}.\end{aligned}$$

Now, for any radial function u, we have with the radial function $v=D^{-1}_{N,d,p}u$, that is $u=D_{N,d,p}v$, that

$$\begin{aligned}&d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}\frac{\left(\int\limits_{\mathbb{R}^N}\frac{|u|^r}{|x|^{N+sd-Nd}}dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{\frac{a}{p}}\left(\int\limits_{\mathbb{R}^N}\frac{|u|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}}\\
&=\frac{\left(\int\limits_{\mathbb{R}^N}|v|^r\frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N}\left(\mathscr{R}_{\frac{1}{d^2}}(v)\right)^p\frac{dx}{|x|^\mu}\right)^{\frac{a}{p}}\left(\int\limits_{\mathbb{R}^N}|v|^q\frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}\\
&=\frac{\left(\int\limits_{\mathbb{R}^N}|v|^r\frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N}|\nabla v|^p\frac{dx}{|x|^\mu}\right)^{\frac{a}{p}}\left(\int\limits_{\mathbb{R}^N}|v|^q\frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}\leqslant\frac{\left(\int\limits_{\mathbb{R}^N}|V|^r\frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N}|\nabla V|^p\frac{dx}{|x|^\mu}\right)^{\frac{a}{p}}\left(\int\limits_{\mathbb{R}^N}|V|^q\frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}\end{aligned}$$

$$
= d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\left(\int_{\mathbb{R}^N} \frac{|U|^r}{|x|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla U|^p \, dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} \frac{|U|^q}{|x|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}}.
$$

As a consequence

$$
\begin{aligned}
CKN(N,\mu,\theta,s,p,q,r) &= \frac{\left(\int_{\mathbb{R}^N} |V|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla V|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |V|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}} \\
&= \frac{\left(\int_{\mathbb{R}^N} |V|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} \left(\mathscr{R}_{\frac{1}{d^2}}(V)\right)^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |V|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}} \\
&= d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\left(\int_{\mathbb{R}^N} \frac{|U|^r}{|x|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla U|^p dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} \frac{|U|^q}{|x|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}} \\
&= d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} GN(N,p,q,r,\mu,\theta,s) \\
&= ICKN(N,\mu,\theta,s,p,q,r).
\end{aligned}
$$

Now, let W be a nonradial optimizer for $GN(N,p,q,r,\mu,\theta,s)$. Hence $Z = D_{N,d,p}^{-1} W$ is a nonradial optimizer for $ICKN(N,\mu,\theta,s,p,q,r)$. Then

$$
ICKN(N,\mu,\theta,s,p,q,r) = \frac{\left(\int_{\mathbb{R}^N} |Z|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} \left(\mathscr{R}_{\frac{1}{d^2}}(Z)\right)^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |Z|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}
$$

$$
< \frac{\left(\int_{\mathbb{R}^N} |Z|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla Z|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |Z|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}
$$

$$
\leqslant CKN(N,\mu,\theta,s,p,q,r)
$$

and we get a contradiction.

3.3. Some special cases

When $N+sd-Nd = N+\theta d - Nd = 0$, then we have that $GN(N,p,q,r,\mu,\theta,s)$ can be attained by both radial and nonradial optimizers (see [14, 15, 33, 46]). Hence in this case we have: (1) If $d>1$, $CKN(N,\mu,\theta,s,p,q,r)$ is attained by radial maximizers only. (2) If $d=1$, $CKN(N,\mu,\theta,s,p,q,r)$ can be achieved by both radial and nonradial optimizers and (3) If $0<d<1$, extremal functions for $CKN(N,\mu,\theta,s,p,q,r)$, if exist, are nonradial.

Moreover, in the following very special case:

$$
\begin{aligned}
&1<p< \ p+\mu<N,\ \theta=\frac{N\mu}{N-p}=s<N \\
&1\leqslant q<r<\frac{Np}{N-p};\ a=\frac{[(N-\theta)r-(N-s)q]p}{[(N-\theta)p-(N-\mu-p)q]r} \\
&r=p\frac{q-1}{p-1},
\end{aligned}
$$

then by the result of [14, 15]:

$$
\begin{aligned}
&CKN(N,\mu,\theta,s,p,q,r) \\
=&ICKN(N,\mu,\theta,s,p,q,r) \\
=&\left(\frac{N-p}{N-p-\mu}\right)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \\
&\times\left(\frac{q-p}{p\sqrt{\pi}}\right)^{a}\left(\frac{pq}{N(q-p)}\right)^{\frac{a}{p}}\left(\frac{\delta}{pq}\right)^{\frac{1}{r}}\left(\frac{\Gamma\left(q\frac{p-1}{q-p}\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{p-1}{p}\frac{\delta}{q-p}\right)\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}}
\end{aligned}
$$

with $\delta = Np-q(N-p)$. Moreover, all the maximizers have the form

$$
V_0(x) = A\left(1+B|x|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}} \text{ for some } A\in\mathbb{R},\ B>0.
$$

Similarly, in the class

$$
1<p< \ p+\mu<N,\ \theta=\frac{N\mu}{N-p}=s<N
$$

$$1 \leqslant q < r < \frac{Np}{N-p};\ a = \frac{[(N-\theta)r-(N-s)q]p}{[(N-\theta)p-(N-\mu-p)q]r}$$
$$q = p\frac{r-1}{p-1},\ r > 2-\frac{1}{p},$$

we have with $\delta = Np - r(N-p)$

$$CKN(N,\mu,\theta,s,p,q,r)$$
$$= \left(\frac{N-p}{N-p-\mu}\right)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}$$
$$\times \left(\frac{p-r}{p\sqrt{\pi}}\right)^{a}\left(\frac{pr}{N(p-r)}\right)^{\frac{a}{p}}\left(\frac{pr}{\delta}\right)^{\frac{1-a}{q}}\left(\frac{\Gamma\left(\frac{p-1}{p}\frac{\delta}{p-r}+1\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(r\frac{p-1}{p-r}+1\right)\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}}.$$

Also all the maximizers have the form

$$V_0(x) = A\left(1-B|x|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)_+^{-\frac{p-1}{r-p}} \text{ for some } A \in \mathbb{R},\ B > 0.$$

3.4. Proof of Theorem 1.2

Proof of Theorem 1.2. We will follow [12]. First, we note that

$$\int_{\mathbb{R}^N}\left|\frac{(\mathscr{R}_0(u))x}{|x|^{b+1}}+t\frac{x}{|x|^{a+1}}u\right|^2dx$$
$$=\int_{\mathbb{R}^N}\frac{|\mathscr{R}_0(u)|^2}{|x|^{2b}}dx+t^2\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx+2t\int_{\mathbb{R}^N}u\frac{x\cdot\nabla u}{|x|^{a+b+1}}dx \geqslant 0 \text{ for every } t \in \mathbb{R}.$$

If we choose

$$t = -\frac{\int_{\mathbb{R}^N}u\frac{x\cdot\nabla u}{|x|^{a+b+1}}dx}{\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx},$$

then

$$\int_{\mathbb{R}^N}\left|\frac{(\mathscr{R}_0(u))x}{|x|^{b+1}}-\frac{\int_{\mathbb{R}^N}u\frac{x\cdot\nabla u}{|x|^{a+b+1}}dx}{\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx}\frac{x}{|x|^{a+1}}u\right|^2dx=\int_{\mathbb{R}^N}\frac{|\mathscr{R}_0(u)|^2}{|x|^{2b}}dx-\frac{\left(\int_{\mathbb{R}^N}u\frac{x\cdot\nabla u}{|x|^{a+b+1}}dx\right)^2}{\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx}.$$

In other words,

$$\left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx\right)\left(\int_{\mathbb{R}^N} \frac{|\mathscr{R}_0(u)|^2}{|x|^{2b}}dx\right)-\left(\int_{\mathbb{R}^N} u\frac{x\cdot\nabla u}{|x|^{a+b+1}}dx\right)^2$$

$$=\frac{1}{\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx}\int_{\mathbb{R}^N}\left|\left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx\right)\frac{(\mathscr{R}_0(u))x}{|x|^{b+1}}-\left(\int_{\mathbb{R}^N} u\frac{x\cdot\nabla u}{|x|^{a+b+1}}dx\right)\frac{x}{|x|^{a+1}}u\right|^2 dx$$

Also, by integration by parts, we get

$$\int_{\mathbb{R}^N} u\frac{x\cdot\nabla u}{|x|^{a+b+1}}dx=-\int_{\mathbb{R}^N} u\frac{x\cdot\nabla u}{|x|^{a+b+1}}dx-[N-a-b-1]\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}dx.$$

That is

$$\int_{\mathbb{R}^N} u\frac{x\cdot\nabla u}{|x|^{a+b+1}}dx=-\frac{N-a-b-1}{2}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}dx.$$

Hence

$$\left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx\right)\left(\int_{\mathbb{R}^N} \frac{|\mathscr{R}_0(u)|^2}{|x|^{2b}}dx\right)-\left(\frac{|N-a-b-1|}{2}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}dx\right)^2$$

$$=\frac{\int_{\mathbb{R}^N}\left|\left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx\right)\frac{(\mathscr{R}_0(u))x}{|x|^{b+1}}-\left(\frac{N-a-b-1}{2}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}dx\right)\frac{x}{|x|^{a+1}}u\right|^2 dx}{\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx}\geqslant 0.$$

Also, by [7], it is clear that $\frac{|N-a-b-1|}{2}$ is sharp.

3.5. Proof of Theorem 1.3

Proof of Theorem 1.3. We will first transform the integrals over $\mathbb{R}^N$ to integrals over the cylinder $\mathscr{C}=\mathbb{S}^{N-1}\times\mathbb{R}$ as follows: set

$$v(t,\omega)=e^{-t\frac{N-2-2b}{2}}u\left(e^{-t},\omega\right)$$

we get

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx=\int_0^{\infty}\int_{S^{N-1}}|u|^2(r,\omega)\,r^{N-1-2a}d\omega dr=\int_{\mathscr{C}}|v|^2(t,\omega)\,e^{2(a-b-1)t}d\mu,$$

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx = \int_0^\infty \int_{S^{N-1}} \left[|u_r|^2 (r,\omega) + \frac{1}{r^2} |\nabla_\omega u(r,\omega)|^2 \right] r^{N-1-2b} d\omega dr$$
$$= \int_{\mathscr{C}} |\nabla_\omega v(t,\omega)|^2 + |v_t|^2 (t,\omega) + \left(\frac{2b+2-N}{2}\right)^2 |v|^2 (t,\omega) d\mu$$

and

$$\int_{\mathbb{R}^N} \frac{\left|\frac{x}{|x|} \cdot \nabla u\right|^2}{|x|^{2b}} dx = \int_0^\infty \int_{S^{N-1}} |u_r|^2 (r,\omega) r^{N-1-2b} d\omega dr$$
$$= \int_{\mathscr{C}} |v_t|^2 (t,\omega) + \left(\frac{2b+2-N}{2}\right)^2 |v|^2 (t,\omega) d\mu.$$

Now assume that u is a nonradial optimizer of (1.4). Then v is also nonradial. Hence, if we now decompose v in terms of spherical harmonics

$$v = \sum_{k=0}^\infty v_k = \sum_{k=0}^\infty f_k(t) \phi_k(\omega),$$

there exists some $k \geqslant 1$ such that $f_k(t)$ is not identical to 0.

We observe that

$$\int_{\mathscr{C}} |\nabla_\omega v(t,\omega)|^2 + |v_t|^2 (t,\omega) + \left(\frac{2b+2-N}{2}\right)^2 |v|^2 (t,\omega) d\mu$$
$$= \sum_{k=0}^\infty \int_{\mathbb{R}} \left[|f_k'|^2 (t) + \left(\lambda_k + \left(\frac{2b+2-N}{2}\right)^2 \right) |f_k|^2 (t) \right] dt$$

and

$$\int_{\mathscr{C}} |v_t|^2 (t,\omega) + \left(\frac{2b+2-N}{2}\right)^2 |v|^2 (t,\omega) d\mu$$
$$= \sum_{k=0}^\infty \int_{\mathbb{R}} \left[|f_k'|^2 (t) + \left(\frac{2b+2-N}{2}\right)^2 |f_k|^2 (t) \right] dt.$$

Now, if we set

$$V(t,\omega) = \left(\sum_{k=0}^\infty |f_k|^2 (t) \right)^{\frac{1}{2}} \text{ and } U(r,\omega) = r^{\frac{2b+2-N}{2}} V(-\ln r, \omega),$$

then we have

$$\sum_{k=0}^\infty |f_k'|^2 (t) \geqslant \frac{\left(\sum_{k=0}^\infty f_k'(t) f_k(t) \right)^2}{\sum_{k=0}^\infty |f_k|^2 (t)} = |V_r|^2 (t,\omega)$$

and

$$\frac{\left(\sum_{k=0}^{\infty}\int_{\mathbb{R}}\left[\left|f_k'\right|^2(t)+\left(\frac{2b+2-N}{2}\right)^2|f_k|^2(t)\right]dt\right)\left(\sum_{k=0}^{\infty}\int_{\mathbb{R}}e^{2(a-b-1)t}|f_k|^2(t)\,dt\right)}{\left(\sum_{k=0}^{\infty}\int_{\mathbb{R}}e^{(a-b-1)t}|f_k|^2(t)\,dt\right)^2}$$

$$\geqslant\frac{\left(\int_{\mathbb{R}}|V_r|^2(t,\omega)+\left(\frac{2b+2-N}{2}\right)^2|V|^2(t,\omega)\,dt\right)\left(\int_{\mathbb{R}}e^{2(a-b-1)t}|V|^2(t,\omega)\,dt\right)}{\left(\int_{\mathbb{R}}e^{(a-b-1)t}|V|^2(t,\omega)\,dt\right)^2}$$

$$=\frac{\left(\int_{\mathbb{R}^N}\frac{|U|^2}{|x|^{2a}}dx\right)\left(\int_{\mathbb{R}^N}\frac{|\nabla U|^2}{|x|^{2b}}dx\right)}{\left(\int_{\mathbb{R}^N}\frac{|U|^2}{|x|^{a+b+1}}dx\right)^2}\geqslant\left(\frac{|N-a-b-1|}{2}\right)^2.$$

As a consequence, all the equalities must happen in all the above inequalities. Hence there exists some α such that

$$f_j'(t)=\alpha f_j(t)$$

for all $j\geqslant 0$ such that $f_j(t)$ is not identical to 0. That is

$$f_j(t)=C_je^{\alpha t}\text{ for some }C_j\neq 0.$$

Hence $V(t)=\left(\sum_{k=0}^{\infty}|f_k|^2(t)\right)^{\frac{1}{2}}=\left(\sum_{k=0}^{\infty}|C_k|^2\right)^{\frac{1}{2}}e^{\alpha t}$. On the other hand, U must be an optimizer for (1.5) and so by [7], $U(t)=D\exp\left(\frac{s}{b+1-a}t^{b+1-a}\right)$, $s\neq 0$. Thus, $V(t)=U(e^{-t})e^{t\frac{2b+2-N}{2}}=D\exp\left(\frac{se^{-t(b+1-a)}}{b+1-a}\right)e^{t\frac{2b+2-N}{2}}$ which is impossible.

In conclusion, optimizers for (1.4) must be radially symmetric.

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Tuan Duy Nguyen
Department of Fundamental Sciences
University of Finance-Marketing
2/4 Tran Xuan Soan St., Tan Thuan Tay Ward, Dist. 7, HCM City, Vietnam
e-mail: tuanduy2312@gmail.com, nguyenduy@ufm.edu.vn

Le-Long Phi
Institute of Research and Development
Duy Tan University
Da Nang 550000, Vietnam
e-mail: lelongphi@duytan.edu.vn

Weijia Yin
University of British Columbia
Vancouver, BC V6T1Z4, Canada
e-mail: weijia.ubc@gmail.com