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ON THE REVERSE HARDY–TYPE INTEGRAL INEQUALITIES IN THE WHOLE PLANE WITH THE EXTENDED RIEMANN–ZETA FUNCTION

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Abstract. In the present paper, using weight functions we obtain some equivalent conditions of two kinds of the reverse Hardy-type integral inequalities with a nonhomogeneous kernel in the whole plane. The constant factors, which are related to the extended Riemann zeta function, are proved to be the best possible. In the form of applications, a few equivalent conditions of two kinds of the reverse Hardy-type integral inequalities with the homogeneous kernel in the whole plane are deduced. We also consider some particular cases.

1. Introduction

If $f(x),g(y) \geqslant 0$,

$$0<\int_0^\infty f^2(x)dx<\infty \ \ and \ \ 0<\int_0^\infty g^2(y)dy<\infty,$$

then we have the following Hilbert integral inequality with the best possible constant factor π (see [1]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\pi\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(y)dy\right)^{\frac{1}{2}}. \tag{1}$$

Recently, by using weight functions, several generalizations of (1) were obtained in two books [2], [3]. Some Hilbert-type inequalities with the homogenous kernels and nonhomogenous kernels were established in [4]-[9]. In 2017, Hong [10] also presented an equivalent condition between Hilbert-type inequalities with a homogenous kernel and some parameters. Some other kinds of Hilbert-type inequalities were obtained in [11]-[18]. Most of these inequalities are constructed in the quarter plane of the first quadrant.

In 2007, Yang [19] established the following Hilbert-type integral inequality in the whole plane:

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda}dxdy<B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_{-\infty}^\infty e^{-\lambda x}f^2(x)dx\int_{-\infty}^\infty e^{-\lambda y}g^2(y)dy\right)^{\frac{1}{2}}, \tag{2}$$

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where the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ is the best possible ($\lambda > 0$, $B(u,v)$ stands for the beta function) (see [35]). He et al. [20]-[33] proved some new Hilbert-type integral inequalities in the whole plane with the best possible constant factors.

In this paper, we obtain a few equivalent conditions of two kinds of the reverse Hardy-type integral inequalities with a nonhomogeneous kernel in the whole plane. The constant factors related to the extended Riemann zeta function are proved to be the best possible. In the form of applications, a few equivalent conditions of two kinds of the reverse Hardy-type integral inequalities with a homogeneous kernel in the whole plane are deduced (see also [34]). We also consider some particular cases.

2. Two lemmas and an example

EXAMPLE 1. Setting

$$h(xy) := \frac{|\ln|xy||^{\beta}}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \quad (x,y \in \mathbf{R}),$$

and then

$$h(u) = \frac{|\ln|u||^{\beta}}{(\max\{|u|,1\})^{\lambda-1}|u-1|} \quad (u \in \mathbf{R}),$$

for $\beta,\sigma > 0, \lambda \in \mathbf{R}$, it follows that

$$\begin{aligned} K^{(1)}(\sigma) &:= \int_0^1 \frac{|\ln u|^{\beta} u^{\sigma-1}}{(\max\{u,1\})^{\lambda-1}} \left(\frac{1}{u+1} + \frac{1}{|u-1|}\right) du \\ &= \int_0^1 (-\ln u)^{\beta} \left(\frac{1}{u+1} + \frac{1}{1-u}\right) u^{\sigma-1} du \\ &= 2\int_0^1 (-\ln u)^{\beta} \frac{u^{\sigma-1}}{1-u^2} du = 2\int_0^1 (-\ln u)^{\beta} \sum_{k=0}^{\infty} u^{2k+\sigma-1} du. \end{aligned}$$

By the Lebesgue term by term integration theorem (cf. [37]), we have

$$\begin{aligned} K^{(1)}(\sigma) &= 2\sum_{k=0}^{\infty} \int_0^1 (-\ln u)^{\beta} u^{2k+\sigma-1} du = 2\sum_{k=0}^{\infty} \frac{1}{(2k+\sigma)^{\beta+1}} \int_0^{\infty} v^{\beta} e^{-v} dv \\ &= \frac{\Gamma(\beta+1)}{2^{\beta}} \zeta(\beta+1,\frac{\sigma}{2}) \in \mathbf{R}_+, \end{aligned} \tag{3}$$

where

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (Re(s) > 1; a > 0)$$

is the extended Riemann zeta function. Note that $\zeta(s,1) = \sum_{k=1}^{\infty} \frac{1}{k^s}$, $Re(s) > 1$ is the Riemann zeta function (cf. [35]).

Similarly, for $\beta > 0, \sigma < \lambda$, we find

$$K^{(2)}(\sigma) := \int_1^{\infty} \frac{|\ln u|^{\beta} u^{\sigma-1}}{(\max\{u,1\})^{\lambda-1}} \left(\frac{1}{u+1} + \frac{1}{|u-1|}\right) du$$

$$= \int_0^1 \frac{|\ln u|^\beta u^{\lambda-\sigma-1}}{(\max\{u,1\})^{\lambda-1}} \left(\frac{1}{u+1} + \frac{1}{|u-1|} \right) du$$
$$= \frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1, \frac{\lambda-\sigma}{2}) \in \mathbf{R}_+. \tag{4}$$

In the sequel, we shall always have that $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $\sigma_1, \sigma, \lambda \in \mathbf{R}=(-\infty,\infty)$, $\beta > 0$, and $M_1, M_2 > 0$.

LEMMA 1. *If $\sigma > 0$ and for any nonnegative measurable functions $f(x)$, $g(y)$ in $\mathbf{R}$, the following inequality*

$$\int_{-\infty}^{\infty} g(y) \left[\int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \frac{|\ln|xy||^\beta f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \right] dy$$
$$\geqslant M_1 \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy \right]^{\frac{1}{q}} \tag{5}$$

holds true, then we have

$$\sigma_1 = \sigma \ \textit{and} \ K^{(1)}(\sigma) \geqslant M_1.$$

Proof. If $\sigma_1 < \sigma$, then for $n \in \mathbf{N}$, we set the following two functions:

$$f_n(x) := \begin{cases} |x|^{\sigma+\frac{1}{pn}-1}, 0 < |x| \leqslant 1 \\ 0, |x| > 1 \end{cases},$$
$$g_n(y) := \begin{cases} 0, 0 < |y| < 1 \\ |y|^{\sigma_1-\frac{1}{qn}-1}, y \geqslant 1 \end{cases},$$

and derive that

$$J_1 := \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f_n^p(x)dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g_n^q(y)dy \right]^{\frac{1}{q}}$$
$$= \left[\int_{-1}^{1} |x|^{p(1-\sigma)-1} |x|^{p(\sigma+\frac{1}{pn}-1)} dx \right]^{\frac{1}{p}} \times \left[\int_{\{y;|y|\geqslant 1\}} |y|^{q(1-\sigma_1)-1} |y|^{q(\sigma_1-\frac{1}{qn}-1)} dy \right]^{\frac{1}{q}}$$
$$= \left(2\int_0^1 x^{\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left(2\int_1^\infty y^{-\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = 2n.$$

We obtain

$$I_1 := \int_{-\infty}^{\infty} g_n(y) \left[\int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \frac{|\ln|xy||^\beta f_n(x)}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} dx \right] dy$$
$$= \int_{-\infty}^{-1} \left[\int_{\frac{1}{y}}^{\frac{-1}{y}} \frac{|\ln|xy||^\beta |x|^{\sigma+\frac{1}{pn}-1}}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} dx \right] (-y)^{\sigma_1-\frac{1}{qn}-1} dy$$

$$
\begin{aligned}
&+\int_1^{\infty}\left[\int_{\frac{-1}{y}}^{\frac{1}{y}}\frac{|\ln|xy||^{\beta}|x|^{\sigma+\frac{1}{pn}-1}}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}dx\right]y^{\sigma_1-\frac{1}{qn}-1}dy\\
&=\int_1^{\infty}\left[\int_{\frac{-1}{y}}^{\frac{1}{y}}(h(-xy)+h(xy))|x|^{\sigma+\frac{1}{pn}-1}dx\right]y^{\sigma_1-\frac{1}{qn}-1}dy\\
&=\int_1^{\infty}\left[\int_{-1}^{1}(h(-u)+h(u))|u|^{\sigma+\frac{1}{pn}-1}du\right]y^{\sigma_1-\sigma-\frac{1}{n}-1}dy, \qquad (6)
\end{aligned}
$$

and then by (5) we have

$$
\begin{aligned}
&2K^{(1)}\left(\sigma+\frac{1}{pn}\right)\frac{1}{\sigma-\sigma_1+\frac{1}{n}}\\
&=\left[\int_{-1}^{1}(h(-u)+h(u))|u|^{(\sigma+\frac{1}{pn})-1}du\right]\int_1^{\infty}y^{\sigma_1-\sigma-\frac{1}{n}-1}dy\\
&=I_1\geqslant M_1J_1=2M_1n. \qquad (7)
\end{aligned}
$$

Since $\{(h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}\}_{n=1}^{\infty}$ $(u\in(0,1))$ is a nonnegative and increasing sequence, by Levi's theorem (cf. [37]), it follows that

$$
\begin{aligned}
K^{(1)}\left(\sigma+\frac{1}{pn}\right)&=\int_0^1(h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du\\
&\to\int_0^1(h(-u)+h(u))u^{\sigma-1}du=K^{(1)}(\sigma)(n\to\infty),
\end{aligned}
$$

and thus by (7) we get that

$$
\infty>\frac{2K^{(1)}(\sigma)}{\sigma-\sigma_1}\geqslant\infty,
$$

which is a contradiction.

If $\sigma_1>\sigma$, then for $n\geqslant\frac{1}{(\sigma_1-\sigma)|q|}$ $(n\in\mathbf{N})$ we set the following two functions:

$$
\widetilde{f}_n(x):=\begin{cases}0,0<|x|<1\\ |x|^{\sigma-\frac{1}{pn}-1},|x|\geqslant 1\end{cases},
$$

$$
\widetilde{g}_n(y):=\begin{cases}|y|^{\sigma_1+\frac{1}{qn}-1},0<|y|\leqslant 1\\ 0,|y|>1\end{cases},
$$

and derive that

$$
\begin{aligned}
\widetilde{J}_1&:=\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}\widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}y^{q(1-\sigma_1)-1}\widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}}\\
&=\left[2\int_1^{\infty}x^{p(1-\sigma)-1}x^{p(\sigma-\frac{1}{pn}-1)}dx\right]^{\frac{1}{p}}\left[2\int_0^1y^{q(1-\sigma_1)-1}y^{q(\sigma_1+\frac{1}{qn}-1)}dy\right]^{\frac{1}{q}}
\end{aligned}
$$

$$= \left(2\int_1^\infty x^{-\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(2\int_0^1 y^{\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = 2n.$$

We obtain

$$\begin{aligned}
\widetilde{I}_1 &:= \int_{-\infty}^{\infty} \widetilde{f}_n(x) \left[\int_{-\frac{1}{|x|}}^{\frac{1}{|x|}} \frac{|\ln|xy||^\beta \widetilde{g}_n(y)}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} dy\right] dx \\
&= \int_{-\infty}^{-1} \left[\int_{\frac{1}{x}}^{\frac{-1}{x}} \frac{|\ln|xy||^\beta |y|^{\sigma_1+\frac{1}{qn}-1}}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} dy\right] (-x)^{\sigma-\frac{1}{pn}-1} dx \\
&\quad + \int_1^\infty \left[\int_{-\frac{1}{x}}^{\frac{1}{x}} \frac{|\ln|xy||^\beta |y|^{\sigma_1+\frac{1}{qn}-1}}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} dy\right] x^{\sigma-\frac{1}{pn}-1} dx \\
&= \int_1^\infty \left[\int_{-\frac{1}{x}}^{\frac{1}{x}} (h(-xy)+h(xy))|y|^{\sigma_1+\frac{1}{qn}-1} dy\right] x^{\sigma-\frac{1}{pn}-1} dx \\
&= \int_1^\infty \left[\int_{-1}^{1} (h(-u)+h(u))|u|^{\sigma_1+\frac{1}{qn}-1} du\right] x^{(\sigma-\sigma_1)-\frac{1}{n}-1} dx, \qquad (8)
\end{aligned}$$

and then by Fubini's theorem (cf. [37]) and (5), it follows that

$$\begin{aligned}
2K^{(1)}\left(\sigma_1+\frac{1}{qn}\right)\frac{1}{\sigma_1-\sigma+\frac{1}{n}} &= \left[\int_{-1}^{1} (h(-u)+h(u))|u|^{\sigma_1+\frac{1}{qn}-1} du\right] \int_1^\infty x^{\sigma-\sigma_1-\frac{1}{n}-1} dx \\
&= \widetilde{I}_1 = \int_{-\infty}^{\infty} \widetilde{g}_n(y) \left[\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{|\ln|xy||^\beta \widetilde{f}_n(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right] dy \\
&\geqslant M_1 \widetilde{J}_1 = 2M_1 n. \qquad (9)
\end{aligned}$$

Since $\sigma_1 + \frac{1}{qn} \geqslant \sigma$, we have that

$$\begin{aligned}
0 \leqslant K^{(1)}\left(\sigma_1+\frac{1}{qn}\right) &= \int_0^1 (h(-u)+h(u))u^{\sigma_1+\frac{1}{qn}-1} du \\
&\leqslant \int_0^1 (h(-u)+h(u))u^{\sigma-1} du = K^{(1)}(\sigma) < \infty,
\end{aligned}$$

and then by (9), for $n \to \infty$, it follows that

$$\infty > \frac{2K^{(1)}(\sigma)}{\sigma_1-\sigma} \geqslant \infty,$$

which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we reduce (7) as follows:

$$K^{(1)}\left(\sigma+\frac{1}{pn}\right) = \int_0^1 (h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1} du \geqslant M_1.$$

Still by Levi's theorem, for $n \to \infty$, we have $K^{(1)}(\sigma) \geqslant M_1$.
This completes the proof of the lemma.

LEMMA 2. *If* $\sigma < \lambda$ *and for any nonnegative measurable functions* $f(x)$, $g(y)$ *in* $\mathbf{R}$, *the following inequality*

$$\int_{-\infty}^{\infty} g(y)\left[\int_{\{x;|x|\geqslant\frac{1}{|y|}\}} \frac{|\ln|xy||^{\beta} f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]dy$$

$$\geqslant M_2\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}} \tag{10}$$

holds true, then we have

$$\sigma_1=\sigma \;\; and \;\; K^{(2)}(\sigma)\geqslant M_2.$$

Proof. If $\sigma_1>\sigma$, then for $n\in\mathbf{N}$ we set two functions $\widetilde{f}_n(x)$ and $\widetilde{g}_n(y)$ as in Lemma 1 and find

$$\widetilde{J}_1=\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}\widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma_1)-1}\widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}}=2n.$$

We obtain

$$\begin{aligned}
\widetilde{I}_2 &:= \int_{-\infty}^{\infty}\widetilde{g}_n(y)\left[\int_{\{x;|x|\geqslant\frac{1}{|y|}\}}\frac{|\ln|xy||^{\beta}\widetilde{f}_n(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]dy\\
&= \int_{-1}^{0}\left[\int_{\{x;|x|\geqslant\frac{-1}{y}\}}\frac{|\ln|xy||^{\beta}|x|^{\sigma-\frac{1}{pn}-1}dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right](-y)^{\sigma_1+\frac{1}{qn}-1}dy\\
&\quad+\int_{0}^{1}\left[\int_{\{x;|x|\geqslant\frac{1}{y}\}}\frac{|\ln|xy||^{\beta}|x|^{\sigma-\frac{1}{pn}-1}dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]y^{\sigma_1+\frac{1}{qn}-1}dy\\
&= \int_{0}^{1}\left[\int_{\{x;|x|\geqslant\frac{1}{y}\}}(h(-xy)+h(xy))|x|^{\sigma-\frac{1}{pn}-1}dx\right]y^{\sigma_1+\frac{1}{qn}-1}dy\\
&= \int_{0}^{1}\left[\int_{\{u;|u|\geqslant 1\}}(h(-u)+h(u))|u|^{\sigma-\frac{1}{pn}-1}du\right]y^{(\sigma_1-\sigma)+\frac{1}{n}-1}dy,
\end{aligned}$$

and then by (10), it follows that

$$2K^{(2)}\left(\sigma-\frac{1}{pn}\right)\frac{1}{\sigma_1-\sigma+\frac{1}{n}}=\widetilde{I}_2\geqslant M_2\widetilde{J}_1=2M_2n. \tag{11}$$

Since $\{(h(-u)+h(u))u^{\sigma-\frac{1}{pn}-1}\}_{n=1}^{\infty}$ $(u\in(1,\infty))$ is a nonnegative and increasing function sequence, by Levi's theorem (cf. [37]), it follows that

$$K^{(2)}\left(\sigma-\frac{1}{pn}\right)=\int_{1}^{\infty}(h(-u)+h(u))u^{\sigma-\frac{1}{pn}-1}du$$

$$\to \int_1^\infty (h(-u)+h(u))u^{\sigma-1}du = K^{(2)}(\sigma)(n\to\infty),$$

and then by (11), we deduce that

$$\infty > \frac{2K^{(2)}(\sigma)}{\sigma_1-\sigma} \geqslant \infty,$$

which is a contradiction.

If $\sigma_1 < \sigma$, then for

$$n \geqslant \frac{1}{(\sigma-\sigma_1)|q|} \quad (n\in \mathbf{N}),$$

we set two functions $f_n(x)$ and $g_n(y)$ as in Lemma 1 and deduce that

$$J_1 = \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g_n^q(y)dy\right]^{\frac{1}{q}} = 2n.$$

We obtain

$$\begin{aligned}
I_2 &:= \int_{-\infty}^{\infty} f_n(x)\left[\int_{\{y;|y|\geqslant\frac{1}{|x|}\}} \frac{|\ln|xy||^\beta g_n(y)dy}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]dx \\
&= \int_{-1}^{0}\left[\int_{\{y;|y|\geqslant\frac{-1}{x}\}} \frac{|\ln|xy||^\beta |y|^{\sigma_1-\frac{1}{qn}-1}dy}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right](-x)^{\sigma+\frac{1}{pn}-1}dx \\
&\quad + \int_0^1\left[\int_{\{y;|y|\geqslant\frac{1}{x}\}} \frac{|\ln|xy||^\beta |y|^{\sigma_1-\frac{1}{qn}-1}dy}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]x^{\sigma+\frac{1}{pn}-1}dx \\
&= \int_0^1\left[\int_{\{y;|y|\geqslant\frac{1}{x}\}} (h(-xy)+h(xy))|y|^{\sigma_1-\frac{1}{qn}-1}dy\right]x^{\sigma+\frac{1}{pn}-1}dx \\
&= \int_{\{u;|u|\geqslant 1\}} (h(-u)+h(u))|u|^{\sigma_1-\frac{1}{qn}-1}du \int_0^1 x^{\sigma-\sigma_1+\frac{1}{n}-1}dx,
\end{aligned}$$

and then by Fubini's theorem (cf. [37]) and (8), it follows that

$$\begin{aligned}
&2K_2\left(\sigma_1-\frac{1}{qn}\right)\frac{1}{\sigma-\sigma_1+\frac{1}{n}} \\
&= \int_0^1 x^{\sigma-\sigma_1+\frac{1}{n}-1}dx \int_{\{u;|u|\geqslant 1\}} (h(-u)+h(u))|u|^{\sigma_1-\frac{1}{qn}-1}du \\
&= I_2 = \int_0^\infty g_n(y)\left[\int_{\{x;|x|\geqslant\frac{1}{|y|}\}} \frac{|\ln|xy||^\beta f_n(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]dy \\
&\geqslant M_2 J_1 = 2M_2 n.
\end{aligned} \tag{12}$$

Since $\sigma_1 - \frac{1}{qn} \leqslant \sigma$, we have

$$0 \leqslant K^{(2)}(\sigma_1 - \frac{1}{qn}) = \int_1^\infty (h(-u)+h(u))u^{\sigma_1-\frac{1}{qn}-1}du$$

$$\leqslant \int_1^\infty (h(-u)+h(u))u^{\sigma-1}du = K^{(2)}(\sigma) < \infty,$$

and then by (12), for $n \to \infty$, it follows that

$$\infty > 2K^{(2)}(\sigma)\frac{1}{\sigma-\sigma_1} \geqslant \infty,$$

which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we reduce (11) as follows:

$$K^{(2)}\left(\sigma - \frac{1}{pn}\right) = \int_1^\infty (h(-u)+h(u))u^{\sigma-\frac{1}{pn}-1}du \geqslant M_2. \tag{13}$$

Since $\{(h(-u)+h(u))u^{\sigma-\frac{1}{pn}-1}\}_{n=1}^\infty$ $(u \in [1,\infty))$ is a nonnegative and increasing function sequence, still by Levi's theorem (cf. [37]), we have

$$\begin{aligned} K^{(2)}(\sigma) &= \int_1^\infty \lim_{n\to\infty}(h(-u)+h(u))u^{\sigma-\frac{1}{pn}-1}du \\ &= \lim_{n\to\infty}\int_1^\infty (h(-u)+h(u))u^{\sigma-\frac{1}{pn}-1}du \geqslant M_2. \end{aligned}$$

This completes the proof of the lemma.

3. Reverse Hardy-type integral inequalities of the first kind

THEOREM 1. *If $\sigma > 0$, then the following conditions are equivalent:*
(i) For any $f(x) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following reverse Hardy-type integral inequality of the first kind with a nonhomogeneous kernel:

$$\begin{aligned} J &:= \left\{\int_{-\infty}^\infty |y|^{p\sigma_1-1}\left[\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{|\ln|xy||^\beta f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]^p dy\right\}^{\frac{1}{p}} \\ &> M_1\left[\int_{-\infty}^\infty |x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned} \tag{14}$$

(ii) For any $g(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^\infty |y|^{q(1-\sigma_1)-1} g^q(y)dy < \infty,$$

we have the following reverse Hardy-type integral inequality of the first kind with a nonhomogeneous kernel:

$$J_1 := \left\{ \int_{-\infty}^{\infty} |x|^{q\sigma-1} \left[\int_{\frac{-1}{|x|}}^{\frac{1}{|x|}} \frac{|\ln|xy||^{\beta} g(y)dy}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \right]^q dx \right\}^{\frac{1}{q}}$$
$$> M_1 \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy \right]^{\frac{1}{q}}. \quad (15)$$

(iii) For any $f(x), g(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty \text{ and } 0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy < \infty,$$

we have the following inequality:

$$I := \int_{-\infty}^{\infty} g(y) \left[\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{|\ln|xy||^{\beta} f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \right] dy$$
$$> M_1 \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy \right]^{\frac{1}{q}}. \quad (16)$$

(iv) $\sigma_1 = \sigma$, and $K^{(1)}(\sigma) \geqslant M_1$.

If Condition (iv) holds true, then the constant factor $M_1 = K^{(1)}(\sigma)$ $(\in \mathbf{R}_+)$ in (14), (15) and (16) (for $\sigma_1 = \sigma$) is the best possible.

Proof. $(i) \Rightarrow (iii)$. By the reverse Hölder inequality (cf. [36]), we have

$$I = \int_{-\infty}^{\infty} \left[|y|^{\sigma_1 - \frac{1}{p}} \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{|\ln|xy||^{\beta} f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \right] \left(|y|^{\frac{1}{p}-\sigma_1} g(y) \right) dy$$
$$\geqslant J \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy \right]^{\frac{1}{q}}. \quad (17)$$

Then by (14), we derive (15).

$(ii) \Rightarrow (iii)$. By the reverse Hölder inequality (cf. [36]), we have

$$I = \int_{-\infty}^{\infty} \left(|x|^{\frac{1}{q}-\sigma} f(x) \right) \left(|x|^{\sigma - \frac{1}{q}} \int_{\frac{-1}{|x|}}^{\frac{1}{|x|}} \frac{|\ln|xy||^{\beta} g(y)dy}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \right) dx$$
$$\geqslant \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}} J_1. \quad (18)$$

Then by (14), we deduce (16).

$(iii) \Rightarrow (iv)$. By Lemma 1, we get that $\sigma_1 = \sigma$, and $K^{(1)}(\sigma) \geqslant M_1$.

$(iv) \Rightarrow (i)$. We obtain the following weight function: For $y \neq 0$,

$$\begin{aligned}
\omega_1(\sigma,y) &:= |y|^\sigma \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{|\ln|xy||^\beta |x|^{\sigma-1}dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \\
&= |y|^\sigma \int_{\frac{-1}{|y|}}^{0} h(xy)(-x)^{\sigma-1}dx + |y|^\sigma \int_0^{\frac{1}{|y|}} h(xy)x^{\sigma-1}dx \\
&= |y|^\sigma \int_0^{\frac{1}{|y|}} h(-xy)x^{\sigma-1}dx + |y|^\sigma \int_0^{\frac{1}{|y|}} h(xy)x^{\sigma-1}dx \\
&= |y|^\sigma \int_0^{\frac{1}{|y|}} (h(-x|y|)+h(x|y|)x^{\sigma-1}dx \\
&= \int_0^1 (h(-u)+h(u))u^{\sigma-1}du = K^{(1)}(\sigma). \qquad (19)
\end{aligned}$$

By the reverse Hölder inequality with weight and (19), we have

$$\begin{aligned}
&\left[\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{|\ln|xy||^\beta f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]^p \\
&= \left\{\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} h(xy)\left[\frac{|y|^{(\sigma-1)/p}}{|x|^{(\sigma-1)/q}}f(x)\right]\left[\frac{|x|^{(\sigma-1)/q}}{|y|^{(\sigma-1)/p}}\right]dx\right\}^p \\
&\geqslant \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} h(xy)\frac{|y|^{\sigma-1}f^p(x)}{|x|^{(\sigma-1)p/q}}dx \left[\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} h(xy)\frac{|x|^{\sigma-1}}{|y|^{(\sigma-1)q/p}}dx\right]^{p-1} \\
&= \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} h(xy)\frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}}f^p(x)dx \cdot \left[\omega_1(\sigma,y)|y|^{q(1-\sigma)-1}\right]^{p-1} \\
&= (K^{(1)}(\sigma))^{p-1}|y|^{-p\sigma+1} \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} h(xy)\frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}}f^p(x)dx. \qquad (20)
\end{aligned}$$

If (20) assumes the form of equality for some $y \in \mathbf{R}\setminus\{0\}$, then (cf. [36]) there exist constants A and B, such that they are not all zero, and

$$A\frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}}f^p(x) = B\frac{|x|^{\sigma-1}}{|y|^{(\sigma-1)q/p}} \text{ a.e. in } \mathbf{R}.$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). It follows that

$$|x|^{p(1-\sigma)-1}f^p(x) = |y|^{q(1-\sigma)}\frac{B}{A|x|} \text{ a.e. in } \mathbf{R},$$

which contradicts the fact that

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1}f^p(x)dx < \infty.$$

Hence, (20) takes the form of strict inequality.

For $\sigma_1 = \sigma$, by the above results and Fubini's theorem (cf. [37]), we have

$$J > (K^{(1)}(\sigma))^{\frac{1}{q}} \left\{ \int_{-\infty}^{\infty} \left[\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x)dx \right] dy \right\}^{\frac{1}{p}}$$

$$= (K^{(1)}(\sigma))^{\frac{1}{q}} \left\{ \int_{-\infty}^{\infty} \left[\int_{\frac{-1}{|x|}}^{\frac{1}{|x|}} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)(p-1)}} dy \right] f^p(x)dx \right\}^{\frac{1}{p}}$$

$$= (K^{(1)}(\sigma))^{\frac{1}{q}} \left[\int_{-\infty}^{\infty} \omega_1(\sigma,x)|x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}}$$

$$= K^{(1)}(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}}.$$

Since $K^{(1)}(\sigma) \geqslant M_1$, we have

$$J > K^{(1)}(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}} \geqslant M_1 \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}},$$

namely, (14) follows.

$(iv) \Rightarrow (ii)$. Similarly, for $\sigma_1 = \sigma$, and $K^{(1)}(\sigma) \geqslant M_1$, by the reverse Hölder inequality, we have

$$\left[\int_{\frac{-1}{|x|}}^{\frac{1}{|x|}} \frac{|\ln|xy||^{\beta} g(y)dy}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \right]^q < \frac{(K^{(1)}(\sigma))^{q-1}}{|x|^{q\sigma-1}} \int_{\frac{-1}{|x|}}^{\frac{1}{|x|}} h(xy) \frac{|x|^{\sigma-1} g^q(y)}{|y|^{(\sigma-1)q/p}} dy,$$

from which we can deduce (15).

Therefore, conditions (i), (ii), (iii) and (iv) are equivalent.

When Condition (iv) is satisfied, if there exists a constant $M_1 \geqslant K^{(1)}(\sigma)$, such that (16) is valid, then for $K^{(1)}(\sigma) \geqslant M_1$ the constant factor $M_1 = K^{(1)}(\sigma)$ in (16) is the best possible.

The constant factor $M_1 = K^{(1)}(\sigma)(\in \mathbf{R}_+)$ in (14) is still the best possible. Otherwise, by (17) (for $\sigma_1 = \sigma$), we can conclude that the constant factor $M_1 = K^{(1)}(\sigma)$ in (16) is not the best possible. Similarly, by (18) (for $\sigma_1 = \sigma$), we can prove that the constant factor $M_1 = K^{(1)}(\sigma)$ in (15) is the best possible.

In particular, for $\sigma = \sigma_1 = \frac{1}{p} > 0$ in Theorem 1, we have

COROLLARY 1. *The following conditions are equivalent:*
(i) For any $f(x) \geqslant 0$, *satisfying*

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left\{\int_{-\infty}^{\infty}\left[\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}}\frac{|\ln|xy||^{\beta}f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]^{p}dy\right\}^{\frac{1}{p}}>M_1\left(\int_{-\infty}^{\infty}|x|^{p-2}f^{p}(x)dx\right)^{\frac{1}{p}}.(21)$$

(ii) For any $g(y)\geqslant 0,$ *satisfying*

$$0<\int_{-\infty}^{\infty}g^{q}(y)dy<\infty,$$

we have the following inequality:

$$\left\{\int_{-\infty}^{\infty}|x|^{q-2}\left[\int_{\frac{-1}{|x|}}^{\frac{1}{|x|}}\frac{|\ln|xy||^{\beta}g(y)dy}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]^{q}dx\right\}^{\frac{1}{q}}>M_1\left(\int_{-\infty}^{\infty}g^{q}(y)dy\right)^{\frac{1}{q}}.(22)$$

(iii) For any $f(x),g(y)\geqslant 0,$ *satisfying*

$$0<\int_{-\infty}^{\infty}|x|^{p-2}f^{p}(x)dx<\infty \text{ and } 0<\int_{-\infty}^{\infty}g^{q}(y)dy<\infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty}g(y)\left[\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}}\frac{|\ln|xy||^{\beta}f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}\right]dy$$
$$>M_1\left(\int_{-\infty}^{\infty}|x|^{p-2}f^{p}(x)dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}g^{q}(y)dy\right)^{\frac{1}{q}}. \quad (23)$$

(iv) $K^{(1)}(\frac{1}{p})\geqslant M_1.$
If Condition (iv) holds true, then the constant

$$M_1=K^{(1)}\left(\frac{1}{p}\right)=\frac{\Gamma(\beta+1)}{2^{\beta}}\zeta\left(\beta+1,\frac{1}{2p}\right)$$

in (21), (22) and (23) is the best possible.

Setting

$$y=\frac{1}{Y},\ G(Y)=g\left(\frac{1}{Y}\right)\frac{1}{Y^{2}}$$

in Theorem 1, and then replacing Y by $y,$ we obtain the following corollary:

COROLLARY 2. *The following conditions are equivalent:*
(i) For any $f(x)\geqslant 0,$ *satisfying*

$$0<\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^{p}(x)dx<\infty,$$

we have the following inequality:

$$\left\{\int_{-\infty}^{\infty}|y|^{-p\sigma_1-1}\left[\int_{-|y|}^{|y|}\frac{|\ln|x/y||^{\beta}f(x)dx}{(\max\{|x/y|,1\})^{\lambda-1}|x/y-1|}\right]^p dy\right\}^{\frac{1}{p}}$$
$$> M_1\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}. \tag{24}$$

(ii) For any $g(y) \geqslant 0$*, satisfying*

$$0<\int_{-\infty}^{\infty}|y|^{q(1+\sigma_1)-1}g^q(y)dy<\infty,$$

we have the following integral inequality:

$$\left\{\int_{-\infty}^{\infty}|x|^{q\sigma-1}\left[\int_{\frac{-1}{|x|}}^{\frac{1}{|x|}}\frac{|\ln|x/y||^{\beta}G(y)dy}{(\max\{|x/y|,1\})^{\lambda-1}|x/y-1|}\right]^q dx\right\}^{\frac{1}{q}}$$
$$> M_1\left[\int_{-\infty}^{\infty}|y|^{q(1+\sigma_1)-1}G^q(y)dy\right]^{\frac{1}{q}}. \tag{25}$$

(iii) For any $f(x), G(y) \geqslant 0$*, satisfying*

$$0<\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)dx<\infty \text{ and } 0<\int_{-\infty}^{\infty}y^{q(1+\sigma_1)-1}G^q(y)dy<\infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty}G(y)\left[\int_{-|y|}^{|y|}\frac{|\ln|x/y||^{\beta}f(x)dx}{(\max\{|x/y|,1\})^{\lambda-1}|x/y-1|}\right]dy$$
$$< M_1\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1+\sigma_1)-1}G^q(y)dy\right]^{\frac{1}{q}}. \tag{26}$$

(iv) $\sigma_1 = \sigma$*, and* $K^{(1)}(\sigma) \geqslant M_1$*.*

If Condition (iv) is satisfied, then the constant $M_1 = K^{(1)}(\sigma)\,(\in \mathbf{R}_+)$ *in (24), (25) and (26) (for* $\sigma_1 = \sigma$*) is the best possible.*

Setting $g(y) = y^{\lambda}G(y)$ and $\mu = \lambda - \sigma_1$ in Corollary 2, we derive the following:

COROLLARY 3. *The following conditions are equivalent:*
(i) For any $f(x) \geqslant 0$*, satisfying*

$$0<\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)dx<\infty,$$

we have the following reverse Hardy-type integral inequality of the first kind with a homogeneous kernel:

$$\left\{\int_{-\infty}^{\infty} y^{p\mu-1}\left[\int_{-|y|}^{|y|}\frac{|\ln|x/y||^{\beta}f(x)dx}{(\max\{|x|,|y|\})^{\lambda-1}|x-y|}\right]^{p}dy\right\}^{\frac{1}{p}}$$
$$> M_1\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}. \tag{27}$$

(ii) For any $g(y)\geqslant 0$, *satisfying*

$$0<\int_{-\infty}^{\infty}|y|^{q(1-\mu)-1}g^{q}(y)dy<\infty,$$

we have the following reverse Hardy-type integral inequality of the first kind with a homogeneous kernel:

$$\left\{\int_{-\infty}^{\infty}|x|^{q\sigma-1}\left[\int_{-|x|}^{|x|}\frac{|\ln|x/y||^{\beta}g(y)dy}{(\max\{|x|,|y|\})^{\lambda-1}|x-y|}\right]^{q}dx\right\}^{\frac{1}{q}}$$
$$> M_1\left[\int_{-\infty}^{\infty}|y|^{q(1-\mu)-1}g^{q}(y)dy\right]^{\frac{1}{q}}. \tag{28}$$

(iii) For any $f(x),g(y)\geqslant 0$, *satisfying*

$$0<\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^{p}(x)dx<\infty \ \text{ and } \ 0<\int_{-\infty}^{\infty}|y|^{q(1-\mu)-1}g^{q}(y)dy<\infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty}g(y)\left[\int_{-|y|}^{|y|}\frac{|\ln|x/y||^{\beta}f(x)dx}{(\max\{|x|,|y|\})^{\lambda-1}|x-y|}\right]dy$$
$$> M_1\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\mu)-1}g^{q}(y)dy\right]^{\frac{1}{q}}; \tag{29}$$

(iv) $\mu+\sigma=\lambda$, *and* $K^{(1)}(\sigma)\geqslant M_1$.

If Condition (iv) holds true, then the constant $M_1=K^{(1)}(\sigma)\,(\in\mathbf{R}_+)$ *in (27), (28) and (29) is the best possible.*

In particular, for $\lambda=1,\sigma=\frac{1}{p},\mu=\frac{1}{q}$ in Corollary 3, we derive the corollary below:

COROLLARY 4. *The following conditions are equivalent:*
(i) For any $f(x)\geqslant 0$, *satisfying*

$$0<\int_{-\infty}^{\infty}|x|^{p-2}f^{p}(x)dx<\infty,$$

we have the following inequality:

$$\left\{\int_{-\infty}^{\infty}|y|^{p-2}\left[\int_{-|y|}^{|y|}\frac{|\ln|x/y||^{\beta}f(x)dx}{|x-y|}\right]^{p}dy\right\}^{\frac{1}{p}} > M_1\left(\int_{-\infty}^{\infty}|x|^{p-2}f^{p}(x)dx\right)^{\frac{1}{p}}. \quad (30)$$

(ii) For any $g(x)\geqslant 0,$ *satisfying*

$$0<\int_{-\infty}^{\infty}|y|^{q-2}g^{q}(y)dy<\infty,$$

we have the following inequality:

$$\left\{\int_{-\infty}^{\infty}|y|^{q-2}\left[\int_{-|x|}^{|x|}\frac{|\ln|x/y||^{\beta}g(y)dy}{|x-y|}\right]^{q}dy\right\}^{\frac{1}{q}} > M_1\left(\int_{-\infty}^{\infty}|y|^{q-2}g^{q}(y)dy\right)^{\frac{1}{q}}. \quad (31)$$

(iii) For any $f(x),g(y)\geqslant 0,$ *satisfying*

$$0<\int_{-\infty}^{\infty}f^{p}(x)dx<\infty \ \text{ and } \ 0<\int_{-\infty}^{\infty}g^{q}(y)dy<\infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty}g(y)\left[\int_{-|y|}^{|y|}\frac{|\ln|x/y||^{\beta}f(x)dx}{|x-y|}\right]dy$$
$$> M_1\left(\int_{-\infty}^{\infty}|x|^{p-2}f^{p}(x)dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|y|^{q-2}g^{q}(y)dy\right)^{\frac{1}{q}}. \quad (32)$$

(iv) $\frac{\Gamma(\beta+1)}{2^{\beta}}\zeta\left(\beta+1,\frac{1}{2p}\right)\geqslant M_1.$
If Condition (iv) holds true, then the constant

$$M_1=\frac{\Gamma(\beta+1)}{2^{\beta}}\zeta\left(\beta+1,\frac{1}{2p}\right)$$

in (30), (31) and (32) is the best possible.

4. Reverse Hardy-type integral inequalities of the second kind

In view of Lemma 2, we obtain the following weight function: For $y\neq 0,$

$$\omega_2(\sigma,y):=|y|^{\sigma}\int_{\{x;|x|\geqslant\frac{1}{|y|}\}}\frac{|\ln|xy||^{\beta}|x|^{\sigma-1}}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}dx$$
$$=\int_{1}^{\infty}(h(-u)+h(u))u^{\sigma-1}du=K^{(2)}(\sigma),$$

and similarly, we get the theorem below.

THEOREM 2. *If $\sigma < \lambda$, then the following conditions are equivalent:*
(i) For any $f(x) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following reverse Hardy-type integral inequality of the second kind with a nonhomogeneous kernel:

$$\left\{ \int_{-\infty}^{\infty} |y|^{p\sigma_1 - 1} \left[\int_{\{x;|x| \geqslant \frac{1}{|y|}\}} \frac{|\ln |xy||^\beta f(x) dx}{(\max\{|xy|,1\})^{\lambda-1} |xy-1|} \right]^p dy \right\}^{\frac{1}{p}}$$
$$> M_2 \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \tag{33}$$

(ii) For any $g(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following reverse Hardy-type integral inequality of the second kind with a nonhomogeneous kernel:

$$\left\{ \int_{-\infty}^{\infty} |x|^{q\sigma - 1} \left[\int_{\{y;|y| \geqslant \frac{1}{|x|}\}} \frac{|\ln |xy||^\beta g(y) dy}{(\max\{|xy|,1\})^{\lambda-1} |xy-1|} \right]^q dx \right\}^{\frac{1}{q}}$$
$$> M_2 \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \tag{34}$$

(iii) For any $f(x), g(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty \text{ and } 0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty} g(y) \left[\int_{\{x;|x| \geqslant \frac{1}{|y|}\}} \frac{|\ln |xy||^\beta f(x) dx}{(\max\{|xy|,1\})^{\lambda-1} |xy-1|} \right] dy$$
$$> M_2 \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \tag{35}$$

(iv) $\sigma_1 = \sigma$, and $K^{(2)}(\sigma) \geqslant M_2$.

If Condition (iv) holds true, then the constant $M_2 = K^{(2)}(\sigma)$ in (33), (34) and (35) (for $\sigma_1 = \sigma$) is the best possible.

In particular, for $\sigma = \sigma_1 = \frac{1}{p}$ in Theorem 2, we derive the corollary below:

COROLLARY 5. *If $\frac{1}{p} < \lambda$, then the following conditions are equivalent:*
(i) For any $f(x) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left\{ \int_{-\infty}^{\infty} \left[\int_{\{x;|x| \geqslant \frac{1}{|y|}\}} \frac{|\ln|xy||^{\beta} f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \right]^p dy \right\}^{\frac{1}{p}} > M_2 \left(\int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx \right)^{\frac{1}{p}}. \tag{36}$$

(ii) For any $g(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} g^q(y)dy < \infty,$$

we have the following inequality:

$$\left\{ \int_{-\infty}^{\infty} |x|^{q-2} \left[\int_{\{y;|y| \geqslant \frac{1}{|x|}\}} \frac{|\ln|xy||^{\beta} g(y)dy}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \right]^q dx \right\}^{\frac{1}{q}} > M_2 \left(\int_{-\infty}^{\infty} g^q(y)dy \right)^{\frac{1}{q}}. \tag{37}$$

(iii) For any $f(x), g(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx < \infty \ \text{ and } \ 0 < \int_{-\infty}^{\infty} g^q(y)dy < \infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty} g(y) \left[\int_{\{x;|x| \geqslant \frac{1}{|y|}\}} \frac{|\ln|xy||^{\beta} f(x)dx}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|} \right] dy > M_2 \left(\int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} g^q(y)dy \right)^{\frac{1}{q}}. \tag{38}$$

(iv) $K^{(2)}(\frac{1}{p}) \geqslant M_2$.

If Condition (iv) holds true, then the constant factor

$$M_2 = K^{(2)}\left(\frac{1}{p}\right) = \frac{\Gamma(\beta+1)}{2^{\beta}} \zeta\left(\beta+1, \frac{p\lambda-1}{2p}\right)$$

in (36), (37) and (38) is the best possible.

Setting

$$y = \frac{1}{Y}, \ \ G(Y) = g\left(\frac{1}{Y}\right)\frac{1}{Y^2}$$

in Theorem 2, and then replacing Y by y, we get the corollary below:

COROLLARY 6. *If $\sigma < \lambda$, then the following conditions are equivalent:*
(i) For any $f(x) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left\{\int_{-\infty}^{\infty} |y|^{-p\sigma_1-1}\left[\int_{\{x;|x|\geqslant|y|\}} \frac{|\ln|x/y||^{\beta} f(x)dx}{(\max\{|x/y|,1\})^{\lambda-1}|x/y-1|}\right]^p dy\right\}^{\frac{1}{p}}$$

$$> M_2\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{39}$$

(ii) For any $g(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |y|^{q(1+\sigma_1)-1} g^q(y)dy < \infty,$$

we have the following integral inequality:

$$\left\{\int_{-\infty}^{\infty} |x|^{q\sigma-1}\left[\int_{\{y;|y|\geqslant|x|\}} \frac{|\ln|x/y||^{\beta} G(y)dy}{(\max\{|x/y|-1\})^{\lambda-1}|x/y-1|}\right]^q dx\right\}^{\frac{1}{q}}$$

$$> M_2\left[\int_{-\infty}^{\infty} |y|^{q(1+\sigma_1)-1} G^q(y)dy\right]^{\frac{1}{q}}. \tag{40}$$

(iii) For any $f(x), G(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty \ \ and \ \ 0 < \int_{-\infty}^{\infty} |y|^{q(1+\sigma_1)-1} G^q(y)dy < \infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty} G(y)\left[\int_{\{x;|x|\geqslant|y|\}} \frac{|\ln|x/y||^{\beta} f(x)dx}{(\max\{|x/y|,1\})^{\lambda-1}|x/y-1|}\right] dy$$

$$> M_2\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} |y|^{q(1+\sigma_1)-1} G^q(y)dy\right]^{\frac{1}{q}}. \tag{41}$$

(iv) $\sigma_1 = \sigma$, and $K^{(2)}(\sigma) \geqslant M_2$.

If Condition (iv) holds true, then the constant $M_2 = K^{(2)}(\sigma)$ in (39), (40) and (41) (for $\sigma_1 = \sigma$) is the best possible.

For $g(y) = y^{\lambda} G(y)$ and $\mu = \lambda - \sigma_1$ in Corollary 6, we obtain the corollary below:

COROLLARY 7. *If $\sigma < \lambda$, then the following conditions are equivalent:*
(i) For any $f(x) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following reverse Hardy-type integral inequality of the second kind with a homogeneous kernel:

$$\left\{ \int_{-\infty}^{\infty} y^{p\mu-1} \left[\int_{\{x;|x| \geqslant |y|\}} \frac{|\ln|x/y||^{\beta} f(x) dx}{(\max\{|x|,|y|\})^{\lambda-1}|x-y|} \right]^p dy \right\}^{\frac{1}{p}}$$
$$> M_2 \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \tag{42}$$

(ii) For any $g(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\mu)-1} g^q(y) dy < \infty,$$

we have the following reverse Hardy-type integral inequality of the second kind with a homogeneous kernel:

$$\left\{ \int_{-\infty}^{\infty} |x|^{q\sigma-1} \left[\int_{\{y;|y| \geqslant |x|\}} \frac{|\ln|x/y||^{\beta} g(y) dy}{(\max\{|x|,|y|\})^{\lambda-1}|x-y|} \right]^q dx \right\}^{\frac{1}{q}}$$
$$> M_2 \left[\int_{-\infty}^{\infty} |y|^{q(1-\mu)-1} g^q(y) dy \right]^{\frac{1}{q}}. \tag{43}$$

(iii) For any $f(x), g(y) \geqslant 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty \ \text{and} \ 0 < \int_{-\infty}^{\infty} |y|^{q(1-\mu)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty} g(y) \left(\int_{\{x;|x| \geqslant |y|\}} \frac{|\ln|x/y||^{\beta} f(x) dx}{(\max\{|x|,|y|\})^{\lambda-1}|x-y|} \right) dy$$
$$> M_2 \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\mu)-1} g^q(y) dy \right]^{\frac{1}{q}}. \tag{44}$$

(iv) $\mu + \sigma = \lambda$, and $K^{(2)}(\sigma) \geqslant M_2$.

If Condition (iv) holds true, then the constant $M_2 = K^{(2)}(\sigma)$ in (42), (43) and (44) is the best possible.

In particular, for $\lambda = 1, \sigma = \frac{1}{q}, \mu = \frac{1}{p}$ in Corollary 7, we obtain the corollary below:

COROLLARY 8. *The following conditions are equivalent:*
(i) For any $f(x) \geqslant 0$, *satisfying*

$$0 < \int_{-\infty}^{\infty} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left[\int_{-\infty}^{\infty}\left(\int_{\{x;|x|\geqslant|y|\}} \frac{|\ln|x/y||^{\beta} f(x)dx}{|x-y|}\right)^p dy\right]^{\frac{1}{p}} > M_2\left(\int_{-\infty}^{\infty} f^p(x)dx\right)^{\frac{1}{p}}. \tag{45}$$

(ii) For any $g(y) \geqslant 0$, *satisfying*

$$0 < \int_{-\infty}^{\infty} g^q(y)dy < \infty,$$

we have the following inequality:

$$\left[\int_{-\infty}^{\infty}\left(\int_{\{y;|y|\geqslant|x|\}} \frac{|\ln|x/y||^{\beta} g(y)dy}{|x-y|}\right)^q dx\right]^{\frac{1}{q}} > M_2\left(\int_{-\infty}^{\infty} g^q(y)dy\right)^{\frac{1}{q}}. \tag{46}$$

(iii) For any $f(x), g(y) \geqslant 0$, *satisfying*

$$0 < \int_{-\infty}^{\infty} f^p(x)dx < \infty \ \text{ and } \ 0 < \int_{-\infty}^{\infty} g^q(y)dy < \infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty} g(y)\left(\int_{\{x;|x|\geqslant|y|\}} \frac{|\ln|x/y||^{\beta} f(x)dx}{|x-y|}\right)dy$$
$$> M_2\left(\int_{-\infty}^{\infty} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty} g^q(y)dy\right)^{\frac{1}{q}}. \tag{47}$$

(iv) $\frac{\Gamma(\beta+1)}{2^{\beta}}\zeta(\beta+1, \frac{1}{2p}) \geqslant M_2$.
If Condition (iv) holds true, then the constant

$$M_2 = \frac{\Gamma(\beta+1)}{2^{\beta}}\zeta\left(\beta+1, \frac{1}{2p}\right)$$

in (45), (46) and (47) is the best possible.

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