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A TRUDINGER–MOSER TYPE INEQUALITY AND ITS EXTREMAL FUNCTIONS IN DIMENSION TWO

XIANFENG SU

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Abstract. Let Ω be a smooth bounded domain in $\mathbb{R}^2$, $W_0^{1,2}(\Omega)$ be the usual Sobolev space and $\lambda(\Omega)$ be the first eigenvalue of the Laplace-Beltrami operator, say

$$\lambda(\Omega) = \inf_{u\in W_0^{1,2}(\Omega),\, \int_\Omega u^2dx=1} \int_\Omega |\nabla u|^2 dx.$$

Using blow-up analysis, we prove that for real numbers $\alpha < \lambda(\Omega)$ and $\beta < 4\pi$, the supremum

$$\sup_{u\in W_0^{1,2}(\Omega),\, \int_\Omega |\nabla u|^2dx-\alpha\int_\Omega u^2dx\leqslant 1} \int_\Omega (e^{4\pi u^2} - \beta u^2)dx$$

can be attained by some function $u \in W_0^{1,2}(\Omega)$ with $\int_\Omega |\nabla u|^2dx - \alpha \int_\Omega u^2dx = 1$. In the case $\beta = 0$, this is reduced to a result of Yang [24].

1. Introduction and main result

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $W_0^{1,2}(\Omega)$ be the usual Sobolev space. The classical Trudinger-Moser inequality [27, 17, 16, 20, 15] states the following:

$$\sup_{u\in W_0^{1,2}(\Omega),\, \|\nabla u\|_2^2\leqslant 1} \int_\Omega e^{\gamma u^2}dx < +\infty, \quad \forall \gamma \leqslant 4\pi; \tag{1}$$

moreover, if $\gamma > 4\pi$, all integrals in (1) are still finite, but the supremum is infinite. Let (u_j) be a function sequence in $W_0^{1,2}(\Omega)$ such that $\|\nabla u_j\|_2 = 1$ and $u_j \rightharpoonup u_0$ weakly in $W_0^{1,2}(\Omega)$. It was proved by Lions [12] that for any $q < 1/(1-\|\nabla u_0\|_2^2)$, there holds

$$\limsup_{j\to\infty} \int_\Omega e^{4\pi q u_j^2}dx < +\infty. \tag{2}$$

If $u_0 \not\equiv 0$, (2) is stronger than (1). While if $u_0 \equiv 0$, (2) gives no information than (1). Nevertheless, Adimurthi-Druet [1] obtained that for any $0 \leqslant \alpha < \lambda(\Omega) = \inf_{u\in W_0^{1,2}(\Omega),\, \|u\|_2=1} \|\nabla u\|_2^2$,

$$\sup_{u\in W_0^{1,2}(\Omega),\, \|\nabla u\|_2^2\leqslant 1} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)}dx < +\infty. \tag{3}$$

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This inequality was generalized by Yang [21, 22] and Zhu [28] to n-dimensional case ($n \geqslant 3$), by Yang [23] to closed Rieman surface case, and by Lu-Yang [13] to the version involving L^p-norm for any $p > 1$. A slightly stronger version of (3) was due to Tintarev [19], say

$$\sup_{u \in W_0^{1,2}(\Omega),\, \|u\|_{1,\alpha} \leqslant 1} \int_\Omega e^{4\pi u^2} dx < +\infty, \quad \forall \alpha < \lambda(\Omega), \tag{4}$$

where $\|\cdot\|_{1,\alpha}^2 = \|\nabla u\|_2^2 - \alpha \|u\|_2^2$. It was recently proved by Yang [24] that extremal functions for the supremum in (4) exist. This result was extended by Nguyen [14] and Yang-Zhu [26] to higher dimensional case. For related works, we refer the reader to Yang-Zhu [25], Li-Yang [8], Li [9] and so on.

Let us briefly recall the history of the problem of extremal functions for Trudinger-Moser inequality. The first result was due to Carleson-Chang [2], who obtained the existence of extremal functions for the supremum in (1) in the case that Ω is the unit disc in $\mathbb{R}^2$, in fact in the n-dimensional case, $\forall n \geqslant 2$. This result was then generalized by Struwe [18] to domains close to the ball in the sense of measure, by Flucher [5] and Lin [11] to general bounded smooth domains. For manifold versions of (1) and their extremal functions, we refer the reader to Fontana [6] and Li [10] respectively.

Let $\alpha < \lambda(\Omega)$ and β be two real numbers. Obviously it follows from (4) that

$$\sup_{u \in W_0^{1,2}(\Omega),\, \|u\|_{1,\alpha} \leqslant 1} \int_\Omega \left(e^{4\pi u^2} - \beta u^2 \right) dx < +\infty. \tag{5}$$

Concerning the extremal functions for the above supremum, we have the following

THEOREM 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $\lambda(\Omega)$ be define as in (3). If $\alpha < \lambda(\Omega)$ and $\beta < 4\pi$, then the supremum*

$$\sup_{u \in W_0^{1,2}(\Omega),\, \|u\|_{1,\alpha} \leqslant 1} \int_\Omega (e^{4\pi u^2} - \beta u^2) dx$$

can be attained by some function $u^* \in W_0^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ with $\|u^*\|_{1,\alpha} = 1$, where $\|\cdot\|_{1,\alpha}$ is defined as in (4).

Note that Theorem 1 is reduced to that of Yang [24]. The proof of Theorem 1 is based on the blow-up analysis, which was originally used by Carleson-Chang [2], Ding-Jost-Li-Wang [4] and Li [10], in particular, we use an argument of Yang [24]. More precisely, on the one hand, by analyzing the asymptotic behavior of the maximizers for subcritical functionals (see Lemma 2 below), we derive an upper bound C_0 of the functional $\int_\Omega e^{4\pi u^2} dx - \beta \int_\Omega u^2 dx$ under the assumption that blow-up occur; On the other hand, by constructing a sequence of test functions, we prove that C_0 is not really an upper bound of the corresponding functional. Combining these two steps, we conclude that blow-up can not occur and that the desired extremal functions would exist.

The remaining part of this paper is organized as follows: In Section 2, we prove the existence of maximizers $u_\varepsilon \in W_0^{1,2}(\Omega)$ for subcritical functionals; In Section 3, we investigate the behavior of the maximizers u_ε by using blow-up analysis; In Section 4, using a result due to Carleson and Chang [2], we obtain an upper bound estimates of the functional $\int_\Omega (e^{4\pi u^2} - \beta u^2)dx$ under the assumption of blow-up analysis; In Section 5, we construct a family of test function to finish the proof of Theorem 1.

2. Maximizers for subcritical Trudinger-Moser functionals

In this section, we will show that maximizers for subcritical functionals exist. This is based on a direct method in the calculus of variation. Let $\alpha < \lambda(\Omega)$ and $\beta < 4\pi$ be fixed. For simplicity, denote

$$\Lambda_{\beta,\varepsilon} = \sup_{u\in W_0^{1,2}(\Omega),\, \|u\|_{1,\alpha}\leqslant 1} \int_\Omega (e^{(4\pi-\varepsilon)u^2} - \beta u^2)dx.$$

Then we have

LEMMA 2. *For any $0<\varepsilon<4\pi$, there exists $u_\varepsilon \in W_0^{1,2}(\Omega)\bigcap C^1(\overline{\Omega})$ with $\|u_\varepsilon\|_{1,\alpha} = 1$ such that*

$$\int_\Omega (e^{(4\pi-\varepsilon)u_\varepsilon^2} - \beta u_\varepsilon^2)dx = \Lambda_{\beta,\varepsilon}. \tag{6}$$

Moreover, in the distributional sense u_ε satisfies the equation

$$\begin{cases} \Delta u_\varepsilon + \alpha u_\varepsilon = -\frac{1}{\lambda_\varepsilon} u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2} + \frac{\beta u_\varepsilon}{\lambda_\varepsilon(4\pi-\varepsilon)} \quad \text{in} \quad \Omega, \\ u_\varepsilon > 0 \quad \text{in} \quad \Omega, \\ \lambda_\varepsilon = \int_\Omega \left(u_\varepsilon^2 e^{(4\pi-\varepsilon)u_\varepsilon^2} - \frac{\beta u_\varepsilon^2}{4\pi-\varepsilon} \right) dx. \end{cases} \tag{7}$$

Proof. For any $0<\varepsilon<4\pi$, we choose a sequence of functions $u_j \in W_0^{1,2}(\Omega)$ such that $\|u_j\|_{1,\alpha} \leqslant 1$ and

$$\lim_{j\to\infty} \int_\Omega (e^{(4\pi-\varepsilon)u_j^2} - \beta u_j^2)dx = \Lambda_{\beta,\varepsilon} \tag{8}$$

Since $\alpha < \lambda(\Omega)$, we get that u_j is bounded in $W_0^{1,2}(\Omega)$. Without loss of generality, we assume $u_j \rightharpoonup u_\varepsilon$ weakly in $W_0^{1,2}(\Omega)$, $u_j \to u_\varepsilon$ strongly in $L^p(\Omega)$ for any $p>1$, and $u_j \to u_\varepsilon$ almost everywhere in Ω. Moreover, we have that $\|u_\varepsilon\|_{1,\alpha} \leqslant \liminf_{j\to\infty} \|u_j\|_{1,\alpha} \leqslant 1$. Note that

$$\begin{aligned} \int_\Omega |\nabla u_j - \nabla u_\varepsilon|^2 dx &= \int_\Omega |\nabla u_j|^2 dx - \int_\Omega |\nabla u_\varepsilon|^2 dx + o_j(1) \\ &= \int_\Omega \left(|\nabla u_j|^2 - \alpha u_j^2\right) dx - \int_\Omega \left(|\nabla u_\varepsilon|^2 - \alpha u_\varepsilon^2\right) dx + o_j(1) \\ &\leqslant 1 - \|u_\varepsilon\|_{1,\alpha}^2 + o_j(1). \end{aligned}$$

Using the Lions' inequality (2), we get $e^{(4\pi-\varepsilon)u_j^2}$ is bounded in $L^s(\Omega)$ for some $s>1$. Since

$$|e^{(4\pi-\varepsilon)u_j^2}-e^{(4\pi-\varepsilon)u_\varepsilon^2}|\leqslant(4\pi-\varepsilon)\left(e^{(4\pi-\varepsilon)u_j^2}+e^{(4\pi-\varepsilon)u_\varepsilon^2}\right)|u_j^2-u_\varepsilon^2|$$

and $u_j\to u_\varepsilon$ strongly in $L^p(\Omega)$ for any $p>1$ as $j\to\infty$, we conclude that

$$\lim_{j\to\infty}\int_\Omega(e^{(4\pi-\varepsilon)u_j^2}-\beta u_j^2)dx=\int_\Omega(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx.$$

This together with (8) immediately leads to (6). Obviously $u_\varepsilon\not\equiv 0$. Suppose $\|u_\varepsilon\|_{1,\alpha}<1$. Since $\beta<4\pi$, we get

$$\Lambda_{\beta,\varepsilon}=\int_\Omega(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx<\int_\Omega\left(e^{(4\pi-\varepsilon)\frac{u_\varepsilon^2}{\|u_\varepsilon\|_{1,\alpha}^2}}-\beta\frac{u_\varepsilon^2}{\|u_\varepsilon\|_{1,\alpha}^2}\right)dx\leqslant\Lambda_{\beta,\varepsilon},$$

which is a contradiction. Hence, we have $\|u_\varepsilon\|_{1,\alpha}=1$. A straightforward calculation shows u_ε satisfies the Euler-Lagrange equation (7). Applying elliptic estimates to (7), we have $u_\varepsilon\in C^1(\overline{\Omega})$. □

3. Blow-up analysis

In view of (7), we will prove that λ_ε has a positive lower bound, which is necessary in the subsequent analysis.

LEMMA 3. *Let λ_ε be as in (7), then*

$$\liminf_{\varepsilon\to0}\lambda_\varepsilon>0.$$

Proof. For any $u\in W_0^{1,2}(\Omega)$ with $\|u\|_{1,\alpha}\leqslant1$, we get

$$\int_\Omega(e^{4\pi u^2}-\beta u^2)dx=\lim_{\varepsilon\to0}\int_\Omega(e^{(4\pi-\varepsilon)u^2}-\beta u^2)dx\leqslant\lim_{\varepsilon\to0}\int_\Omega(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx.$$

This leads to

$$\sup_{u\in W_0^{1,2}(\Omega),\,\|u\|_{1,\alpha}\leqslant1}\int_\Omega(e^{4\pi u^2}-\beta u^2)dx\leqslant\Lambda_{\beta,\varepsilon}=\lim_{\varepsilon\to0}\int_\Omega(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx.$$

One can easily see that

$$\int_\Omega(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx=\Lambda_{\beta,\varepsilon}\leqslant\sup_{u\in W_0^{1,2}(\Omega),\,\|u\|_{1,\alpha}\leqslant1}\int_\Omega(e^{4\pi u^2}-\beta u^2)dx.$$

Using an elementary inequality $te^t\geqslant e^t-1$ for $t\geqslant0$, we have

$$\lambda_\varepsilon\geqslant\frac{1}{4\pi-\varepsilon}\int_\Omega(e^{(4\pi-\varepsilon)u_\varepsilon^2}-1-\beta u_\varepsilon^2)dx.$$

Therefore

$$\liminf_{\varepsilon\to 0}\lambda_\varepsilon \geqslant \frac{1}{4\pi}\liminf_{\varepsilon\to 0}\int_\Omega (e^{(4\pi-\varepsilon)u_\varepsilon^2}-1-\beta u_\varepsilon^2)dx$$
$$=\frac{1}{4\pi}\left(\sup_{u\in W_0^{1,2}(\Omega),\,\|u\|_{1,\alpha}\leqslant 1}\int_\Omega (e^{4\pi u^2}-\beta u^2)dx-|\Omega|\right)>0.$$

This ends the proof of the lemma. □

Denote

$$c_\varepsilon = u_\varepsilon(x_\varepsilon)=\max_\Omega u_\varepsilon.$$

If c_ε is bounded, then applying elliptic estimates to (7), we can find some $u^*\in W_0^{1,2}(\Omega)$ such that $u_\varepsilon\to u^*$ in $C^1(\overline{\Omega})$. Clearly, $\|u_\varepsilon\|_{1,\alpha}=1$. Moreover, we get

$$\int_\Omega \left(e^{4\pi u^{*2}}-\beta u^{*2}\right)dx=\lim_{\varepsilon\to 0}\int_\Omega\left(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2\right)dx$$
$$=\sup_{u\in W_0^{1,2}(\Omega),\,\|u\|_{1,\alpha}\leqslant 1}\int_\Omega (e^{4\pi u^2}-\beta u^2)dx.$$

Hence u^* is the desired extremal function and Theorem 1 holds.

In the sequel, we assume $c_\varepsilon=u_\varepsilon(x_\varepsilon)\to+\infty$ and $x_\varepsilon\to x_0\in\overline{\Omega}$ as $\varepsilon\to 0$. By a result of Gidas-Ni-Nirenberg [7], one has $x_0\notin\partial\Omega$. Since u_ε is bounded in $W_0^{1,2}(\Omega)$, we can assume without loss of generality, $u_\varepsilon\rightharpoonup u_0$ weakly in $W_0^{1,2}(\Omega)$, $u_\varepsilon\to u_0$ strongly in $L^q(\Omega)$ for any $q>1$, $u_\varepsilon\to u_0$ almost everywhere in Ω. The following energy concentration phenomenon is crucial in blow-up analysis. Namely

LEMMA 4. *$u_0\equiv 0$ and $|\nabla u_\varepsilon|^2dx\rightharpoonup\delta_{x_0}$ weakly in sense of measure as $\varepsilon\to 0$, where δ_{x_0} is the usual Dirac measure centered at x_0.*

Proof. Suppose $u_0\not\equiv 0$, then we obtain

$$\int_\Omega |\nabla(u_\varepsilon-u_0)|^2dx=1-\left(\int_\Omega|\nabla u_0|^2dx-\alpha\int_\Omega u_0^2dx\right)+o_\varepsilon(1). \tag{9}$$

In view of (9), Lions' inequality (2) implies that $e^{(4\pi-\varepsilon)u_\varepsilon^2}$ is bound in $L^s(\Omega)$ for some $s>1$. Applying elliptic estimates to (7), we have u_ε is bounded in $W^{2,s}(\Omega)$. Then the Sobolev embedding theorem implies that u_ε is bounded in $C^0(\overline{\Omega})$, which contradicts $c_\varepsilon\to+\infty$ as $\varepsilon\to 0$. Therefore $u_0\equiv 0$. Consequently, we have $\lim_{\varepsilon\to 0}\int_\Omega|\nabla u_\varepsilon|^2dx=1$.

We next proof $|\nabla u_\varepsilon|^2dx\rightharpoonup\delta_{x_0}$. For otherwise, we can find $r_0>0$ and $\eta>0$ such that $\mathbb{B}_{r_0}(x_0)\subset\Omega$ and

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{B}_{r_0}(x_0)}|\nabla u_\varepsilon|^2dx\leqslant 1-\eta.$$

One may choose a cut-off function $\phi \in C_0^\infty(\mathbb{B}_{r_0}(x_0))$ verifying that $\phi(x) \equiv 1$ on $\mathbb{B}_{r_0/2}(x_0)$, $0 \leqslant \phi(x) \leqslant 1$ on $\mathbb{B}_{r_0}(x_0)$, and

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{B}_{r_0}(x_0)} |\nabla(\phi u_\varepsilon)|^2 dx \leqslant 1-\eta.$$

The classical Trudinger-Moser inequality (1) implies that $e^{(4\pi-\varepsilon)\phi^2 u_\varepsilon^2}$ is bounded in $L^{\frac{2}{2-\eta}}(\mathbb{B}_{r_0/2}(x_0))$. Applying elliptic estimates to (7), we get u_ε is bounded in $C^0(\overline{\mathbb{B}_{r_0/4}(x_0)})$ contradicting $c_\varepsilon \to +\infty$ again. This completes the proof of the lemma. $\square$

To proceed, we set

$$r_\varepsilon = \sqrt{\lambda_\varepsilon} c_\varepsilon^{-1} e^{-(2\pi-\varepsilon/2)c_\varepsilon^2}.$$

Then we have

LEMMA 5. *For any $\gamma < 2\pi$, there holds $r_\varepsilon e^{\gamma c_\varepsilon^2} \to 0$ as $\varepsilon \to 0$ and consequently $r_\varepsilon \to 0$ as $\varepsilon \to 0$.*

Proof. By the definition of r_ε, we obtain

$$\begin{aligned} r_\varepsilon^2 e^{2\gamma c_\varepsilon^2} &= c_\varepsilon^{-2} e^{-(4\pi-\varepsilon-2\gamma)c_\varepsilon^2} \int_\Omega u_\varepsilon^2 \left(e^{(4\pi-\varepsilon)u_\varepsilon^2} - \frac{\beta}{4\pi-\varepsilon} \right) dx \\ &\leqslant c_\varepsilon^{-2} \int_\Omega u_\varepsilon^2 e^{2\gamma u_\varepsilon^2} dx + o_\varepsilon(1). \end{aligned} \tag{10}$$

Since $\gamma < 2\pi$, we can choose $p_1 > 1$ such that $\gamma p_1 < 2\pi$. In view of the classical Trudinger-Moser inequality (1), we have by the Hölder inequality

$$\int_\Omega u_\varepsilon^2 e^{2\gamma u_\varepsilon^2} dx \leqslant \left(\int_\Omega e^{2\gamma p_1 u_\varepsilon^2} dx \right)^{\frac{1}{p_1}} \left(\int_\Omega u_\varepsilon^{2p_2} dx \right)^{\frac{1}{p_2}} = o_\varepsilon(1), \tag{11}$$

where $1/p_1 + 1/p_2 = 1$. Combining (10) and (11), we obtain $r_\varepsilon e^{\gamma c_\varepsilon^2} \to 0$ as $\varepsilon \to 0$. It is not difficult to see that $r_\varepsilon \to 0$ as $\varepsilon \to 0$. $\square$

Denote

$$\Omega_\varepsilon = \{x \in \mathbb{R}^2 : x_\varepsilon + r_\varepsilon x \in \Omega\}.$$

Define two blow-up functions

$$\begin{cases} \psi_\varepsilon(x) = c_\varepsilon^{-1} u_\varepsilon(x_\varepsilon + r_\varepsilon x), & x \in \Omega_\varepsilon, \\ \varphi_\varepsilon(x) = c_\varepsilon(u_\varepsilon(x_\varepsilon + r_\varepsilon x) - c_\varepsilon), & x \in \Omega_\varepsilon. \end{cases} \tag{12}$$

We now investigate the convergence behavior of ψ_ε and φ_ε. More precisely, we have

LEMMA 6. *$\psi_\varepsilon \to 1$ in $C^{1,\theta}_{\mathrm{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$; $\varphi_\varepsilon \to \varphi$ in $C^{1,\theta}_{\mathrm{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$, where*

$$\varphi(x) = -\frac{1}{4\pi} \log(1 + \pi |x|^2).$$

Proof. A direct calculation shows

$$\Delta\psi_\varepsilon = -\alpha r_\varepsilon^2\psi_\varepsilon + \frac{\beta}{4\pi-\varepsilon} r_\varepsilon^2\lambda_\varepsilon^{-1}\psi_\varepsilon - c_\varepsilon^{-2}\psi_\varepsilon e^{(4\pi-\varepsilon)(u_\varepsilon^2(x_\varepsilon+r_\varepsilon x)-c_\varepsilon^2)}. \tag{13}$$

Since $|\psi_\varepsilon| \leqslant 1$, $u_\varepsilon^2 \leqslant c_\varepsilon^2$ and $r_\varepsilon \to 0$ as $\varepsilon \to 0$, we have by applying elliptic estimate to (13) that $\psi_\varepsilon \to \psi$ in $C^{1,\theta}_{\text{loc}}(\mathbb{R}^2)$, where ψ is a bounded harmonic function in $\mathbb{R}^2$. The Liouville theorem leads to $\psi \equiv 1$. Also we have

$$\Delta\varphi_\varepsilon = -\alpha c_\varepsilon^2 r_\varepsilon^2\psi_\varepsilon + \frac{\beta}{4\pi-\varepsilon} c_\varepsilon^2 r_\varepsilon^2\lambda_\varepsilon^{-1}\psi_\varepsilon - \psi_\varepsilon e^{(4\pi-\varepsilon)(u_\varepsilon^2(x_\varepsilon+r_\varepsilon x)-c_\varepsilon^2)}. \tag{14}$$

In view of Lemma 5, we have by applying elliptic estimates to (14) that $\varphi_\varepsilon \to \varphi$ in $C^{1,\theta}_{\text{loc}}(\mathbb{R}^2)$, where φ satisfies

$$\begin{cases} -\Delta\varphi = e^{8\pi\varphi} \quad \text{in} \quad \mathbb{R}^2 \\ \varphi(0) = 0 = \sup_{\mathbb{R}^2}\varphi \\ \int_{\mathbb{R}^2} e^{8\pi\varphi}dx \leqslant 1. \end{cases}$$

By a classification result of Chen-Li [3], we conclude that

$$\varphi(x) = -\frac{1}{4\pi}\log(1+\pi|x|^2)$$

and

$$\int_{\mathbb{R}^2} e^{8\pi\varphi}dx = 1. \tag{15}$$

□

Now, we will consider the convergence behavior of u_ε away from the concentration point x_0. Similar to [10, 1], define

$$u_{\varepsilon,\gamma} = \min\{\gamma c_\varepsilon, u_\varepsilon\},$$

then we have

LEMMA 7. *For any $0 < \gamma < 1$, there holds*

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_{\varepsilon,\gamma}|^2 dx = \gamma.$$

Proof. Testing the equation (7) by $(u_\varepsilon - \gamma c_\varepsilon)^+$, we obtain for any fixed $R > 0$,

$$\int_\Omega |\nabla(u_\varepsilon-\gamma c_\varepsilon)^+|^2 dx = \alpha\int_\Omega u_\varepsilon(u_\varepsilon-\gamma c_\varepsilon)^+ dx - \beta\int_\Omega \frac{u_\varepsilon(u_\varepsilon-\gamma c_\varepsilon)^+}{\lambda_\varepsilon(4\pi-\varepsilon)}dx + \frac{1}{\lambda_\varepsilon}\int_\Omega u_\varepsilon(u_\varepsilon-\gamma c_\varepsilon)^+ e^{(4\pi-\varepsilon)u_\varepsilon^2}dx$$

$$\geqslant \frac{1}{\lambda_\varepsilon}\int_{\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} u_\varepsilon(u_\varepsilon-\gamma c_\varepsilon)^+ e^{(4\pi-\varepsilon)u_\varepsilon^2}dx+o_\varepsilon(1)$$
$$\geqslant (1-\gamma)(1+o_\varepsilon(1))\int_{\mathbb{B}_R(0)} e^{8\pi\varphi}dx+o_\varepsilon(1).$$

Letting $\varepsilon\to 0$ first and then $R\to+\infty$ in the above inequality, we have by (15) that

$$\liminf_{\varepsilon\to 0}\|\nabla(u_\varepsilon-\gamma c_\varepsilon)^+)\|_2^2\geqslant 1-\gamma. \tag{16}$$

Similarly as above, testing (7) by $u_{\varepsilon,\gamma}$, we obtain

$$\liminf_{\varepsilon\to 0}\|\nabla u_{\varepsilon,\gamma}\|_2^2 dx\geqslant \gamma. \tag{17}$$

Note that

$$\|\nabla u_{\varepsilon,\gamma}\|_2^2+\|\nabla(u_\varepsilon-\gamma c_\varepsilon)^+\|_2^2=\|u_\varepsilon\|_{1,\alpha}^2+\alpha\|u_\varepsilon\|_2^2=1+o_\varepsilon(1). \tag{18}$$

Combining (16), (17) and (18), we finish the proof of the lemma. □

As a consequence of Lemma 7, we have the following:

LEMMA 8. *There holds*

$$\lim_{\varepsilon\to 0}\int_\Omega (e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx\leqslant |\Omega|+\limsup_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}.$$

Proof. Let $0<\gamma<1$, we have

$$\begin{aligned}\int_\Omega (e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx &= \int_\Omega\left(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\frac{\beta}{4\pi-\varepsilon}\right)dx+\int_\Omega\frac{\beta}{4\pi-\varepsilon}dx+o_\varepsilon(1)\\ &=\int_\Omega\frac{\beta}{4\pi-\varepsilon}dx+\int_{u_\varepsilon\leqslant\gamma c_\varepsilon}\left(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\frac{\beta}{4\pi-\varepsilon}\right)dx\\ &\quad+\int_{u_\varepsilon>\gamma c_\varepsilon}\left(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\frac{\beta}{4\pi-\varepsilon}\right)dx+o_\varepsilon(1)\\ &\leqslant \int_\Omega\frac{\beta}{4\pi-\varepsilon}dx+\int_\Omega\left(e^{(4\pi-\varepsilon)u_{\varepsilon,\gamma}^2}-\frac{\beta}{4\pi-\varepsilon}\right)dx+\frac{\lambda_\varepsilon}{\gamma^2c_\varepsilon^2}+o_\varepsilon(1)\\ &=\int_\Omega e^{(4\pi-\varepsilon)u_{\varepsilon,\gamma}^2}dx+\frac{\lambda_\varepsilon}{\gamma^2c_\varepsilon^2}+o_\varepsilon(1).\end{aligned} \tag{19}$$

Note that $u_{\varepsilon,\gamma}$ converges to 0 almost everywhere. Hence $\int_\Omega e^{(4\pi-\varepsilon)u_{\varepsilon,\gamma}^2}dx$ converges to $|\Omega|$. Passing to the limit $\varepsilon\to 0$ in (19), we have

$$\lim_{\varepsilon\to 0}\int_\Omega (e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx\leqslant |\Omega|+\frac{1}{\gamma^2}\limsup_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}.$$

Letting $\gamma\to 1$, we get the desired result. □

Clearly, Lemma 8 implies that

$$\lim_{\varepsilon\to 0}\frac{c_\varepsilon}{\lambda_\varepsilon}=0. \tag{20}$$

This result will be applied to prove the following:

LEMMA 9. *For any* $\phi\in C^2(\overline{\Omega})$, *we have*

$$\lim_{\varepsilon\to 0}\int_\Omega \lambda_\varepsilon^{-1}c_\varepsilon u_\varepsilon\left(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\frac{\beta}{4\pi-\varepsilon}\right)dx=\phi(x_0).$$

Proof. Let $\phi\in C^2(\overline{\Omega})$ be fixed. Write for simplicity $g_\varepsilon=\lambda_\varepsilon^{-1}c_\varepsilon u_\varepsilon\left(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\frac{\beta}{4\pi-\varepsilon}\right)$. For any fixed γ, $0<\gamma<1$, there holds

$$\int_\Omega g_\varepsilon\phi dx=\int_{u_\varepsilon<\gamma c_\varepsilon}g_\varepsilon\phi dx+\int_{\{u_\varepsilon\geqslant\gamma c_\varepsilon\}\setminus\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)}g_\varepsilon\phi dx+\int_{\{u_\varepsilon\geqslant\gamma c_\varepsilon\}\cap\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)}g_\varepsilon\phi dx. \tag{21}$$

We will estimate the right-hand integrals of (21). Obviously

$$\int_{u_\varepsilon<\gamma c_\varepsilon}g_\varepsilon\phi dx=\frac{c_\varepsilon}{\lambda_\varepsilon}\int_{u_\varepsilon<\gamma c_\varepsilon}\phi u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}dx-\frac{c_\varepsilon}{\lambda_\varepsilon}\int_{u_\varepsilon<\gamma c_\varepsilon}\frac{\beta\phi u_\varepsilon}{4\pi-\varepsilon}dx. \tag{22}$$

Let $1<s<1/\gamma$ be fixed and $1/s+1/t=1$. Using Hölder inequality and the classical Trudinger-Moser inequlity (1), we obtain

$$\begin{aligned}\left|\int_{u_\varepsilon<\gamma c_\varepsilon}\phi u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2}dx\right| &\leqslant \sup_\Omega|\phi|\int_\Omega u_{\varepsilon,\gamma}e^{(4\pi-\varepsilon)u_{\varepsilon,\gamma}^2}dx\\ &\leqslant \sup_\Omega|\phi|\left(\int_\Omega u_{\varepsilon,\gamma}^t dx\right)^{1/t}\left(\int_\Omega e^{(4\pi-\varepsilon)su_{\varepsilon,\gamma}^2}dx\right)^{1/s}\\ &= o_\varepsilon(1).\end{aligned} \tag{23}$$

It is easy to see that

$$\left|\int_{u_\varepsilon<\gamma c_\varepsilon}\frac{\beta\phi u_\varepsilon}{4\pi-\varepsilon}dx\right|\leqslant C\sup_\Omega|\phi|\int_\Omega u_\varepsilon dx=o_\varepsilon(1). \tag{24}$$

Here we apply the fact that $u_\varepsilon\to 0$ in $L^q(\Omega)$ for any $q>0$. Inserting (20), (23) and (24) into (22), we get

$$\int_{u_\varepsilon<\gamma c_\varepsilon}g_\varepsilon\phi dx=o_\varepsilon(1). \tag{25}$$

It follows from Lemma 5 that $\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)\subset\{u_\varepsilon\geqslant\gamma c_\varepsilon\}$ for $\varepsilon>0$ sufficiently small. Hence we have

$$\int_{\{u_\varepsilon\geqslant\gamma c_\varepsilon\}\cap\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)}g_\varepsilon\phi dx=\phi(x_0)(1+o_\varepsilon(1))\int_{\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)}\lambda_\varepsilon^{-1}c_\varepsilon u_\varepsilon\left(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\frac{\beta}{4\pi-\varepsilon}\right)dx$$

$$= \phi(x_0)(1+o_\varepsilon(1))\left(\int_{\mathbb{B}_R(0)} e^{8\pi\varphi}dx + o_\varepsilon(1)\right)$$
$$= \phi(x_0)(1+o_\varepsilon(1)+o_R(1)).$$

Letting $\varepsilon \to 0$ first, then $R \to +\infty$, we have

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{\{u_\varepsilon \geqslant \gamma c_\varepsilon\}\cap \mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} g_\varepsilon \phi dx = \phi(x_0). \tag{26}$$

For any $\phi \in C^2(\overline{\Omega})$, we calculate

$$\int_{\{u_\varepsilon \geqslant \gamma c_\varepsilon\}\setminus \mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} g_\varepsilon \phi dx \leqslant \frac{1}{\gamma}\sup_\Omega |\phi| \int_{\{u_\varepsilon \geqslant \gamma c_\varepsilon\}\setminus \mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} \lambda_\varepsilon^{-1} u_\varepsilon^2 \left(e^{(4\pi-\varepsilon)u_\varepsilon^2} - \frac{\beta}{4\pi-\varepsilon}\right) dx$$
$$\leqslant \frac{1}{\gamma}\sup_\Omega |\phi| \left(1 - \int_{\mathbb{B}_R(0)} e^{8\pi\varphi}dx + o_\varepsilon(1)\right) = o_\varepsilon(1) + o_R(1).$$

This implies that

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{\{u_\varepsilon \geqslant \gamma c_\varepsilon\}\setminus \mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} g_\varepsilon \phi dx = 0. \tag{27}$$

Inserting (25) - (27) into (21), we finish the proof of Lemma 9. □

LEMMA 10. *For any* $1<q<2$, $c_\varepsilon u_\varepsilon \rightharpoonup G$ *weakly in* $W_0^{1,q}(\Omega)$, *where* G *is a distributional solution to*

$$\begin{cases} -\Delta G = \delta_{x_0} + \alpha G, \\ G = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{28}$$

Moreover, $c_\varepsilon u_\varepsilon \to G$ *in* $C^1_{\mathrm{loc}}(\overline{\Omega}\setminus\{x_0\})$.

Proof. Multiplying both sides of the equation (7) by c_ε, one has

$$-\Delta(c_\varepsilon u_\varepsilon) - \alpha c_\varepsilon u_\varepsilon = \lambda_\varepsilon^{-1} c_\varepsilon u_\varepsilon \left(e^{(4\pi-\varepsilon)u_\varepsilon^2} - \frac{\beta}{4\pi-\varepsilon}\right). \tag{29}$$

By Lemma 9, $g_\varepsilon = \lambda_\varepsilon^{-1} c_\varepsilon u_\varepsilon \left(e^{(4\pi-\varepsilon)u_\varepsilon^2} - \frac{\beta}{4\pi-\varepsilon}\right)$ is bounded in $L^1(\Omega)$. We claim that $c_\varepsilon u_\varepsilon$ is bounded in $L^1(\Omega)$. Suppose not, we assume that $\|c_\varepsilon u_\varepsilon\|_1 \to +\infty$ as $\varepsilon \to 0$. Set a new sequence of functions

$$w_\varepsilon = \frac{c_\varepsilon u_\varepsilon}{\|c_\varepsilon u_\varepsilon\|_1}.$$

Then we have $\|w_\varepsilon\|_1 = 1$ and

$$-\Delta w_\varepsilon = \alpha w_\varepsilon + \frac{1}{\|c_\varepsilon u_\varepsilon\|_1}\frac{c_\varepsilon u_\varepsilon}{\lambda_\varepsilon}\left(e^{(4\pi-\varepsilon)u_\varepsilon^2} - \frac{\beta}{4\pi-\varepsilon}\right). \tag{30}$$

We can derive from Lemma 9 and the definition of w_ε that Δw_ε is bounded in $L^1(\Omega)$. Using an argument of Struwe ([18], Theorem 2.2) to (30), we have w_ε is bounded in $W_0^{1,q}(\Omega)$ for any $1<q<2$. Assume $w_\varepsilon \rightharpoonup w$ weakly in $W_0^{1,q}(\Omega)$. In particular, $w_\varepsilon \to w$ in $L^1(\Omega)$. Since $g_\varepsilon/\|c_\varepsilon u_\varepsilon\|_1 \to 0$ in $L^1(\Omega)$, we conclude that w is a distribution solution to the equation

$$-\Delta w - \alpha w = 0.$$

Further, we have $w \equiv 0$. This contradicts $\|w\|_1 = \lim_{\varepsilon\to 0}\|w_\varepsilon\|_1 = 1$ and confirms our claim.

Since $g_\varepsilon + \alpha c_\varepsilon u_\varepsilon$ is bounded in $L^1(\Omega)$, again by Theorem 2.2 in [18], we have $c_\varepsilon u_\varepsilon$ is bounded in $W_0^{1,q}(\Omega)$ for $1<q<2$. Hence we obtain

$$\begin{aligned} c_\varepsilon u_\varepsilon &\rightharpoonup G \quad \textit{weakly} \quad \textit{in} \quad W_0^{1,q}(\Omega) \quad (1<q<2), \\ c_\varepsilon u_\varepsilon &\to G \quad \textit{strongly} \quad \textit{in} \quad L^p(\Omega) \quad (\forall p>1). \end{aligned}$$

For any fixed $r>0$, choose a cut-off function $\eta \in C_0^1(\Omega\backslash\mathbb{B}_r(x_0))$ such that $\eta \equiv 1$ on $\Omega\backslash\mathbb{B}_{2r}(x_0)$. Then one has $\|\nabla(\eta u_\varepsilon)\|_2 \to 0$ as $\varepsilon \to 0$. Therefore $e^{(4\pi-\varepsilon)(\eta u_\varepsilon)^2}$ is bounded in $L^s(\Omega\setminus\mathbb{B}_r(x_0))$ for any $s>1$, and $e^{(4\pi-\varepsilon)u_\varepsilon^2}$ is bounded in $L^s(\Omega\backslash\mathbb{B}_{2r}(x_0))$. Applying the elliptic estimate to (29), we have $c_\varepsilon u_\varepsilon \to G$ in $C^1(\overline{\Omega}\backslash\mathbb{B}_{4r}(x_0))$. By Lemma 9, we obtain $g_\varepsilon \rightharpoonup \delta_{x_0}$ in sense of measure, where δ_{x_0} means the Dirac measure centered at x_0. In view of (29), G is a distributional solution to (28). □

By elliptic estimates, G takes the form:

$$G = -\frac{1}{2\pi}\log|x-x_0| + A_{x_0} + \widetilde{\psi}(x), \tag{31}$$

where A_{x_0} is a constant depending on x_0, $\widetilde{\psi}(x) \in C^1(\overline{\Omega})$ and $\widetilde{\psi}(x_0) = 0$.

4. Upper bound estimate

In this section, we need the following Carleson-Chang's result to derive an upper bound of the integral $\int_\Omega (e^{(4\pi-\varepsilon)u_\varepsilon^2} - \beta u_\varepsilon^2)dx$. Namely

LEMMA 11. *(Carleson-Chang [2]) Let $\mathbb{B}$ be the unit disc in $\mathbb{R}^2$. Assume v_ε is a sequence of functions in $W_0^{1,2}(\mathbb{B})$. If $\int_\mathbb{B}|\nabla v_\varepsilon|^2dx = 1$ and $|\nabla v_\varepsilon|^2dx \rightharpoonup \delta_0$ as $\varepsilon \to 0$ weakly in sense of measure. Then*

$$\limsup_{\varepsilon\to 0}\int_\mathbb{B}(e^{4\pi v_\varepsilon^2}-1)dx \leqslant e\pi.$$

In view of (28) and (31), we have by the divergence theorem

$$\begin{aligned} \int_{\Omega\backslash\mathbb{B}_\delta(x_0)}|\nabla G|^2dx &= -\int_{\Omega\backslash\mathbb{B}_\delta(x_0)} G\Delta G dx + \int_{\partial(\Omega\backslash\mathbb{B}_\delta(x_0))} G\frac{\partial G}{\partial\nu}ds \\ &= -\frac{1}{2\pi}\log\delta + A_{x_0} + \alpha\int_\Omega G^2dx + o_\delta(1). \end{aligned}$$

Since $c_\varepsilon u_\varepsilon \to G$ in $C^1_{\text{loc}}(\Omega\backslash\{x_0\})$, we consequently get

$$\int_{\Omega\backslash\mathbb{B}_\delta(x_0)} |\nabla u_\varepsilon|^2 dx = \frac{1}{c_\varepsilon^2}\left(-\frac{1}{2\pi}\log\delta + A_{x_0} + \alpha\int_\Omega G^2 dx + o_\delta(1) + o_\varepsilon(1)\right). \quad (32)$$

Let $s_\varepsilon = \sup_{\partial\mathbb{B}_\delta(x_0)} u_\varepsilon$ and $\widetilde{u}_\varepsilon = (u_\varepsilon - s_\varepsilon)^+$, the positive part of $u_\varepsilon - s_\varepsilon$. Then $\widetilde{u}_\varepsilon \in W_0^{1,2}(\mathbb{B}_\delta(x_0))$. Since $\int_{\mathbb{B}_\delta(x_0)} |\nabla u_\varepsilon|^2 dx = 1 - \int_{\Omega\backslash\mathbb{B}_\delta(x_0)} |\nabla u_\varepsilon|^2 dx + \alpha\int_\Omega u_\varepsilon^2 dx$ and $\int_\Omega u_\varepsilon^2 dx = o_\varepsilon(1)$, we have by (32) that

$$\int_{\mathbb{B}_\delta(x_0)} |\nabla \widetilde{u}_\varepsilon|^2 dx \leqslant 1 - \frac{1}{c_\varepsilon^2}\left(-\frac{1}{2\pi}\log\delta + A_{x_0} + \alpha\int_\Omega G^2 dx + o_\delta(1) + o_\varepsilon(1)\right).$$

By Lemma 11, we get

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{B}_\delta(x_0)} (e^{4\pi\widetilde{u}_\varepsilon^2/\tau_\varepsilon} - 1)dx \leqslant \pi e\delta^2, \quad (33)$$

where $\tau_\varepsilon = \int_{\mathbb{B}_\delta(x_0)} |\nabla u_\varepsilon|^2 dx$. Moreover, we know from Lemma 6 that $u_\varepsilon = c_\varepsilon + o_\varepsilon(1)$ on $\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)$. Hence, there holds on $\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)$

$$\begin{aligned}(4\pi-\varepsilon)u_\varepsilon^2 &\leqslant (4\pi-\varepsilon)(\widetilde{u}_\varepsilon + s_\varepsilon)^2 \leqslant 4\pi\widetilde{u}_\varepsilon^2 + 8\pi\widetilde{u}_\varepsilon s_\varepsilon + o_\varepsilon(1)\\ &\leqslant 4\pi\widetilde{u}_\varepsilon^2 - 4\log\delta + 8\pi A_{x_0} + o_\delta(1) + o_\varepsilon(1)\\ &\leqslant 4\pi\widetilde{u}_\varepsilon^2/\tau_\varepsilon - 2\log\delta + 4\pi A_{x_0} + o(1),\end{aligned}$$

where $o(1)\to 0$ as $\varepsilon\to 0$ first and next $\delta\to 0$. Therefore

$$\begin{aligned}\int_{\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} (e^{(4\pi-\varepsilon)u_\varepsilon^2} - \beta u_\varepsilon^2)dx &\leqslant \int_{\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} e^{(4\pi-\varepsilon)u_\varepsilon^2}dx + o(1)\\ &\leqslant \frac{e^{4\pi A_{x_0}+o(1)}}{\delta^2}\int_{\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} (e^{4\pi\widetilde{u}_\varepsilon^2/\tau_\varepsilon} - 1)dx + o(1)\\ &\leqslant \frac{e^{4\pi A_{x_0}+o(1)}}{\delta^2}\int_{\mathbb{B}_\delta(x_0)} (e^{4\pi\widetilde{u}_\varepsilon^2/\tau_\varepsilon} - 1)dx + o(1). \quad (34)\end{aligned}$$

Combining (33) and (34), one concludes for any fixed $R>0$

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} (e^{(4\pi-\varepsilon)u_\varepsilon^2} - \beta u_\varepsilon^2)dx \leqslant \pi e^{4\pi A_{x_0}+1}. \quad (35)$$

On the other hand, we have

$$\begin{aligned}\int_{\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)} (e^{(4\pi-\varepsilon)u_\varepsilon^2} - \beta u_\varepsilon^2)dx &= r_\varepsilon^2 e^{(4\pi-\varepsilon)c_\varepsilon^2}\left(\int_{\mathbb{B}_R(0)} e^{8\pi\varphi}dy + o_\varepsilon(1)\right) + o_\varepsilon(1)\\ &= \frac{\lambda_\varepsilon}{c_\varepsilon^2}(1 + o_R(1) + o_\varepsilon(1)) + o_\varepsilon(1).\end{aligned}$$

It follows that

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\mathbb{B}_{Rr_\varepsilon}(x_\varepsilon)}(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx=\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2} \tag{36}$$

In view of Lemma 8, (35) and (36), we have

$$\begin{aligned}&\sup_{u\in W_0^{1,2}(\Omega),\,\|u\|_{1,\alpha}\leqslant 1}\int_\Omega(e^{4\pi u^2}-\beta u^2)dx\\ &=\limsup_{\varepsilon\to 0}\int_\Omega(e^{(4\pi-\varepsilon)u_\varepsilon^2}-\beta u_\varepsilon^2)dx\leqslant|\Omega|+\pi e^{4\pi A_{x_0}+1}.\end{aligned} \tag{37}$$

5. Test functions

In this section, we will construct a family of test functions $\phi_\varepsilon\in W_0^{1,2}(\Omega)$ such that $\|\phi_\varepsilon\|_{1,\alpha}=1$ and

$$\limsup_{\varepsilon\to 0}\int_\Omega(e^{4\pi\phi_\varepsilon^2}-\beta\phi_\varepsilon^2)dx>|\Omega|+\pi e^{4\pi A_{x_0}+1} \tag{38}$$

for sufficiently small $\varepsilon>0$. This leads to a contradiction between (37) and (38). Then we immediately conclude that c_ε is bounded. Therefore, Theorem 1 holds by elliptic estimates. For this purpose, we write $r=|x-x_0|$ and set

$$\phi_\varepsilon=\begin{cases}C+\frac{1}{C}\left(-\frac{1}{4\pi}\log\left(1+\pi\frac{r^2}{\varepsilon^2}\right)+B\right) & \text{if}\quad r\leqslant R\varepsilon\\ \frac{G-\eta\widetilde{\psi}}{C} & \text{if}\quad R\varepsilon<r<2R\varepsilon\\ \frac{G}{C} & \text{if}\quad r\geqslant 2R\varepsilon,\end{cases}$$

where $\eta\in C_0^\infty(\mathbb{B}_{2R\varepsilon}(x_0))$ satisfying $\eta\equiv 1$ on $\mathbb{B}_{R\varepsilon}(x_0)$ and $\|\nabla\eta\|_{L^\infty}=O(\frac{1}{R\varepsilon})$, G is given as in (31), $R=-\log\varepsilon$, B and C are constants depending only on ε to be determined later. To ensure $\phi_\varepsilon\in W_0^{1,2}(\Omega)$, let

$$C+\frac{1}{C}\left(-\frac{1}{4\pi}\log\left(1+\pi R^2\right)+B\right)=\frac{1}{C}\left(-\frac{1}{2\pi}\log(R\varepsilon)+A_{x_0}\right),$$

which gives that

$$4\pi C^2=-\log\varepsilon^2+\log\pi+4\pi A_{x_0}-4\pi B+O\left(\frac{1}{R^2}\right). \tag{39}$$

A straightforward calculation shows

$$\int_\Omega(|\nabla\phi_\varepsilon|^2-\alpha\phi_\varepsilon^2)dx=\frac{1}{4\pi C^2}\left(-\log\varepsilon^2+\log\pi+4\pi A_{x_0}-1+O(R\varepsilon\log(R\varepsilon))+O\left(\frac{1}{R^2}\right)\right).$$

Setting $\|\phi_\varepsilon\|_{1,\alpha}=1$, we get

$$4\pi C^2=-1+4\pi A_{x_0}+\log\pi-\log\varepsilon^2+O(R\varepsilon\log(R\varepsilon))+O\left(\frac{1}{R^2}\right). \tag{40}$$

Combining (39) and (40) , we get

$$B = \frac{1}{4\pi} + O(R\varepsilon \log(R\varepsilon)) + O\left(\frac{1}{R^2}\right). \tag{41}$$

For all $x \in \mathbb{B}_{R\varepsilon}(x_0)$, we have by (39) and (41) that

$$\begin{aligned} 4\pi\phi_\varepsilon^2 &\geqslant 4\pi C^2 + 8\pi B - 2\log\left(1 + \pi\frac{r^2}{\varepsilon^2}\right) \\ &= -2\log\left(1+\pi\frac{r^2}{\varepsilon^2}\right) - \log\varepsilon^2 + \log\pi + 4\pi A_{x_0} + 1 + O\left(\frac{1}{R^2}\right). \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathbb{B}_{R\varepsilon}(x_0)} (e^{4\pi\phi_\varepsilon^2} - \beta\phi_\varepsilon^2)dx &\geqslant e^{\log\pi + 4\pi A_0 + 1 + O\left(\frac{1}{R^2}\right)} \int_{\mathbb{B}_R(0)} \frac{1}{(1+\pi|x|^2)^2} dx + O\left(\frac{1}{R^2}\right) \\ &= \pi e^{4\pi A_{x_0}+1} + O\left(\frac{1}{R^2}\right). \end{aligned} \tag{42}$$

Moreover, on $\Omega \backslash \mathbb{B}_{R\varepsilon}(x_0)$, we have the estimate

$$\begin{aligned} \int_{\Omega\backslash\mathbb{B}_{R\varepsilon}(x_0)} (e^{(4\pi-\varepsilon)\phi_\varepsilon^2} - \beta\phi_\varepsilon^2)dx &\geqslant \int_{\Omega\backslash\mathbb{B}_{2R\varepsilon}(x_0)} (e^{4\pi\phi_\varepsilon^2} - \beta\phi_\varepsilon^2)dx \\ &\geqslant \int_{\Omega\backslash\mathbb{B}_{2R\varepsilon}(x_0)} (1 + 4\pi\phi_\varepsilon^2 - \beta\phi_\varepsilon^2)dx \\ &\geqslant |\Omega| + \frac{4\pi-\beta}{C^2}\left(\int_\Omega G^2 dx + o_\varepsilon(1)\right). \end{aligned} \tag{43}$$

Since $\beta < 4\pi$ and $C^2/R^2 = o_\varepsilon(1)$, we have by (42) and (43) that

$$\int_\Omega (e^{4\pi\phi_\varepsilon^2} - \beta\phi_\varepsilon^2)dx \geqslant |\Omega| + \pi e^{4\pi A_{x_0}+1} + \frac{4\pi-\beta}{C^2}\left(\int_\Omega G^2 dx + o_\varepsilon(1)\right).$$

This implies that (38) holds provided that $\varepsilon > 0$ is chosen sufficiently small.

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Xianfeng Su
School of Information
Huaibei Normal University
Huaibei, 235000, P. R. China
e-mail: suxf2006@ruc.edu.cn