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EXPONENTIAL TYPE LOCALLY GENERALIZED STRICTLY DOUBLE DIAGONALLY TENSORS AND EIGENVALUE LOCALIZATION

JIANZHOU LIU AND LIANG XIONG

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Abstract. In this paper, we introduce exponential type locally generalized strictly double diagonally dominant tensors. This concept extends the concept of strictly diagonally dominant tensors. It is shown that exponential type locally generalized strictly double diagonally dominant tensors must be H-tensors. Furthermore, as applications of exponential type locally generalized strictly double diagonally dominant tensors, we present some new eigenvalue localization sets and checkable sufficient condition for the positive definiteness of even-order real symmetric tensors. Appropriate numerical examples are proposed to illustrate that our new tensors eigenvalue localization sets are more precise than some existing sets in some cases.

1. Introduction

Tensors (also known as multidimensional arrays) and their eigenvalues have become increasingly significant issue in several diverse fields of applied mathematics and computational mathematics, and promoted the development of numerical multilinear algebra. On the other side, they have a great diversity of practical applications, such as higher Markov chain [14], best-rank one approximation in data analysis [12, 13] and positive definiteness of even-order multivariate forms in automatical control [15].

For a positive integer $n \geqslant 2$, let $N = \{1,2,\cdots,n\}$. The set of all real (complex) numbers is denoted by $\mathbb{R}$ $(\mathbb{C})$. We call $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ a complex (real) tensor of order m dimension n, denoted by $\mathbb{C}^{[m\times n]}(\mathbb{R}^{[m\times n]})$, if

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{C}(\mathbb{R}),$$

where $i_j = 1,2,\cdots,n$ for $j = 1,2,\cdots,m$. Apparently, a vector is a tensor of order 1 and a matrix is a tensor of order 2. Given a tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m\times n]}$, $\mathscr{A}$ is nonnegative if every its entry $a_{i_1 i_2 \cdots i_m} \geqslant 0$. A real tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ is called symmetric if

$$a_{i_1 i_2 \cdots i_m} = a_{\pi(i_1 i_2 \cdots i_m)}, \forall \pi \in \Pi_m,$$

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where Π_m is the permutation group of m indices. Particularly, a real tensor of order m dimension n is called the unit tensor , if its entries are $\delta_{i_1 i_2 \cdots i_m}$ for $i_1, \cdots, i_m \in N$, where

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

In 2005, Qi [1] and Lim [2] introduced the concept of eigenvalues for higher order tensors, independently. For a complex tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ of order m dimension n, and an n dimension vector $x = (x_1, x_2, \cdots, x_n)^\top$, $\mathscr{A}x^{m-1}$ is an n dimension vector in $\mathbb{C}^n$, whose i-th component is

$$(\mathscr{A}x^{m-1})_i = \sum_{i_2, i_3, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m},$$

Moreover, if there are a complex number λ and a nonzero complex vector $x = (x_1, x_2, \cdots, x_m)^\top$ such that

$$\mathscr{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathscr{A}$ and x an eigenvector of $\mathscr{A}$ associated with λ, where

$$x^{[m-1]} = \left(x_1^{m-1}, x_2^{m-1}, \cdots, x_n^{m-1}\right)^\top.$$

If λ and x are all real, then λ is called an H-eigenvalue of $\mathscr{A}$ and x is called an H-eigenvector of $\mathscr{A}$ associated with λ [1, 2]. Owing to the vitally significant theoretical significance and extensive practical application of tensor eigenvalues, an increasing number of scholars devote themselves to the study of tensor spectral theory. In particular, eigenvalues of nonnegative tensors develop rapidly in theory and algorithms [16, 17, 18, 19, 4, 5, 14, 25]. In recent years, there is a good deal of literatures on the survey of eigenvalue inclusion sets [1, 20, 6, 21, 10] for general tensors. These eigenvalue inclusion theorems provide abundant conditions for us to identifying the positive definiteness of an even-order real symmetric tensors.

For an mth-degree homogeneous polynomial form of n variables $f(x)$ can be denoted as

$$f(x) = \mathscr{A}x^m = \sum_{i_1, i_2, \cdots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \tag{1.1}$$

where $x \in \mathbb{R}^n$. When m is even, $f(x)$ is called positive definite if

$$f(x) > 0 \quad \forall x \in \mathbb{R}^n, x \neq 0.$$

The positive definiteness of multivariate polynomial $f(x)$ plays a significant role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control [15]. Unfortunately, for $n \geqslant 3$ and $m \geqslant 4$, this issue is a hard problem in mathematics. Furthermore, Qi [1] pointed out that $f(x)$ defined by (1.1) is positive definite if and only if the real symmetric tensor $\mathscr{A}$ is positive definite. Recently, M-tensors that extended from M-matrices was introduced by Wei [11] and Zhang [9], and the spectral theory of M-tensors provided a new method for certifying positive definiteness of a multivariate form. Moreover, Wei et al [11] extended H-matrices to H-tensors as a special case of M-tensors. Li [7] gave some criteria for identifying the positive definiteness of even-order real symmetric tensors by considering the class of nonsingular H-tensors. Since then, more and more researchers have begun to study tensors with special structures, and to study the properties of their eigenvalues and applications in other fields. There are much more works about structured tensors, see [22, 23, 25, 26, 24, 27, 28, 29] for more details.

In this paper, we investigate exponential type locally generalized strictly double diagonally dominant tensors and its application, which is further extension discussion of strictly diagonally dominant tensors. We will prove that exponential type locally generalized strictly double diagonally dominant tensors are H-tensors, and give some criteria for positive definiteness of even-order real symmetric tensors. Finally, as an important application of exponential locally generalized strictly double diagonally dominant tensors, some new tensors eigenvalues inclusion sets are given. Meanwhile, several proper numerical examples are given to show that these new tensors eigenvalue inclusion regions are more precise than some existing results in some cases.

The remainder of this paper is as follows. In Section 2, some preliminaries, that is, definitions and useful lemmas are proposed. In Section 3, we define exponential type locally generalized strictly double diagonally dominant tensors and establish the relationship between exponential type locally generalized strictly double diagonally dominant tensors and H-tensors. Furthermore, some checkable sufficient conditions for the positive definiteness of even-order real symmetric tensors are provided. In Section 4, we present some new eigenvalue localization regions for tensors based on the third part of the analysis and conclusion.

2. Preliminaries

In this section, we start with some definitions and useful lemmas for eigenvalue of general tensor and H-tensors.

DEFINITION 2.1. [7, Definition 2] Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$. $\mathscr{A}$ is called an H-tensor if there is an entrywise positive vector $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ such that for all $i \in N$,

$$|a_{i\cdots i}| x_i^{m-1} > \sum_{\substack{i_2,\cdots,i_m \in N, \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| x_{i_2} \cdots x_{i_m}. \tag{2.1}$$

DEFINITION 2.2. [9, Definition 3.14] A tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$ is called a (strictly) diagonally dominant tensor if for $i \in N$,

$$|a_{i\cdots i}| \geqslant (>) \sum_{\substack{i_2,\cdots,i_m \in N, \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|. \tag{2.2}$$

DEFINITION 2.3. [16, Definition 2.1] Let $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. Then $\mathscr{A}$ is called reducible, if there exists a nonempty proper index subset $I\subset N$ such that

$$a_{i_1i_2\cdots i_m}=0,\quad \forall i_1\in I,\quad \forall i_2\cdots i_m\notin I.$$

If $\mathscr{A}$ is not reducible, then $\mathscr{A}$ is called irreducible.

DEFINITION 2.4. [34, Definition 2.5] An order m dimension n tensor $\mathscr{A}$ is called weakly reducible, if there exists a nonempty proper index subset $I\subset N$ such that

$$\forall a_{i_1i_2\cdots i_m}=0,\quad i_1\in I,\quad \textit{and at least an } i_j\in N\setminus I, j=2,\cdots,m.$$

If $\mathscr{A}$ is not weakly reducible, then we call $\mathscr{A}$ weakly irreducible.

In the case of matrices, we note that there is no difference between irreducible and weakly irreducible. However, an irreducible tensor must be a weakly irreducible tensor, which is not valid in turn. Hence, for the weakly irreducible tensors, the results can also be applied to irreducible tensors.

DEFINITION 2.5. [10, Definition 7] Let $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. For some $i,j\in N,\quad i\neq j$, if there exist indices $k_1,k_2,\cdots,k_r$ with

$$\sum_{\substack{i_2,\cdots,i_m\in N,\\ \delta_{k_si_2\cdots i_m}=0,\\ k_{s+1}\in\{i_2,\cdots,i_m\}}}|a_{k_si_2\cdots i_m}|\neq 0,\quad s=0,1,\cdots,r,$$

where $k_0=i,k_{r+1}=j$, we call there is a nonzero elements chain from i to j.

DEFINITION 2.6. [8, Definition 1.1] Let $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$, $D=diag(d_1,d_2,\cdots,d_n)$. Denote

$$\mathscr{B}=(b_{i_1i_2\cdots i_m})=\mathscr{A}D^{m-1},\quad b_{i_1i_2\cdots i_m}=a_{i_1i_2\cdots i_m}d_{i_2}d_{i_3}\cdots d_{i_m},\quad i_1,i_2,\cdots,i_m\in N,$$

we call $\mathscr{B}$ as the product of the tensor $\mathscr{A}$ and the matrix D.

LEMMA 2.1. [11, Proposition 29] *Let $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. Then $\mathscr{A}$ is an H-tensor if and only if exists a positive diagonal matrix $D\in\mathbb{C}^{n,n}$ such that $\mathscr{A}D^{m-1}$ is strictly diagonally dominant tensor.*

LEMMA 2.2. [31, Theorem 2] *Let $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. Then $\mathscr{A}$ is an H-tensor if and only if $\mathscr{A}D$ is an H-tensor, where $D\in\mathbb{C}^{n,n}$ is an arbitrary positive diagonal matrix.*

LEMMA 2.3. [30, Theorem 3.4] *Let $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. If $\mathscr{A}$ is a weakly irreducible diagonally dominant tensor such that $|a_{i\cdots i}|>\sum_{\substack{i_2,\cdots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|$ for at least one i, then $\mathscr{A}$ is H-tensor.*

LEMMA 2.4. [10, Theorem 6] *Let* $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. *If*

(i) $|a_{i\cdots i}|\geqslant \sum\limits_{\substack{i_2,\cdots,i_m\in N;\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|, \quad \forall i\in N;$

(ii) $N_1=\left\{ i\in N: |a_{i\cdots i}|>\sum\limits_{\substack{i_2,\cdots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|\right\}\neq\emptyset;$

(iii) For any $i\notin N_1$, *there exists a non-zero elements chain from* i *to* j *such that* $j\in N_1$, *then* $\mathscr{A}$ *is a* H*-tensor.*

LEMMA 2.5. [7, Theorem 9] *Let* $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}^{[m\times n]}$ *is an even-order symmetric tensor,* $a_{i\cdots i}>0,\forall i\in N$. *If* $\mathscr{A}$ *is a* H*-tensor,* $\mathscr{A}$ *is positive definite.*

3. Exponential type locally generalized double diagonally dominant tensors and some properties

In this section, we introduce exponential type locally generalized (strictly) double diagonally dominant tensor, which is a generalization for (strictly) diagonally dominant tensors, and the relationship between exponential type locally generalized (strictly) double diagonally dominant tensor and H-tensor are also established. In addition, we present some checkable sufficient condition for the positive definiteness of even-order real symmetric tensors at the end of this section.

Let $N_i\subset N, N=\bigcup\limits_{i=1}^{k}N_i, N_i\bigcap N_j=\emptyset, 1\leqslant i,j\leqslant k, i\neq j$. And $\sigma=(\sigma_1,\sigma_2,\cdots,\sigma_k)$ is a permutation of $(1,2,\cdots,k)$, then $\forall i\in N$, there exist some $1\leqslant\sigma_t\leqslant k$ such that $i\in N_{\sigma_t}$.

For a tensor $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$, we denote

$$\Delta_i^N := \{(i_2,i_3,\cdots,i_m):\delta_{ii_2\cdots i_m}=0, i_s\in N, s=2,3,\cdots,m\},$$
$$\Delta_i^{N_{\sigma_t}} := \{(i_2,i_3,\cdots,i_m):\delta_{ii_2\cdots i_m}=0, i_s\in N_{\sigma_t}, s=2,3,\cdots,m\}.$$

It is evident that $\Delta_i^{N_{\sigma_p}}\bigcap\Delta_i^{N_{\sigma_q}}=\emptyset,\forall i\in N, 1\leqslant p\neq q\leqslant k$. Without loss of generality, we always assume $i\in N_{\sigma_u}, j\in N_{\sigma_v}$ for $1\leqslant u\neq v\leqslant k$ and $1\leqslant i\neq j\leqslant n$. We use the following notation for the rest of the paper.

$$r_i(\mathscr{A})=\sum_{\substack{i_2,\cdots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|=\sum_{i_2,\cdots,i_m\in N}|a_{ii_2\cdots i_m}|-|a_{i\cdots i}|,$$

$$\alpha_i^{\Delta_i^{N_{\sigma_u}}}(\mathscr{A})=\sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N_{\sigma_u}},\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|=\sum_{\substack{i_2,\cdots,i_m\in N_{\sigma_u},\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|,$$

$$\beta_i^{\Delta_i^{N_{\sigma_u}}}(\mathscr{A})=r_i(\mathscr{A})-\alpha_i^{\Delta_i^{N_{\sigma_u}}}(\mathscr{A}),$$

$$N^+ = \left\{ i \in N : |a_{ii\cdots i}| > \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right\},\ N^0 = \left\{ i \in N : |a_{ii\cdots i}| = \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right\},$$

$$J_1 = J_1(\mathscr{A}) = \{ i \in N : |a_{ii\cdots i}| > r_i(\mathscr{A}) \},$$

$$J_2 = J_2(\mathscr{A}) = \{ i \in N : |a_{ii\cdots i}| \leqslant r_i(\mathscr{A}) \},$$

$$\tilde{J}_1 = \left\{ i \in N : \left(|a_{ii\cdots i}| - \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right)^{\frac{1}{m-1}} \left(|a_{jj\cdots j}| - \alpha_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}) \right) > \left(\beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right)^{\frac{1}{m-1}} \beta_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}), i \in N_{\sigma_u}, j \in N_{\sigma_v}, 1 \leqslant \sigma_u \neq \sigma_v \leqslant k \right\},$$

$$\tilde{J}_0 = \left\{ i \in N : \left(|a_{ii\cdots i}| - \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right)^{\frac{1}{m-1}} \left(|a_{jj\cdots j}| - \alpha_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}) \right) = \left(\beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right)^{\frac{1}{m-1}} \beta_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}), i \in N_{\sigma_u}, j \in N_{\sigma_v}, 1 \leqslant \sigma_u \neq \sigma_v \leqslant k \right\}.$$

DEFINITION 3.1. Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$. $\mathscr{A}$ is exponential type generalized locally double diagonally dominant tensor if the following inequality is holds for $\forall i, j \in N, i \neq j$,

$$\left(|a_{ii\cdots i}| - \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right)^{\frac{1}{m-1}} \left(|a_{jj\cdots j}| - \alpha_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}) \right) \geqslant \left(\beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right)^{\frac{1}{m-1}} \beta_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}), \tag{3.1}$$

where $i \in N_{\sigma_u}, j \in N_{\sigma_v}, 1 \leqslant \sigma_u \neq \sigma_v \leqslant k$. $\mathscr{A}$ is exponential type locally generalized strictly double diagonally dominant if the strict inequality holds in (3.1).

Obviously, if $\mathscr{A}$ is exponential type locally generalized double diagonally dominant tensor and $J_1 \neq \emptyset$, then $N = N^0 \bigcup N^+$ holds. And if $\mathscr{A}$ is exponential type locally generalized strictly double diagonally dominant tensor and $J_1 \neq \emptyset$, then $N = N^+$ is satisfied.

THEOREM 3.1. *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$. If $\mathscr{A}$ is exponential type locally generalized strictly double diagonally dominant tensor and $J_1 \neq \emptyset$, then $\mathscr{A}$ is a H-tensor.*

Proof. Since $\mathscr{A}$ is exponential type locally generalized strictly double diagonally dominant tensor and $J_1 \neq \emptyset$, then $N = N^+$ is valid. That is, there exists at most one $1 \leqslant k_0 \leqslant k$ such that

$$|a_{ii\cdots i}| - \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) > \beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}), \quad \forall i \in N_{\sigma_u}, \sigma_u \neq k_0. \tag{3.2}$$

In other words, we have $N \setminus N_{k_0} \subseteq J_1$. If $N_{k_0} = \emptyset$, then $\mathscr{A}$ is strictly diagonally dominant tensor, and $\mathscr{A}$ must be a H-tensor in this situation. Therefore, we always assume

$N_{k_0} \setminus J_1 \neq \emptyset$. Next, we take appropriate positive integer d_{k_0} such that

$$\max_{i\in N\setminus N_{k_0}} \left(\frac{\beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A})}{|a_{ii\cdots i}| - \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A})} \right)^{\frac{1}{m-1}} < d_{k_0} < \min_{j\in N_{k_0}\setminus J_1} \frac{|a_{jj\cdots j}| - \alpha_j^{\Delta_j^{N\sigma_v}}(\mathscr{A})}{\beta_j^{\Delta_j^{N\sigma_v}}(\mathscr{A})}, \tag{3.3}$$

Since $N{=}N^+$ and $i\in N\setminus N_{k_0}, j\in N_{k_0}\setminus J_1$, then we have $|a_{ii\cdots i}|-\alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A})>0, \beta_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}) > 0$. From the inequality of (3.3), one has $0 < d_{k_0} < 1$. Hence, we construct the following positive diagonal matrix $D \in \mathbb{C}^{n,n}$ with diagonal entries

$$d_i = \begin{cases} 1, & i \in N_{k_0}, \\ d_{k_0}, & i \in N \setminus N_{k_0}. \end{cases}$$

Now, we consider tensor $\mathscr{B} = \mathscr{A} D^{m-1} = (b_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$. In view of $\alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) = \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N\sigma_u},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|$ and $\beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) = r_i(\mathscr{A}) - \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A})$, then there is at least $i_l \notin N_{\sigma_u}$, $l = 2,\cdots,m$ for $(i_2,\cdots,i_m) \in \Delta_i^N \setminus \Delta_i^{N\sigma_u}$. That is to say, when one has $(i_2,\cdots,i_m) \in \Delta_i^N \setminus \Delta_i^{N\sigma_u}$, then at least one of $d_{i_2},\cdots,d_{i_m}$ is equal to d_{k_0}.

Thus, for any $i \in N_{k_0} \setminus J_1$, we have $|a_{ii\cdots i}| > \alpha_i^{\Delta_i^{N_{k_0}}}(\mathscr{A}) + \beta_i^{\Delta_i^{N_{k_0}}}(\mathscr{A}) d_{k_0}$ from the right inequality of (3.3), and $|a_{ii\cdots i}| = |b_{ii\cdots i}|$. Then

$$\begin{aligned} & r_i(\mathscr{B}) \\ = & \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N_{k_0}},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| d_{i_2}\cdots d_{i_m} + \sum_{(i_2,\cdots,i_m)\in\Delta_i^N\setminus\Delta_i^{N_{k_0}}} |a_{ii_2\cdots i_m}| d_{i_2}\cdots d_{i_m} \\ = & \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N_{k_0}},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{(i_2,\cdots,i_m)\in\Delta_i^N\setminus\Delta_i^{N_{k_0}}} |a_{ii_2\cdots i_m}| d_{i_2}\cdots d_{i_m} \\ \leqslant & \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N_{k_0}},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{(i_2,\cdots,i_m)\in\Delta_i^N\setminus\Delta_i^{N_{k_0}}} |a_{ii_2\cdots i_m}| d_{k_0} \\ = & \alpha_i^{\Delta_i^{N_{k_0}}}(\mathscr{A}) + \beta_i^{\Delta_i^{N_{k_0}}}(\mathscr{A}) d_{k_0} < |a_{ii\cdots i}| = |b_{ii\cdots i}|. \end{aligned} \tag{3.4}$$

For any $i \in N_{k_0} \cap J_1$, we obtain $|a_{ii\cdots i}| > \alpha_i^{\Delta_i^{N_{k_0}}}(\mathscr{A}) + \beta_i^{\Delta_i^{N_{k_0}}}(\mathscr{A})$ and $|a_{ii\cdots i}| = |b_{ii\cdots i}|$. Then

$$\begin{aligned} & r_i(\mathscr{B}) \\ = & \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N_{k_0}},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| d_{i_2}\cdots d_{i_m} + \sum_{(i_2,\cdots,i_m)\in\Delta_i^N\setminus\Delta_i^{N_{k_0}}} |a_{ii_2\cdots i_m}| d_{i_2}\cdots d_{i_m} \end{aligned}$$

$$\leqslant \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N_{k_0}},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{(i_2,\cdots,i_m)\in\Delta_i^N\setminus\Delta_i^{N_{k_0}}} |a_{ii_2\cdots i_m}|d_{k_0} \tag{3.5}$$
$$= \alpha_i^{\Delta_i^{N_{k_0}}}(\mathscr{A}) + \beta_i^{\Delta_i^{N_{k_0}}}(\mathscr{A})d_{k_0} < \alpha_i^{\Delta_i^{N_{k_0}}}(\mathscr{A}) + \beta_i^{\Delta_i^{N_{k_0}}}(\mathscr{A}) < |a_{ii\cdots i}| = |b_{ii\cdots i}|.$$

And for any $i \in N \setminus N_{k_0}$, we get $|a_{ii\cdots i}|d_{k_0}^{m-1} > \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A})d_{k_0}^{m-1} + \beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A})$ from the left inequality of (3.3), and $|a_{ii\cdots i}|d_{k_0}^{m-1} = |b_{ii\cdots i}|$. Then

$$\begin{aligned}
&r_i(\mathscr{B})\\
&= \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N\sigma_u},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|d_{i_2}\cdots d_{i_m} + \sum_{(i_2,\cdots,i_m)\in\Delta_i^N\setminus\Delta_i^{N\sigma_u}} |a_{ii_2\cdots i_m}|d_{i_2}\cdots d_{i_m}\\
&= \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N\sigma_u},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|d_{k_0}^{m-1} + \sum_{(i_2,\cdots,i_m)\in\Delta_i^N\setminus\Delta_i^{N\sigma_u}} |a_{ii_2\cdots i_m}|d_{i_2}\cdots d_{i_m}\\
&\leqslant \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_i^{N\sigma_u},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|d_{k_0}^{m-1} + \sum_{(i_2,\cdots,i_m)\in\Delta_i^N\setminus\Delta_i^{N\sigma_u}} |a_{ii_2\cdots i_m}|\\
&= \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A})d_{k_0}^{m-1} + \beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) < |a_{ii\cdots i}|d_{k_0}^{m-1} = |b_{ii\cdots i}|.
\end{aligned} \tag{3.6}$$

Therefore, we have $|b_{ii\cdots i}| > r_i(\mathscr{B})$ for all $i \in N$ from inequalities (3.4), (3.5) and (3.6), that is to say, tensor $\mathscr{B} = \mathscr{A}D^{m-1}$ is strictly diagonally dominant. Furthermore, $\mathscr{A}$ is a H-tensor by Lemma 2.1. The proof is completed.

When $k = 2$, so that $N = N_{\sigma_1} \bigoplus N_{\sigma_2}$, and taking $J_1 = N_{\sigma_u}, u = 1,2$, the following conclusion is obtained immediately.

COROLLARY 3.1. *Let $\mathscr{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m\times n]}$. If there exists a nonempty subset N_{σ_1} of N such that:*

(i) $|a_{ii\cdots i}| > r_i(\mathscr{A})$ for all $i \in N_{\sigma_1}$;

(ii) $\left(|a_{ii\cdots i}| - \alpha_i^{\Delta_i^{N\sigma_1}}(\mathscr{A})\right)^{\frac{1}{m-1}} \left(|a_{jj\cdots j}| - \alpha_j^{\Delta_j^{N\sigma_2}}(\mathscr{A})\right) > \left(\beta_i^{\Delta_i^{N\sigma_1}}(\mathscr{A})\right)^{\frac{1}{m-1}} \beta_j^{\Delta_j^{N\sigma_2}}(\mathscr{A})$ *for all $i \in N_{\sigma_1}, j \in N_{\sigma_2}$, where $N_{\sigma_1} \bigoplus N_{\sigma_2} = N$,*

then $\mathscr{A}$ is a H-tensor.

What is worth noticing here is that one of $N_{\sigma_u}, u = 1,2$ must be a subset of J_1 in Corollary 3.1. If that's not the case, there must be $i_0 \in N_{\sigma_1}, j_0 \in N_{\sigma_2}$ such that

$$|a_{i_0 i_0\cdots i_0}| - \alpha_{i_0}^{\Delta_{i_0}^{N\sigma_1}}(\mathscr{A}) \leqslant \beta_{i_0}^{\Delta_{i_0}^{N\sigma_1}}(\mathscr{A}), |a_{j_0 j_0\cdots j_0}| - \alpha_{j_0}^{\Delta_{j_0}^{N\sigma_2}}(\mathscr{A}) \leqslant \beta_{j_0}^{\Delta_{j_0}^{N\sigma_2}}(\mathscr{A}).$$

Since $N_{\sigma_1} \cup N_{\sigma_2} = N = N^+$, the following inequality is obtained immediately.

$$\left(|a_{i_0 i_0 \cdots i_0}| - \alpha_{i_0}^{\Delta_{i_0}^{N\sigma_1}}(\mathscr{A})\right)^{\frac{1}{m-1}} \left(|a_{j_0 j_0 \cdots j_0}| - \alpha_{j_0}^{\Delta_{j_0}^{N\sigma_2}}(\mathscr{A})\right) \leqslant \left(\beta_{i_0}^{\Delta_{i_0}^{N\sigma_1}}(\mathscr{A})\right)^{\frac{1}{m-1}} \beta_{j_0}^{\Delta_{j_0}^{N\sigma_2}}(\mathscr{A}).$$

This is contrary to Corollary 3.1.

When $k = n$ and $N_i = \{i\}$, so that $N = N_{\sigma_1} \bigoplus N_{\sigma_2} \bigoplus \cdots \bigoplus N_{\sigma_n}$, then

$$\alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) = 0, \quad \beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) = r_i(\mathscr{A}), \quad i \in N,$$

there is the following conclusion in a moment.

COROLLARY 3.2. *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$. If the following inequality holds for all $i, j \in N, i \neq j$*

$$\left(|a_{ii\cdots i}|\right)^{\frac{1}{m-1}} |a_{jj\cdots j}| > \left(r_i(\mathscr{A})\right)^{\frac{1}{m-1}} r_j(\mathscr{A}).$$

Then $\mathscr{A}$ is a H-tensor.

THEOREM 3.2. *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$. $\mathscr{A}$ is exponential type locally generalized double diagonally dominant tensor. If $\mathscr{A}$ is weakly irreducible and $J_1 \neq \emptyset, \tilde{J}_1 \neq \emptyset$, then $\mathscr{A}$ is a H-tensor.*

Proof. Since $\mathscr{A}$ is exponential type locally generalized double diagonally dominant tensor and $J_1 \neq \emptyset$, then $N = N^0 \bigcup N^+$. Furthermore, we prove $N = N^+$, that is $N^0 = \emptyset$. If $N^0 \neq \emptyset$, then there exists $i_0 \in N^0$ such that $|a_{i_0 i_0 \cdots i_0}| \leqslant r_{i_0}(\mathscr{A})$.

(i) For the case that $|a_{i_0 i_0 \cdots i_0}| < r_{i_0}(\mathscr{A})$, it is evident that $|a_{i_0 i_0 \cdots i_0}| - \alpha_{i_0}^{\Delta_{i_0}^{N\sigma_{i_0}}}(\mathscr{A}) = 0$ and $\beta_{i_0}^{\Delta_{i_0}^{N\sigma_{i_0}}}(\mathscr{A}) > 0$. From the (3.1), we obtain $\beta_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}) = 0$ for $\forall j \in N \setminus N_{\sigma_v}$. Thus, we have

$$a_{j i_2 \cdots i_m} = 0, \quad \forall j \in N_{\sigma_v}, \quad \exists i_2 \cdots i_m \notin N_{\sigma_v},$$

equivalently, $\mathscr{A}$ is weakly reducible. This leads to a contradiction.

(ii) For the case that $|a_{i_0 i_0 \cdots i_0}| = r_{i_0}(\mathscr{A})$, that is $\beta_{i_0}^{\Delta_{i_0}^{N\sigma_{i_0}}}(\mathscr{A}) = 0$, then there exists a nonempty proper index subset $i_0 \in N_{\sigma_{i_0}} \subset N$ such that

$$a_{i_0 i_2 \cdots i_m} = 0, \quad \forall i_0 \in N_{\sigma_{i_0}}, \quad \exists i_2 \cdots i_m \notin N_{\sigma_{i_0}},$$

which is in contradiction with the weak irreducibility of $\mathscr{A}$. From the above (i) and (ii), we can get $N^0 = \emptyset$. From the above analysis in Theorem 3.1, there exists at most one $1 \leqslant k_0 \leqslant k$ such that

$$|a_{ii\cdots i}| - \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \geqslant \beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}), \quad \forall i \in N_{\sigma_u}, \sigma_u \neq k_0.$$

In other words, we have $N\setminus N_{k_0}\subseteq J_1$. If $N_{k_0}=\emptyset$, then $\mathscr{A}$ is weakly irreducible diagonally dominant tensor, and there exists at least an $i\in N$ such that $|b_{ii\cdots i}|>r_i(\mathscr{B})$ by reason of $\tilde{J}_1\neq\emptyset$. Then, $\mathscr{A}$ must be a H-tensor by Lemma 2.3. Therefore, we suppose $N_{k_0}\setminus J_0\neq\emptyset$. We construct the positive diagonal matrix $D\in\mathbb{C}^{n,n}$ by using the same method with Theorem 3.1, and denote $\mathscr{B}=\mathscr{A}D^{m-1}=(b_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. Similar to the proof of Theorem 3.1, we conclude that $|b_{ii\cdots i}|\geqslant r_i(\mathscr{B})$ for all $i\in N$. In addition, there is at least an $i\in N$ such that $|b_{ii\cdots i}|>r_i(\mathscr{B})$ on account of $\tilde{J}_1\neq\emptyset$.

Since $D\in\mathbb{C}^{n,n}$ is a positive diagonal matrix, we know that tensor $\mathscr{B}$ has the same weak irreducibility with tensor $\mathscr{A}$ from definition 2.6. Moreover, $\mathscr{A}$ is weakly irreducible, and so is $\mathscr{B}$. Hence, we see that $\mathscr{B}$ is an H-tensor by the Lemma 2.3. Furthermore, $\mathscr{A}$ is also an H-tensor in the light of Lemma 2.2. The proof is completed. Since the irreducibility of tensors implies the weak irreducibility of tensors, we obtain the following conclusion at once.

COROLLARY 3.3. *Let $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. $\mathscr{A}$ is exponential type locally generalized double diagonally dominant tensor. If $\mathscr{A}$ is irreducible and $J_1\neq\emptyset,\tilde{J}_1\neq\emptyset$, then $\mathscr{A}$ is a H-tensor.*

THEOREM 3.3. *Let $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. $\mathscr{A}$ is exponential type locally generalized double diagonally dominant tensor. If $\forall i\in N\setminus\tilde{J}_1=\tilde{J}_0\neq\emptyset$, there exists a non-zero elements chain from i to j such that $j\in\tilde{J}_1\neq\emptyset$, then $\mathscr{A}$ is a H-tensor.*

Proof. First of all, it is the same as theorem 3.2, we construct the positive diagonal matrix $D\in\mathbb{C}^{n,n}$ and denote $\mathscr{B}=\mathscr{A}D^{m-1}=(b_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$. Similar to the proof of Theorem 3.2, we can get $|b_{ii\cdots i}|\geqslant r_i(\mathscr{B})$ for all $i\in N$, and there is at least an $i\in N$ such that $|b_{ii\cdots i}|>r_i(\mathscr{B})$.

Because $D\in\mathbb{C}^{n,n}$ is a positive diagonal matrix, there exists a non-zero elements chain of $\mathscr{B}$ from i to j, if and only if there exists a non-zero elements chain of $\mathscr{A}$ from i to j at same time. Furthermore, if $|b_{ii\cdots i}|=r_i(\mathscr{B})$, then $i\in N\setminus\tilde{J}_1=\tilde{J}_0$; by the assumption, we see that there exists a non-zero elements chain of $\mathscr{A}$ from i to j, such that $j\in\tilde{J}_1$. Then, there is a non-zero elements chain of $\mathscr{B}$ from i to j, such that $j\in\tilde{J}_1$ satisfies $|b_{jj\cdots j}|>r_j(\mathscr{B})$.

Based on above discussion, we realize that the tensor $\mathscr{B}$ meets the condition of Lemma 2.4, so $\mathscr{B}$ is an H-tensor. Furthermore, $\mathscr{A}$ is also an H-tensor on the basis of Lemma 2.2. The proof is completed.

According to Lemma 2.5 and preceding Theorem 3.1-3.3, we can obtain some checkable sufficient condition for the positive definiteness of an even-order real symmetric tensor.

THEOREM 3.4. *Let $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}^{[m\times n]}$ is an even-order symmetric tensor, and $a_{i\cdots i}>0$ for all $i\in N$. If $\mathscr{A}$ satisfies one of the following conditions:*

(i) all the conditions of Theorem 3.1;

(ii) all the conditions of Theorem 3.2;

(iii) all the conditions of Theorem 3.3,

then $\mathscr{A}$ is positive definite.

EXAMPLE 3.1. We consider the following a 4th-degree homogeneous polynomial.

$$f(x)=\mathscr{A}x^4=8x_1^4+2x_2^4+2x_3^4+2x_4^4+4x_1^3x_2+4x_1^3x_3+4x_1^3x_4$$

Obviously, we can obtain an order 4 dimension 4 real symmetric tensor $\mathscr{A}=(a_{i_1i_2i_3i_4})$, where

$$\begin{aligned}
&a_{1111}=8, \quad a_{2222}=a_{3333}=a_{4444}=2,\\
&a_{1211}=a_{1121}=a_{1112}=1, \quad a_{1113}=a_{1131}=a_{1311}=1,\\
&a_{1114}=a_{1141}=a_{1411}=1, \quad a_{2111}=a_{3111}=a_{4111}=1,
\end{aligned}$$

and other elements are $a_{i_1i_2i_3i_4}=0$. By the calculation, we get

$$|a_{1111}|=8<9=r_1(\mathscr{A})$$

and

$$|a_{2222}|(|a_{1111}|-r_1(\mathscr{A})+|a_{1222}|)=-2<0=r_2(\mathscr{A})|a_{1222}|.$$

Apparently, $\mathscr{A}$ is neither a strictly diagonally dominant tenor nor a quasi-doubly strictly diagonally dominant tenor, so in this situation, it does not work that we identify the positive definiteness of $\mathscr{A}$ by using Theorem 3 in [23] and Theorem 4 in [24].

Let $k=3, N=\bigcup_{i=1}^{3}N_i, N_1=\{1,2\}, N_2=\{3\}, N_3=\{4\}$, $\sigma=(1,2,3)$.
By computations, we get

$$\begin{aligned}
&(|a_{1111}|-\alpha_1^{\Delta_1^{N_1}}(\mathscr{A}))^{\frac{1}{3}}(|a_{3333}|-\alpha_3^{\Delta_3^{N_2}}(\mathscr{A}))=(8-3)^{\frac{1}{3}}(2-0)>\left(\beta_1^{\Delta_1^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\beta_3^{\Delta_3^{N_2}}(\mathscr{A})=6^{\frac{1}{3}},\\
&(|a_{2222}|-\alpha_2^{\Delta_2^{N_1}}(\mathscr{A}))^{\frac{1}{3}}(|a_{3333}|-\alpha_3^{\Delta_3^{N_2}}(\mathscr{A}))=(2-1)^{\frac{1}{3}}(2-0)>\left(\beta_2^{\Delta_2^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\beta_3^{\Delta_3^{N_2}}(\mathscr{A})=0^{\frac{1}{3}},\\
&(|a_{1111}|-\alpha_1^{\Delta_1^{N_1}}(\mathscr{A}))^{\frac{1}{3}}(|a_{4444}|-\alpha_4^{\Delta_4^{N_3}}(\mathscr{A}))=(8-3)^{\frac{1}{3}}(2-0)>\left(\beta_1^{\Delta_1^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\beta_4^{\Delta_4^{N_3}}(\mathscr{A})=6^{\frac{1}{3}},\\
&(|a_{2222}|-\alpha_2^{\Delta_2^{N_1}}(\mathscr{A}))^{\frac{1}{3}}(|a_{4444}|-\alpha_4^{\Delta_4^{N_3}}(\mathscr{A}))=(2-1)^{\frac{1}{3}}(2-0)>\left(\beta_2^{\Delta_2^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\beta_4^{\Delta_4^{N_3}}(\mathscr{A})=0^{\frac{1}{3}},\\
&(|a_{3333}|-\alpha_3^{\Delta_3^{N_2}}(\mathscr{A}))^{\frac{1}{3}}(|a_{4444}|-\alpha_4^{\Delta_4^{N_3}}(\mathscr{A}))=(2-0)^{\frac{1}{3}}(2-0)>\left(\beta_3^{\Delta_3^{N_2}}(\mathscr{A})\right)^{\frac{1}{3}}\beta_4^{\Delta_4^{N_3}}(\mathscr{A})=1^{\frac{1}{3}}.
\end{aligned}$$

Therefore, $\mathscr{A}$ is exponential type locally strictly double diagonally dominant tensor. Furthermore, we obtain that $\mathscr{A}$ is positive definite by Theorem 3.4.

4. New eigenvalue localization sets for general tensors

In this section, as a significantly important application of exponential type locally generalized strictly double diagonally dominant tensors, we will investigate into eigenvalue estimation of general tensors based on the relationship between the exponential locally generalized strictly double diagonally dominant tensors and H-tensors. Next, several new eigenvalue localization regions for tensors will be given. Before proposing our results, we review the existing eigenvalue localization sets for tensors.

In 2005, Qi [1] have given an eigenvalue inclusion set for real symmetric tensors, which was an extension of the Gersgorin's eigenvalue inclusion theorem from matrices [3]. It is evident that this result can also be extended to general tensors.

THEOREM 4.1. [1, Theorem 6] *Let* $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$, $n\geqslant 2$. *Then*

$$\sigma(\mathscr{A})\subseteq\Gamma(\mathscr{A})=\bigcup_{i\in N}\Gamma_i(\mathscr{A}),$$

where $\sigma(\mathscr{A})$ *is the set of all the eigenvalue of* $\mathscr{A}$ *and*

$$\Gamma_i(\mathscr{A})=\{z\in\mathbb{C}:|z-a_{i\cdots i}|\leqslant r_i(\mathscr{A})\},r_i(\mathscr{A})=\sum_{\substack{i_2,\cdots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|.$$

To obtain tighter sets than $\Gamma(\mathscr{A})$, Li et al. [6] generalized the Brauer's eigenvalue localization set of matrices [3] and presented the following Brauer-type eigenvalue localization sets for tensors.

THEOREM 4.2. [6, Theorem 2.1]*Let* $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}$, $n\geqslant 2$, *Then*

$$\sigma(\mathscr{A})\subseteq\mathscr{K}(\mathscr{A})=\bigcup_{\substack{i,j\in N\\ j\neq i}}\mathscr{K}_{i,j}(\mathscr{A}),$$

where

$$\mathscr{K}_{i,j}(\mathscr{A})=\left\{z\in\mathbb{C}:\left(|z-a_{i\cdots i}|-r_i^j(\mathscr{A})\right)|z-a_{j\cdots j}|\leqslant|a_{ij\cdots j}|r_j(\mathscr{A})\right\}$$

and

$$r_i^j(\mathscr{A})=\sum_{\substack{\delta_{ii_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|=r_i(\mathscr{A})-|a_{ij\cdots j}|.$$

At the same time, to reduce computations, Li et al. [6] gave an S-type eigenvalue localization set by dividing $N=\{1,2,\cdots,n\}$ into disjoint subsets S and $\overline{S}$, where $\overline{S}$ was the complement of S in N. The details are as follows:

Given a nonempty proper subset S of N, we denote

$$\Delta^N:=\{(i_2,i_3,\cdots,i_m):\text{ each } i_j\in N \text{ for } j=2,\cdots,m\},$$
$$\Delta^S:=\{(i_2,i_3,\cdots,i_m):\text{ each } i_j\in S \text{ for } j=2,\cdots,m\},$$

and then

$$\overline{\Delta^S}=\Delta^N\backslash\Delta^S.$$

This indicates that for a tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$, we have that for $i \in S$,

$$r_i(\mathscr{A}) = r_i^{\Delta^S}(\mathscr{A}) + r_i^{\overline{\Delta^S}}(\mathscr{A}),$$

where

$$r_i^{\Delta^S}(\mathscr{A}) = \sum_{\substack{(i_2,\cdots,i_m)\in\Delta^S,\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|, r_i^{\overline{\Delta^S}}(\mathscr{A}) = \sum_{(i_2,\cdots,i_m)\in\overline{\Delta^S}} |a_{ii_2\cdots i_m}|.$$

THEOREM 4.3. [6, Theorem 2.2] *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$, $n \geqslant 2$, and S be a nonempty proper subset of N. Then*

$$\sigma(\mathscr{A}) \subseteq \mathscr{K}^S(\mathscr{A}) = \left(\bigcup_{i\in S, j\in\overline{S}} \mathscr{K}_{i,j}(\mathscr{A})\right) \bigcup \left(\bigcup_{i\in\overline{S}, j\in S} \mathscr{K}_{i,j}(\mathscr{A})\right).$$

After that, some new S-type eigenvalue localization sets for tensors based on previous partitioning method for index set was proposed by Li et al. [21], Huang et al [20] and some other investigators, and it was proved that those new sets are more precise than $\Gamma(\mathscr{A})$, $\mathscr{K}(\mathscr{A})$ and $\mathscr{K}^S(\mathscr{A})$.

THEOREM 4.4. [20, Theorem 3.1] *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$, $n \geqslant 2$, and S be a nonempty proper subset of N. Then*

$$\sigma(\mathscr{A}) \subseteq \Upsilon^S(\mathscr{A}) := \left(\bigcup_{i\in S, j\in\overline{S}} \Upsilon_i^j(\mathscr{A})\right) \bigcup \left(\bigcup_{i\in\overline{S}, j\in S} \Upsilon_i^j \mathscr{A})\right),$$

where

$$\Upsilon_i^j(\mathscr{A}) = \{z \in \mathbb{C} : |(z-a_{i\cdots i})(z-a_{j\cdots j}) - a_{ij\cdots j}a_{ji\cdots i}| \leqslant |z-a_{j\cdots j}| r_i^j(\mathscr{A}) + |a_{ij\cdots j}| r_j^i(\mathscr{A})\}.$$

THEOREM 4.5. [21, Theorem 4] *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$, $n \geqslant 2$, and S be a nonempty proper subset of N. Then*

$$\sigma(\mathscr{A}) \subseteq \Omega^S(\mathscr{A}) := \left(\bigcup_{i\in S, j\in\overline{S}} \Omega_{i,j}^S(\mathscr{A})\right) \bigcup \left(\bigcup_{i\in\overline{S}, j\in S} \Omega_{i,j}^{\overline{S}} \mathscr{A})\right),$$

where

$$\Omega_{ij}^S(\mathscr{A}) := \left\{z \in \mathbb{C} : (|z-a_{i\cdots i}|)\left(|z-a_{j\cdots j}| - r_j^{\overline{\Delta^S}}(\mathscr{A})\right) \leqslant r_i(\mathscr{A}) r_j^{\Delta^S}(\mathscr{A})\right\},$$

$$\Omega_{ij}^{\overline{S}}(\mathscr{A}) := \left\{z \in \mathbb{C} : (|z-a_{i\cdots i}|)\left(|z-a_{j\cdots j}| - r_j^{\overline{\Delta^{\overline{S}}}}(\mathscr{A})\right) \leqslant r_i(\mathscr{A}) r_j^{\Delta^{\overline{S}}}(\mathscr{A})\right\}.$$

In addition, Zhao et.al [10] obtained the following eigenvalue inclusion region for tensors on the basis that H-tensors have no zero eigenvalues.

LEMMA 4.1. [10, Lemma 2] *Let* $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}$. *If* $\mathscr{A}$ *is a* H*-tensor, then* $0 \notin \sigma(\mathscr{A})$.

THEOREM 4.6. [10, Theorem 11] *Let* $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}, n \geqslant 2$. *Then*

$$\sigma(\mathscr{A}) \subseteq \Phi(\mathscr{A}) = \bigcap_{S \subseteq N} \Phi^S(\mathscr{A}),$$

where

$$\Phi^S(\mathscr{A}) = \left(\bigcup_{i \in S} \Phi_i^S(\mathscr{A}) \right) \bigcup \left(\bigcup_{i \in S, j \in \overline{S}} \Phi_{i,j}^S(\mathscr{A}) \right),$$
$$\Phi_i^S(\mathscr{A}) = \{ z \in \mathbb{C} : |z - a_{ii\cdots i}| \leqslant r_i(\mathscr{A}), i \in S \},$$
$$\Phi_{i,j}^S(\mathscr{A}) = \left\{ z \in \mathbb{C} : \left(|z - a_{ii\cdots i}| - r_i^{\Delta^S}(\mathscr{A}) \right) \left(|z - a_{jj\cdots j}| - r_j^{\overline{\Delta^S}}(\mathscr{A}) \right) \leqslant r_i^{\overline{\Delta^S}}(\mathscr{A}) r_j^{\Delta^S}(\mathscr{A}) \right\}.$$

Now, we will present a eigenvalue localization set for tensor on basis of Theorem 3.1.

THEOREM 4.7. *Let* $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m\times n]}, n \geqslant 2$. *Then*

$$\sigma(\mathscr{A}) \subseteq \Psi(\mathscr{A}) = \left(\bigcup_{i \in N_{\sigma_u}, i \neq j} \hat{\Psi}_{i,j}(\mathscr{A}) \right) \bigcup \left(\bigcup_{1 \leqslant \sigma_u \neq \sigma_v \leqslant k} \tilde{\Psi}_{i,j}(\mathscr{A}) \right), 1 \leqslant i, j \leqslant k,$$

where

$$\hat{\Psi}_{i,j}(\mathscr{A}) = \{ z \in \mathbb{C} : |z - a_{ii\cdots i}| \leqslant r_i(\mathscr{A}), i \in N_{\sigma_u}, i \neq j \},$$
$$\tilde{\Psi}_{i,j}(\mathscr{A}) = \left\{ z \in \mathbb{C} : \left(|z - a_{ii\cdots i}| - \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right)^{\frac{1}{m-1}} \left(|z - a_{jj\cdots j}| - \alpha_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}) \right) \right.$$
$$\left. \leqslant \left(\beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) \right)^{\frac{1}{m-1}} \beta_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}), i \in N_{\sigma_u}, j \in N_{\sigma_v}, 1 \leqslant \sigma_u \neq \sigma_v \leqslant k \right\}.$$

Proof. For every eigenvalue $\lambda \in \sigma(\mathscr{A})$, we take

$$\mathscr{B} = (b_{i_1 i_2 \cdots i_m}) = \lambda I - \mathscr{A}.$$

Then

$$0 \in \sigma(\mathscr{B}), r_i(\mathscr{A}) = r_i(\mathscr{B}), \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) = \alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{B}), \beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A}) = \beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{B}), \forall i \in N.$$

We assume $\lambda \notin \Psi(\mathscr{A})$, then there exists a nonempty subset $N_{\sigma_u} \subseteq N, N_{\sigma_u} \bigcap N_{\sigma_v} = \emptyset, 1 \leqslant i \neq j \leqslant k$ such that $\lambda \notin \Psi_{i,j}(\mathscr{A})$. Therefore, we obtain

$$|z - a_{ii\cdots i}| > r_i(\mathscr{A}), \quad i \in N_{\sigma_u}, \forall i \neq j,$$

$$\left(|z-a_{ii\cdots i}|-\alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{A})\right)^{\frac{1}{m-1}}\left(|z-a_{jj\cdots j}|-\alpha_j^{\Delta_j^{N\sigma_v}}(\mathscr{A})\right)>\left(\beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{A})\right)^{\frac{1}{m-1}}\beta_j^{\Delta_j^{N\sigma_v}}(\mathscr{A}),$$

where $i\in N_{\sigma_u}, j\in N_{\sigma_v}, 1\leqslant \sigma_u\neq\sigma_v\leqslant k$.

Equivalently, we have

$$|b_{ii\cdots i}|>r_i(\mathscr{B}),\quad i\in N_{\sigma_u},\forall i\neq j,$$

$$\left(|b_{ii\cdots i}|-\alpha_i^{\Delta_i^{N\sigma_u}}(\mathscr{B})\right)^{\frac{1}{m-1}}\left(|b_{jj\cdots j}|-\alpha_j^{\Delta_j^{N\sigma_v}}(\mathscr{B})\right)>\left(\beta_i^{\Delta_i^{N\sigma_u}}(\mathscr{B})\right)^{\frac{1}{m-1}}\beta_j^{\Delta_j^{N\sigma_v}}(\mathscr{B}),$$

where $i\in N_{\sigma_u}, j\in N_{\sigma_v}, 1\leqslant \sigma_u\neq\sigma_v\leqslant k$.

According to Theorem 3.1, $\mathscr{B}$ is an H-tensor. Furthermore, by Lemma 4.1, we get $0\notin\sigma(\mathscr{B})$. This leads to a contradiction. Therefore, $\lambda\in\sigma(\mathscr{A})$. The proof is completed.

REMARK 4.1. It is worth noting that Theorem 4.7 is established for every $N_{\sigma_u}\subseteq N, 1\leqslant i\leqslant k, 2\leqslant k\leqslant n$. In other words, no matter how to divide the set N, $\Psi(\mathscr{A})$ captures all the eigenvalues of tensor $\mathscr{A}$.

If $k=2$, that is $N_{\sigma_1}\bigoplus N_{\sigma_2}=N$, $\sigma=\{1,2\}$, then we have the following results by the Corollary 3.1.

COROLLARY 4.1. *Let* $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}, n\geqslant 2$. *Then*

$$\sigma(\mathscr{A})\subseteq\widehat{\Psi}(\mathscr{A})=\left(\bigcup_{i\in N_{\sigma_u},i=1,2,i\neq j}\hat{\Psi}_{i,j}(\mathscr{A})\right)\bigcup\left(\bigcup_{1\leqslant\sigma_u\neq\sigma_v\leqslant 2}\tilde{\Psi}_{i,j}(\mathscr{A})\right).$$

If $k=n$ and $N_i=\{i\}$, that is $N=N_{\sigma_1}\bigoplus N_{\sigma_2}\bigoplus\cdots\bigoplus N_{\sigma_n}$, then $\alpha_i^{\Delta_i^{N\sigma_u}}=0, \beta_i^{\Delta_i^{N\sigma_u}}=r_i(\mathscr{A}), i\in N$, there is the following conclusion at this time by the Corollary 3.2.

COROLLARY 4.2. *Let* $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}, n\geqslant 2$. *Then*

$$\sigma(\mathscr{A})\subseteq\mathscr{D}(\mathscr{A})=\bigcup_{\substack{i,j\in N\\ j\neq i}}\mathscr{D}_{i,j}(\mathscr{A}),$$

where

$$\mathscr{D}_{ij}(\mathscr{A})=\left\{z\in\mathbb{C}:|z-a_{i\cdots i}|^{\frac{1}{m-1}}|z-a_{j\cdots j}|\leqslant\left(r_i(\mathscr{A})\right)^{\frac{1}{m-1}}r_j(\mathscr{A})\right\}.$$

There is no doubt that the following result is established.

THEOREM 4.8. *Let* $\mathscr{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m\times n]}, n\geqslant 2$. *Then*

$$\sigma(\mathscr{A})\subseteq\Psi(\mathscr{A})\subseteq\Gamma(\mathscr{A}).$$

Obviously, this new eigenvalue localization region is more precise than Theorem 2.1 [1] from Theorem 4.8. Not only so, we show that $\Psi(\mathscr{A})$ is tighter than some existing results.

EXAMPLE 4.1. Let $\mathscr{A} \in \mathbb{R}^{[4\times 4]}$ be a order 4 dimension 4 real tensor with elements defined as follows;

$$\begin{aligned}
&a_{1111} = 10, \quad a_{2222} = 8, \quad a_{3333} = 7, \quad a_{4444} = 5,\\
&a_{1333} = a_{1444} = 1, \quad a_{1211} = a_{1113} = a_{1141} = 1,\\
&a_{1332} = a_{1442} = a_{1232} = a_{1234} = a_{1321} = a_{1214} = 1,\\
&a_{2333} = a_{2444} = 1, \quad a_{2112} = a_{2234} = a_{2113} = a_{2343} = a_{2123} = 1,\\
&a_{3222} = 1, \quad a_{3111} = 1, a_{3121} = a_{3434} = a_{3123} = 1,\\
&a_{4222} = 1, \quad a_{4111} = 1, a_{4121} = a_{4334} = 1,
\end{aligned}$$

and other elements are $a_{i_1 i_2 i_3 i_4} = 0$.

Let $S = \{1,2\}$. Evidently, $\overline{S} = \{3,4\}$. By computations, we get that

$$\begin{aligned}
&r_1(\mathscr{A}) = 11, r_1^3(\mathscr{A}) = 10, r_1^4(\mathscr{A}) = 10, r_1^{\Delta^S}(\mathscr{A}) = 1, r_1^{\overline{\Delta^S}}(\mathscr{A}) = 10, r_1^{\Delta^{\overline{S}}}(\mathscr{A}) = 2, r_1^{\overline{\Delta^{\overline{S}}}}(\mathscr{A}) = 9,\\
&r_2(\mathscr{A}) = 7, r_2^3(\mathscr{A}) = 6, r_2^4(\mathscr{A}) = 6, r_2^{\Delta^S}(\mathscr{A}) = 1, r_2^{\overline{\Delta^S}}(\mathscr{A}) = 6, r_2^{\Delta^{\overline{S}}}(\mathscr{A}) = 3, r_2^{\overline{\Delta^{\overline{S}}}}(\mathscr{A}) = 4,\\
&r_3(\mathscr{A}) = 5, r_3^1(\mathscr{A}) = 4, r_3^2(\mathscr{A}) = 4, r_3^{\Delta^S}(\mathscr{A}) = 3, r_3^{\overline{\Delta^S}}(\mathscr{A}) = 2, r_3^{\overline{\Delta^{\overline{S}}}}(\mathscr{A}) = 4, r_3^{\Delta^{\overline{S}}}(\mathscr{A}) = 1,\\
&r_4(\mathscr{A}) = 4, r_4^1(\mathscr{A}) = 3, r_4^2(\mathscr{A}) = 3, r_4^{\Delta^S}(\mathscr{A}) = 3, r_4^{\overline{\Delta^S}}(\mathscr{A}) = 1, r_4^{\overline{\Delta^{\overline{S}}}}(\mathscr{A}) = 3, r_4^{\Delta^{\overline{S}}}(\mathscr{A}) = 1.
\end{aligned}$$

By Theorem 4.1(Theorem 6 of [1]), we get

$$\sigma(\mathscr{A}) \subseteq \Gamma(\mathscr{A}) = \{z \in \mathbb{C} : -1 \leqslant z \leqslant 21\},$$

By Theorem 4.3(Theorem 2.2 of [6]), we get

$$\sigma(\mathscr{A}) \subseteq \mathscr{K}^S(\mathscr{A}) = \{z \in \mathbb{C} : -0.7016 \leqslant z \leqslant 20.3739\},$$

By Theorem 4.4(Theorem 3.1 of [20]), we get

$$\sigma(\mathscr{A}) \subseteq \Upsilon^S(\mathscr{A}) = \{z \in \mathbb{C} : -0.7016 \leqslant z \leqslant 20.3739\},$$

By Theorem 4.5(Theorem 4 of [21]), we get

$$\sigma(\mathscr{A}) \subseteq \Omega^S(\mathscr{A}) = \{z \in \mathbb{C} : -0.4641 \leqslant z \leqslant 19.7823\},$$

By Theorem 4.6(Theorem 11 of [10]), we get

$$\sigma(\mathscr{A}) \subseteq \Phi(\mathscr{A}) = \{z \in \mathbb{C} : -0.3723 \leqslant z \leqslant 19.6847\},$$

Next, we will estimate the eigenvalues of this tensor by applying theorem 4.7 and compare it with the previous results.

Let $k=3, N=\bigcup_{i=1}^{3} N_i, N_1=\{3,4\}, N_2=\{1\}, N_3=\{2\}$, $\sigma=(1,2,3)$. When $N_{\sigma_u}=N_1$,that is $\sigma_u=1$, then we have

$$|z-a_{3333}|\leqslant r_3(\mathscr{A}),\quad |z-a_{4444}|\leqslant r_4(\mathscr{A}),$$
$$\left(|z-a_{3333}|-\alpha_3^{\Delta_3^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\left(|z-a_{1111}|-\alpha_1^{\Delta_1^{N_2}}(\mathscr{A})\right)\leqslant\left(\beta_3^{\Delta_3^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\beta_1^{\Delta_1^{N_2}}(\mathscr{A}),$$
$$\left(|z-a_{3333}|-\alpha_3^{\Delta_3^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\left(|z-a_{2222}|-\alpha_2^{\Delta_2^{N_3}}(\mathscr{A})\right)\leqslant\left(\beta_3^{\Delta_3^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\beta_2^{\Delta_2^{N_3}}(\mathscr{A}),$$
$$\left(|z-a_{4444}|-\alpha_4^{\Delta_4^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\left(|z-a_{1111}|-\alpha_1^{\Delta_1^{N_2}}(\mathscr{A})\right)\leqslant\left(\beta_4^{\Delta_4^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\beta_1^{\Delta_1^{N_2}}(\mathscr{A}),$$
$$\left(|z-a_{4444}|-\alpha_4^{\Delta_4^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\left(|z-a_{2222}|-\alpha_2^{\Delta_2^{N_3}}(\mathscr{A})\right)\leqslant\left(\beta_4^{\Delta_4^{N_1}}(\mathscr{A})\right)^{\frac{1}{3}}\beta_2^{\Delta_2^{N_3}}(\mathscr{A}),$$

equivalently,

$$|z-7|\leqslant 5,\quad |z-5|\leqslant 4,$$
$$(|z-7|-1)^{\frac{1}{3}}(|z-10|-0)\leqslant 4^{\frac{1}{3}}*11,\quad (|z-7|-1)^{\frac{1}{3}}(|z-8|-0)\leqslant 4^{\frac{1}{3}}*7,$$
$$(|z-5|-1)^{\frac{1}{3}}(|z-10|-0)\leqslant 3^{\frac{1}{3}}*11,\quad (|z-5|-1)^{\frac{1}{3}}(|z-8|-0)\leqslant 3^{\frac{1}{3}}*7.$$

By Theorem 4.7 in our result, we obtain

$$\bigcup_{i\in N_{\sigma_u}, i\neq j}\hat{\Psi}_{i,j}(\mathscr{A})=\hat{\Psi}_{3,j}(\mathscr{A})\bigcup\hat{\Psi}_{4,j}(\mathscr{A})=\{2\leqslant z\leqslant 12\}\bigcup\{1\leqslant z\leqslant 9\}=\{1\leqslant z\leqslant 12\},$$
$$\tilde{\Psi}_{3,1}(\mathscr{A})=\{0.2521\leqslant z\leqslant 18.0827\},\quad \tilde{\Psi}_{3,2}(\mathscr{A})=\{1.3450\leqslant z\leqslant 14.0861\},$$
$$\tilde{\Psi}_{4,1}(\mathscr{A})=\{0.0032\leqslant z\leqslant 17.1099\},\quad \tilde{\Psi}_{4,2}(\mathscr{A})=\{1.0000\leqslant z\leqslant 13.2227\},$$
$$\bigcup_{1\leqslant\sigma_u\neq\sigma_v\leqslant k}\tilde{\Psi}_{i,j}(\mathscr{A})=\tilde{\Psi}_{3,1}(\mathscr{A})\bigcup\tilde{\Psi}_{3,2}(\mathscr{A})\bigcup\tilde{\Psi}_{4,1}(\mathscr{A})\bigcup\tilde{\Psi}_{4,2}(\mathscr{A})$$
$$=\{0.0032\leqslant z\leqslant 18.0827\}.$$

From the above, we have

$$\Psi(\mathscr{A})=\Big(\bigcup_{i\in N_{\sigma_u}, i\neq j}\hat{\Psi}_{i,j}(\mathscr{A})\Big)\bigcup\Big(\bigcup_{1\leqslant\sigma_u\neq\sigma_v\leqslant k}\tilde{\Psi}_{i,j}(\mathscr{A})\Big)=\{0.0032\leqslant z\leqslant 18.0827\},$$

which implies that eigenvalue inclusion region for tensors in Theorem 4.7 is more precise than some existing results in some situations.

5. Conclusions

We have introduced exponential type locally generalized strictly double diagonally dominant tensors. This concept is a natural extension of strictly diagonally dominant tensors. Moreover, we have shown that exponential type locally generalized strictly double diagonally dominant tensors must be H-tensors. On this basis, some checkable sufficient conditions for the positive definiteness of even-order real symmetric tensors and several new eigenvalues localization sets for general tensors have being presented. And some numerical examples are given to illustrate the effectiveness and superiority of our results.

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Jianzhou Liu
School of Mathematics and Computational Science
Xiangtan University
Xiangtan 411105, Hunan, PR China
e-mail: liujz@xtu.edu.cn
Hunan Key Laboratory for Computation and Simulation in Science and Engineering
School of Mathematics and Computational Science, Xiangtan University
Xiangtan 411105, Hunan,PR China

Liang Xiong
School of Mathematics and Computational Science
Xiangtan University
Xiangtan 411105, Hunan, PR China
e-mail: xiongliang199@163.com