Journal of Mathematical Inequalities
Volume 14, Number 3 (2020), 631–640 doi:10.7153/jmi-2020-14-40

AN ELEMENTARY THREE–VARIABLE INEQUALITY WITH CONSTRAINTS FOR THE POWER FUNCTION OF THE NORMS ON SOME METRIC SPACES

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(*Communicated by J. Matkowski*)

Abstract. A three-variable inequality with constraints for the p-power function of the norms on some metric spaces is established by an elementary technique. The new inequality performs sharper than the classical triangle inequality in the case $p=1$. The performance of the involved corollaries are compared with Jensen's inequality and Clarkson's inequality.

1. Introduction

The importance of inequalities is evident to each researcher. Mathematicians presented a lot of powerful inequalities (see [1]-[12]), and benefited from them in return. Research on the inequality for the power function of norms may not seem very attractive, but it is actually useful. For example, the Clarkson's inequality, which is stated as follows, plays an important role in investigating the convexity of the space $L^p(\Omega)$.

THEOREM 1. *(Clarkson's Inequality)([13, p.44]) Let $u,v\in L^p(\Omega)$. Then*

$$\left\|\frac{u+v}{2}\right\|_p^p+\left\|\frac{u-v}{2}\right\|_p^p\leqslant\frac{\|u\|_p^p+\|v\|_p^p}{2},\ \textit{if}\ \ 2\leqslant p<\infty;$$
$$\left\|\frac{u+v}{2}\right\|_p^p+\left\|\frac{u-v}{2}\right\|_p^p\geqslant\frac{\|u\|_p^p+\|v\|_p^p}{2},\ \textit{if}\ \ 1<p\leqslant 2.$$

In this paper, we propose a three-variable inequality for the power function of the norms. The inequality has the type as

$$\|x+y\|_p^p+\|x+z\|_p^p\geqslant\|x+y+z\|_p^p+\|x\|_p^p,\ \text{under condition 1},$$
$$\|x+y\|_p^p+\|x+z\|_p^p\leqslant\|x+y+z\|_p^p+\|x\|_p^p,\ \text{under condition 2}.$$

Here the function $\|\cdot\|_p^p$ represents the p-power function of the norm $\|\cdot\|_p$ on the given space. We prove that the new inequality holds not only for the case $p=2$ on the real

Mathematics subject classification (2010): 26D15, 39B62.
Keywords and phrases: Three-variable inequality, triangle inequality, Jensen's inequality, Clarkson's inequality, inequality for the power function of the norm.

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inner product space, but also for the case $1 \leqslant p < \infty$ on some other spaces, such as $\mathbb{R}$ and $L^p(\Omega)$. The special case "$p = 1$" of this inequality performs sharper than the well-known triangle inequality. Moreover, as the corollary of the new inequality, a special case of Jensen's inequality, whose performance approximates to the result of Theorem 1, is also given for various cases.

The remaining sections of this article is organized as follows. In Section 2, we prove that the new inequality holds for $\|\cdot\|^2$ (i.e. $p = 2$) on the real inner product space. In Section 3, via an elementary (but effective) procedure we investigate the new inequality on $\mathbb{R}$ for $p \in [1, \infty)$, and get some corollaries. Finally in Section 4 we extend the discussion to the space $L^p(\Omega,\ \mu)$. Based on the primary results of Section 3, we establish the corresponding inequalities on $L^p(\Omega,\ \mu)$. A further investigation for the case $\|\cdot\|_q^p$, $q \neq p$, while norms $\|\cdot\|_q$ and $\|\cdot\|_p$ are equivalent, is researched as well.

2. Results on the real inner product space

Let us look at some enlightening results on the real inner product space X first. Assume the norm $\|\cdot\|$ is induced by the real inner product $\langle \cdot,\ \cdot \rangle$ on X, i.e.,

$$\|x\| = \langle x,\ x \rangle^{1/2}, \quad \forall x \in X. \tag{1}$$

We have following results.

PROPOSITION 1. *Let the norm $\|\cdot\|$ be defined as (1). Then for $\forall x, y, z \in X$, we have*

$$\|x+y\|^2 + \|x+z\|^2 \geqslant \|x+y+z\|^2 + \|x\|^2, \text{ if } \langle y,\ z \rangle \leqslant 0;$$
$$\|x+y\|^2 + \|x+z\|^2 \leqslant \|x+y+z\|^2 + \|x\|^2, \text{ if } \langle y,\ z \rangle \geqslant 0.$$

Proof. Since

$$\begin{aligned}
&\|x+y\|^2 + \|x+z\|^2 - \|x+y+z\|^2 - \|x\|^2 \\
&= \langle x+y,\ x+y \rangle + \langle x+z,\ x+z \rangle - \langle x+y+z,\ x+y+z \rangle - \langle x,\ x \rangle \\
&= \langle x+y,\ x+y \rangle + \langle x,\ x \rangle + 2\langle x,\ z \rangle + \langle z,\ z \rangle - \langle x+y,\ x+y \rangle - 2\langle x+y,\ z \rangle - \langle z,\ z \rangle - \langle x,\ x \rangle \\
&= -2\langle y,\ z \rangle,
\end{aligned}$$

we conclude this proposition. □

COROLLARY 1. *Under the assumption of Proposition 1, if $\langle y,\ z \rangle = 0$, then for $\forall x \in X$, following equality is true.*

$$\|x+y\|^2 + \|x+z\|^2 = \|x+y+z\|^2 + \|x\|^2.$$

There is another equality which is the well-known "parallelogram law" for the special case $z = -y$ of Proposition 1 as follows (see [6, p.568] and [14, p.219]),

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y,\ x+y \rangle + \langle x-y,\ x-y \rangle = 2\|x\|^2 + 2\|y\|^2, \tag{2}$$

which is also a special case of Clarkson's inequality. This beautiful equality is an important reference for the discussion in the remaining sections.

Proposition 1 is essentially an inequality of the inner product, and it enlighten us to search for similar inequalities on a more general space. To this end, we will first prepare some primary theoretical tools for the further study.

3. Results on $\mathbb{R}$

Let $\mathbb{R}$ denote the set of real numbers and $|\cdot|$ represent the absolute value function on $\mathbb{R}$. For the given numbers $p,\lambda \in \mathbb{R}$ which satisfy $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant 1$ respectively, we say $p \in [1,\infty)$ and $\lambda \in [0,1]$.

For the given number $z \in \mathbb{R}$, we define a function from $\mathbb{R}$ to $\mathbb{R}$:

$$F_z(x) = |x+z|^p - |x|^p, \quad p \in [1,\infty). \tag{3}$$

The monotonicity of $F_z(x)$ is described as follows.

LEMMA 1. *The function $F_z(x)$ defined by (3) is non-decreasing if $z \geqslant 0$, and non-increasing if $z \leqslant 0$.*

Proof. If $z \geqslant 0$, then we have

$$F_z(x) = \begin{cases} (x+z)^p - x^p, & \text{if} \quad 0 < x \leqslant x+z; \\ (x+z)^p - (-1)^p x^p, & \text{if} \quad x \leqslant 0 \leqslant x+z; \\ (-1)^p[(x+z)^p - x^p], & \text{if} \quad x \leqslant x+z < 0, \end{cases}$$

and its derivative function for each case is:

$$F_z'(x) = \begin{cases} p[(x+z)^{p-1} - x^{p-1}], & \text{if} \quad 0 < x \leqslant x+z; \\ p[(x+z)^{p-1} + (-x)^{p-1}], & \text{if} \quad x \leqslant 0 \leqslant x+z; \\ -p\{[-(x+z)]^{p-1} - (-x)^{p-1}\}, & \text{if} \quad x \leqslant x+z < 0. \end{cases}$$

It follows that $F_z'(x) \geqslant 0$ for all the three cases, which means $F_z(x)$ is non-decreasing provided $z \geqslant 0$.

Otherwise, if $z \leqslant 0$, we let $y = x+z$, i.e., $x = y-z$, and then

$$F_z(x) = |y|^p - |y-z|^p = -\left(|y-z|^p - |y|^p\right) = -F_{-z}(y).$$

Noticing $(-z) \geqslant 0$, it follows that $F_{-z}(y)$ is non-decreasing, which implies that $F_z(x) = -F_{-z}(y) = -F_{-z}(x+z)$ is non-increasing. Thus we prove this lemma. $\square$

Based on Lemma 1 we obtain some inequalities on $\mathbb{R}$.

PROPOSITION 2. *If $p \in [1, \infty)$, then for $\forall x,\ y,\ z \in \mathbb{R}$, we have*

$$|x+y|^p + |x+z|^p \geqslant |x+y+z|^p + |x|^p, \quad if\ y \cdot z \leqslant 0; \tag{4}$$

$$|x+y|^p + |x+z|^p \leqslant |x+y+z|^p + |x|^p, \quad if\ y \cdot z \geqslant 0. \tag{5}$$

Proof. Observing the two sides of the inequality (4), we have

$$\begin{aligned} left-right &= |x+y|^p + |x+z|^p - (|x+y+z|^p + |x|^p) \\ &= |x+y|^p - |x|^p - (|x+z+y|^p - |x+z|^p), \\ &= F_y(x) - F_y(x+z), \end{aligned}$$

where the function $F_y(x)$ is defined as (3). Hence it follows from Lemma 1 that

$$left-right \begin{cases} \geqslant 0, & \text{if} \quad y \geqslant 0, z \leqslant 0; \text{ or } \quad y \leqslant 0, z \geqslant 0; \\ \leqslant 0, & \text{if} \quad y \geqslant 0, z \geqslant 0; \text{ or } \quad y \leqslant 0, z \leqslant 0. \end{cases}$$

Therefore, we get (4) and (5), and complete the proof. □

REMARK 1. The special case $p = 1$ of Proposition 2 is

$$\begin{cases} |x+y| + |x+z| \geqslant |x+y+z| + |x|, \forall x \in \mathbb{R}, \ y \cdot z \leqslant 0; \\ |x+y| + |x+z| \leqslant |x+y+z| + |x|, \forall x \in \mathbb{R}, \ y \cdot z \geqslant 0. \end{cases} \tag{6}$$

Compared with the corresponding results of triangle inequality

$$\begin{cases} |x+y| + |x+z| \geqslant |2x+y+z|; \\ |2x+y+z| \leqslant |x+y+z| + |x|, \end{cases}$$

inequality (6) is obviously performs sharper.

Some corollaries are derived immediately.

COROLLARY 2. *If* $p \in [1, \infty)$, *then for* $\forall x, y \in \mathbb{R}$ *and* $\lambda \in [0, 1]$, *we have*

$$|x+y|^p + |x-y|^p \geqslant 2\left[\lambda |x|^p + (1-\lambda)|y|^p\right]. \tag{7}$$

Proof. Let $z = -y$ in inequality (4), it yields

$$|x+y|^p + |x-y|^p \geqslant 2|x|^p,$$

and

$$|x+y|^p + |x-y|^p = |y+x|^p + |y-x|^p \geqslant 2|y|^p.$$

Therefore, $|x+y|^p + |x-y|^p$ should not be less than any convex combination of $2|x|^p$ and $2|y|^p$. The proof is completed. □

For arbitrary numbers $u, v \in \mathbb{R}$, there always exist numbers $x, y \in \mathbb{R}$, so that

$$u = x+y, \quad v = x-y,$$

where

$$x = (u+v)/2, \quad y = (u-v)/2.$$

Hence Corollary 2 can be restated as the following equivalent form.

COROLLARY 3. *If* $p \in [1, \infty)$, *then for* $\forall u, v \in \mathbb{R}$ *and* $\lambda \in [0, 1]$, *we have*

$$|u|^p + |v|^p \geqslant 2\left[\lambda \left|\frac{u+v}{2}\right|^p + (1-\lambda)\left|\frac{u-v}{2}\right|^p\right]. \tag{8}$$

REMARK 2. The special case $\lambda = 1$ of Corollary 3 leads to the following inequality

$$\frac{|u|^p + |v|^p}{2} \geqslant \left|\frac{u+v}{2}\right|^p, \quad p \in [1, \infty),$$

which is a special case of Jensen's inequality.

REMARK 3. The special case $p = 1$, $\lambda = 1$ of Corollary 3 coincides with the triangle inequality

$$|u| + |v| \geqslant |u+v|,$$

which indicates that Corollary 3 generalizes the triangle inequality on $\mathbb{R}$.

COROLLARY 4. *If* $p \in [1, \infty)$, *then for* $\forall x, y \in \mathbb{R}$ *and* $\lambda \in [0, 1]$, *following inequality is true.*

$$|x+y|^p \leqslant \frac{1}{2}\left[\lambda |x|^p + (1-\lambda)|y|^p + \lambda |x+2y|^p + (1-\lambda)|y+2x|^p\right]. \tag{9}$$

Proof. Let $z = y$ in inequality (5), it follows that

$$2|x+y|^p \leqslant |x+2y|^p + |x|^p,$$

and symmetrically

$$2|x+y|^p = 2|y+x|^p \leqslant |y+2x|^p + |y|^p.$$

Therefore, we have

$$2|x+y|^p \leqslant \lambda\left(|x+2y|^p + |x|^p\right) + (1-\lambda)\left(|y+2x|^p + |y|^p\right), \quad \forall \lambda \in [0,1],$$

and prove the inequality (9). □

4. Results on $L^p(\Omega, \mu)$

Next we apply the primary results to the well-known metric space. Let (Ω, μ) be a measure space, and $p \in [1, \infty)$. The space $L^p(\Omega, \mu)$ is defined as the collection of equivalence classes of measurable functions for which $\|f\|_{\Omega,\, p} < \infty$, $\forall f \in L^p(\Omega, \mu)$, where

$$\|f\|_{\Omega,\, p} = \left(\int_\Omega |f|^p d\mu\right)^{1/p} \tag{10}$$

is the p-norm on $L^p(\Omega,\mu)$ (see [1], [14]). In this article, for the given $f_1, f_2 \in L^p(\Omega,\mu)$ we define the subsets associated with the value of f_1 and f_2 as follows.

$$E_-(f_1,\ f_2) = \{x\in\Omega|\ f_1(x)f_2(x)<0\}, \tag{11}$$

$$E_+(f_1,\ f_2) = \{x\in\Omega|\ f_1(x)f_2(x)>0\}, \tag{12}$$

$$E_0(f_1,\ f_2) = \{x\in\Omega|\ f_1(x)f_2(x)=0\}, \tag{13}$$

It is obvious that

$$\Omega = E_-(f_1,\ f_2)\bigcup E_+(f_1,\ f_2)\bigcup E_0(f_1,\ f_2),$$

and for arbitrary vectors $f, f_1, f_2 \in L^p(\Omega,\mu)$, the relationship between the p-power functions of the corresponding norms can be expressed as follows:

$$\begin{aligned}\|f\|^p_{\Omega,\,p} &= \|f\|^p_{E_-(f_1,\,f_2),\,p} + \|f\|^p_{E_+(f_1,\,f_2),\,p} + \|f\|^p_{E_0(f_1,\,f_2),\,p}\\ &= \|f\|^p_{E_-(f_1,\,f_2)\bigcup E_0(f_1,\,f_2),\,p} + \|f\|^p_{E_+(f_1,\,f_2),\,p}\\ &= \|f\|^p_{E_-(f_1,\,f_2),\,p} + \|f\|^p_{E_+(f_1,\,f_2)\bigcup E_0(f_1,\,f_2),\,p}\end{aligned}$$

Based on the results on $\mathbb{R}$, we have following results.

PROPOSITION 3. *If $p\in[1,\infty)$, then for $\forall f,\ g,\ h \in L^p(E,\mu)$, following inequalities are true.*

$$\begin{aligned}\|f+g\|^p_{\Omega,\,p} + \|f+h\|^p_{\Omega,\,p} \geqslant\ & \|f+g+h\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p} + \|f\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p}\\ &+ \|f+g\|^p_{E_+(g,\,h),\,p} + \|f+h\|^p_{E_+(g,\,h)\ p},\end{aligned} \tag{14}$$

$$\begin{aligned}\|f+g\|^p_{\Omega,\,p} + \|f+h\|^p_{\Omega,\,p} \leqslant\ & \|f+g+h\|^p_{E_+(g,\,h)\bigcup E_0(g,\,h),\,p} + \|f\|^p_{E_+(g,\,h)\bigcup E_0(g,\,h),\,p}\\ &+ \|f+g\|^p_{E_-(g,\,h),\,p} + \|f+h\|^p_{E_-(g,\,h)\ p}.\end{aligned} \tag{15}$$

Proof. We will mainly prove the inequality (14). It follows from (4) that

$$\begin{aligned}&\|f+g\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p} + \|f+h\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p}\\ =&\int_{E_-(g,\,h)\bigcup E_0(g,\,h)} |f(x)+g(x)|^p d\mu + \int_{E_-(g,\,h)\bigcup E_0(g,\,h)} |f(x)+h(x)|^p d\mu\\ =&\int_{E_-(g,\,h)\bigcup E_0(g,\,h)} [|f(x)+g(x)|^p + |f(x)+h(x)|^p]\, d\mu\\ \geqslant&\int_{E_-(g,\,h)\bigcup E_0(g,\,h)} [|f(x)+g(x)+h(x)|^p + |f(x)|^p]\, d\mu \quad (from\ (4))\\ =&\|f+g+h\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p} + \|f\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p},\end{aligned}$$

Therefore,

$$\begin{aligned}\|f+g\|^p_{\Omega,\,p} + \|f+h\|^p_{\Omega,\,p} =&\|f+g\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p} + \|f+h\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p}\\ &+ \|f+g\|^p_{E_+(g,\,h),\,p} + \|f+h\|^p_{E_+(g,\,h),\,p}\\ \geqslant&\|f+g+h\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p} + \|f\|^p_{E_-(g,\,h)\bigcup E_0(g,\,h),\,p}\\ &+ \|f+g\|^p_{E_+(g,\,h),\,p} + \|f+h\|^p_{E_+(g,\,h),\,p},\end{aligned}$$

the inequality (14) is proved.

The proof for inequality (15) is a simple modification of above process, thus be omitted. □

COROLLARY 5. *If* $p \in [1, \infty)$, *then for* $\forall f,\ g \in L^p(\Omega,\mu)$, $\lambda \in [0,\ 1]$, *following inequalities hold.*

$$\|f+g\|_{\Omega,\,p}^p + \|f-g\|_{\Omega,\,p}^p \geqslant 2\left[\lambda\|f\|_{\Omega,\,p}^p + (1-\lambda)\|g\|_{\Omega,\,p}^p\right], \tag{16}$$

$$\|f\|_{\Omega,\,p}^p + \|g\|_{\Omega,\,p}^p \geqslant 2\left[\lambda\left\|\frac{f+g}{2}\right\|_{\Omega,\,p}^p + (1-\lambda)\left\|\frac{f-g}{2}\right\|_{\Omega,\,p}^p\right], \tag{17}$$

$$\|f+g\|_{\Omega,\,p}^p \leqslant \frac{1}{2}\left[\lambda\|f\|_{\Omega,\,p}^p + (1-\lambda)\|g\|_{\Omega,\,p}^p + \lambda\|f+2g\|_{\Omega,\,p}^p + (1-\lambda)\|2f+g\|_{\Omega,\,p}^p\right]. \tag{18}$$

Proof. Let $h=-g$ in inequality (14). It is obvious that $E_+(g,\ -g)=\emptyset$ and

$$\Omega = E_-(g,\ -g)\bigcup E_0(g,\ -g).$$

Thus, it follows from (14) that

$$\|f+g\|_{\Omega,\,p}^p + \|f-g\|_{\Omega,\,p}^p \geqslant 2\|f\|_{\Omega,\,p}^p, \tag{19}$$

and

$$\|f+g\|_{\Omega,\,p}^p + \|f-g\|_{\Omega,\,p}^p = \|g+f\|_{\Omega,\,p}^p + \|g-f\|_{\Omega,\,p}^p \geqslant 2\|g\|_{\Omega,\,p}^p. \tag{20}$$

Alternately, let $h=g$ in inequality (15). In this case, $E_-(g,\ g)=\emptyset$ and

$$\Omega = E_+(g,\ g)\bigcup E_0(g,\ g),$$

hence it follows from (15) that

$$2\|f+g\|_{\Omega,\,p}^p \leqslant \|f+2g\|_{\Omega,\,p}^p + \|f\|_{\Omega,\,p}^p, \tag{21}$$

and symmetrically

$$2\|f+g\|_{\Omega,\,p}^p \leqslant \|2f+g\|_{\Omega,\,p}^p + \|g\|_{\Omega,\,p}^p. \tag{22}$$

Therefore, according to inequalities (19), (20), (21), (22) and similar to the analysis of Corollary 2, Corollary 3 and Corollary 4 one can deduce inequalities (16), (17) and (18), and complete this proof. □

REMARK 4. The special case "$\lambda = 1/2$" of inequalities (16) and (17) yield

$$\|f+g\|_{\Omega,\,p}^p + \|f-g\|_{\Omega,\,p}^p \geqslant \|f\|_{\Omega,\,p}^p + \|g\|_{\Omega,\,p}^p \geqslant \left\|\frac{f+g}{2}\right\|_{\Omega,\,p}^p + \left\|\frac{f-g}{2}\right\|_{\Omega,\,p}^p,$$

which has following equivalent form

$$\|u\|_{\Omega,\,p}^{p}+\|v\|_{\Omega,\,p}^{p}\geqslant\left\|\frac{u+v}{2}\right\|_{\Omega,\,p}^{p}+\left\|\frac{u-v}{2}\right\|_{\Omega,\,p}^{p}\geqslant\left\|\frac{u}{2}\right\|_{\Omega,\,p}^{p}+\left\|\frac{v}{2}\right\|_{\Omega,\,p}^{p},$$

if let $f=(u+v)/2$, $g=(u-v)/2$. The performance of above inequality approximates (but less precise than) that of Clarkson's inequality. Comparatively speaking, the advantage of Corollary 5 lies in its wider scope of application since it holds for $p\in[1,\infty)$.

We have to point out that the above discussion requires that the norm $\|\cdot\|_{\Omega,\,p}$ depends on the parameter p which is also the exponent of the power function. A natural question is whether there is an appropriate inequality for the p-power function of an arbitrary norm $\|\cdot\|_{\Omega,\,q}$ while $q\neq p$. The complete study of this issue await further research in the future, and here we prepare a narrow answer for the case that the two involved norms are equivalent.

DEFINITION 1. ([14], [15]) We say the two norms $\|\cdot\|_{q}$ and $\|\cdot\|_{p}$ are equivalent on space Y, if there exist real positive constants m and M such that

$$m\|f\|_{p}\leqslant\|f\|_{q}\leqslant M\|f\|_{p},\quad\forall f\in Y. \tag{23}$$

In fact, (23) holds for all the finite-dimensional spaces, such as $Y=\mathbb{R}^{n}$ (see [15]).

COROLLARY 6. *Let $p\in[1,\,\infty)$ and norm $\|\cdot\|_{\Omega,\,p}$ be defined by (10). Suppose norms $\|\cdot\|_{\Omega,\,q}$ and $\|\cdot\|_{\Omega,\,p}$ satisfy the relation as (23), and the constants m and M are defined by (23), then for $\forall f,\ g\in L^{p}(\Omega,\mu)$, $\lambda\in[0,\,1]$, the following inequalities are true.*

$$\|f+g\|_{\Omega,\,q}^{p}+\|f-g\|_{\Omega,\,q}^{p}\geqslant 2\left(\frac{m}{M}\right)^{p}\left[\lambda\|f\|_{\Omega,\,q}^{p}+(1-\lambda)\|g\|_{\Omega,\,q}^{p}\right], \tag{24}$$

$$\|f\|_{\Omega,\,q}^{p}+\|g\|_{\Omega,\,q}^{p}\geqslant 2\left(\frac{m}{M}\right)^{p}\left[\lambda\left\|\frac{f+g}{2}\right\|_{\Omega,\,q}^{p}+(1-\lambda)\left\|\frac{f-g}{2}\right\|_{\Omega,\,q}^{p}\right], \tag{25}$$

$$\|f+g\|_{\Omega,\,q}^{p}\leqslant\frac{1}{2}\left(\frac{M}{m}\right)^{p}\left[\lambda\|f\|_{\Omega,\,q}^{p}+(1-\lambda)\|g\|_{\Omega,\,q}^{p}+\lambda\|f+2g\|_{\Omega,\,q}^{p}+(1-\lambda)\|2f+g\|_{\Omega,\,q}^{p}\right]. \tag{26}$$

Proof. Because

$$\|f+g\|_{\Omega,\,q}^{p}+\|f-g\|_{\Omega,\,q}^{p}\geqslant m^{p}\left[\|f+g\|_{\Omega,\,p}^{p}+\|f-g\|_{\Omega,\,p}^{p}\right]\quad\text{(from (23))}$$
$$\geqslant 2m^{p}\left[\lambda\|f\|_{\Omega,\,p}^{p}+(1-\lambda)\|g\|_{\Omega,\,p}^{p}\right]\quad\text{(from (16))}$$
$$\geqslant 2\left(\frac{m}{M}\right)^{p}\left[\lambda\|f\|_{\Omega,\,q}^{p}+(1-\lambda)\|g\|_{\Omega,\,q}^{p}\right],\quad\text{(from (23))}$$

the inequality (24) is proved.

Inequality (25) is equivalent to inequality (24), and its proof is omitted.

As for the inequality (26), we have

$$\begin{aligned}
&\|f+g\|_{\Omega,q}^p \leqslant M^p\|f+g\|_{\Omega,p}^p \quad (\text{from } (23))\\
\leqslant&\frac{1}{2}M^p\left[\lambda\|f\|_{\Omega,p}^p+(1-\lambda)\|g\|_{\Omega,p}^p+\lambda\|f+2g\|_{\Omega,p}^p+(1-\lambda)\|2f+g\|_{\Omega,p}^p\right]\\
&\qquad\qquad (\text{from } (18))\\
\leqslant&\frac{1}{2}\left(\frac{M}{m}\right)^p\left[\lambda\|f\|_{\Omega,q}^p+(1-\lambda)\|g\|_{\Omega,q}^p+\lambda\|f+2g\|_{\Omega,q}^p+(1-\lambda)\|2f+g\|_{\Omega,q}^p\right].\\
&\qquad\qquad (\text{from } (23))
\end{aligned}$$

Thus we get inequality (26) and complete this proof. □

We recall that $L^2(\Omega,\mu)$ is an inner product space. Therefore, according to equality (2) and similar to the analysis as Corollary 6 we give following corollary.

COROLLARY 7. *Assume norm $\|\cdot\|_{\Omega,2}$ is defined by (10) (p=2). Suppose norms $\|\cdot\|_{\Omega,q}$ and $\|\cdot\|_{\Omega,2}$ satisfy the relation as (23), and the constants m and M are defined by (23), then for $\forall f,\ g\in L^2(\Omega,\ \mu)$, the following inequalities hold.*

$$\|f+g\|_{\Omega,q}^2+\|f-g\|_{\Omega,q}^2 \geqslant 2\left(\frac{m}{M}\right)^2\left(\|f\|_{\Omega,q}^2+\|g\|_{\Omega,q}^2\right), \tag{27}$$

$$\|f\|_{\Omega,q}^2+\|g\|_{\Omega,q}^2 \geqslant 2\left(\frac{m}{M}\right)^2\left(\left\|\frac{f+g}{2}\right\|_{\Omega,q}^2+\left\|\frac{f-g}{2}\right\|_{\Omega,q}^2\right). \tag{28}$$

Proof. Since

$$\begin{aligned}
&\|f+g\|_{\Omega,q}^2+\|f-g\|_{\Omega,q}^2 \geqslant m^2\left(\|f+g\|_{\Omega,2}^2+\|f-g\|_{\Omega,2}^2\right) \quad (\text{from } (23))\\
=&2m^2\left(\|f\|_{\Omega,2}^2+\|g\|_{\Omega,2}^2\right) \quad (\text{from } (2))\\
\geqslant&2\left(\frac{m}{M}\right)^2\left(\|f\|_{\Omega,q}^2+\|g\|_{\Omega,q}^2\right), \quad (\text{from } (23))
\end{aligned}$$

we obtain inequality (27). Inequality (28) can be obtained similarly. □

5. Conclusion

By some elementary technique we establish a three-variable inequality for the p-power function of the norms on some metric spaces, such as the inner product space and $L^p(\Omega,\ \mu)$. However, there are still some problems that need further study, namely,

(i) Does the inequality still hold for $\|\cdot\|_q^p$ where the norm $\|\cdot\|_q$ is not equivalent to the norm $\|\cdot\|_p$?

(ii) Does the inequality hold for a more generalized metric space?

Further research and the improvement or generalization on the existing primary work is undoubtedly the next goal in the future.

Acknowledgement. This work was supported by the National Natural Science Funds of China (No. 11371081, No. 61575090, No. 61775169) and the State Key Laboratory of Synthetical Automation for Process Industries Fundamental Research Funds, China (No. 2013ZCX02).

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(Received March 18, 2019)

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