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# REARRANGEMENTS AND JENSEN TYPE INEQUALITIES RELATED TO CONVEXITY, SUPERQUADRACITY, STRONG CONVEXITY AND 1–QUASICONVEXITY

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*Abstract.* In this paper we derive and discuss some new theorems related to all rearrangements of a given set in $\mathbb{R}^n$, denoted $(\mathbf{x})$ and use the results to prove some new Jensen type inequalities for convex, superquadratic, strongly convex and $1$-quasiconvex functions.

## 1. Introduction

Let $\alpha$, $\beta \in \mathbb{R}$ and let $(\mathbf{x})$ denote the set of all rearrangements of a given set $(x_1,x_2,\dots,x_n)$, $\alpha < x_i < \beta$, $i = 1,\dots,n$. In this paper we obtain new theorems related to the rearrangements of $(\mathbf{x})$ and use the results to get Jensen's type inequalities for convex, superquadratic, strongly convex and $1$-quasiconvex functions.

Our first main result proved in Section 2 is:

THEOREM 1. *Let $F(u,v)$ be differentiable and symmetric real function defined on $\alpha < u,v < \beta$, $-\infty \leqslant \alpha < \beta \leqslant \infty$ and assume that*

$$\frac{\partial F(v,u)}{\partial v} \leqslant \frac{\partial F(v,w)}{\partial v}, \qquad \alpha < u,v,w < \beta \tag{1.1}$$

*and*

$$\frac{\partial F(w,u)}{\partial w} \leqslant \frac{\partial F(w,v)}{\partial w} \qquad \alpha < u,v,w < \beta \tag{1.2}$$

*for $u \leqslant \min(w,v)$.*

*Then, for any set $(\mathbf{x}) = (x_1,x_2,\dots,x_n)$, $\alpha < x_i < \beta$, $i = 1,\dots,n$, given except its arrangements*

$$\sum_{i=1}^{n} F(x_i,x_{i+1}), \qquad x_{n+1} = x_1 \tag{1.3}$$

*is **maximal** if $(\mathbf{x})$ is arranged in circular symmetrical order and **minimal** if $(\mathbf{x})$ is arranged in circular alternating order as defined below.*

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The maximum of (1.3) was proved in [1] in 1967, as a generalization of the special case $F(x,y) = f(|x-y|)$, where $f$ is a concave and decreasing function, which was proved in 1963 in [5].

The minimum of (1.3) for the special case $F(x,y) = xy$ was dealt with recently by H. Yu in [8]. There the author introduced the circular alternating order arrangement of a set $(\mathbf{x})$ and proved that $\sum_{i=1}^{n} x_i x_{i+1}$, $x_{n+1} = x_1$ gets its minimum for this specific arrangement.

This result motivates us to deal again with the behavior of (1.3) under rearrangement of $(\mathbf{x})$.

We continue by giving some definitions missed in the formulation of Theorem 1 and which are important in the sequel. Moreover, we formulate one more main result (Theorem 2) and also some other theorems, lemmas and corollaries we need for the proofs of Theorems 1 and 2.

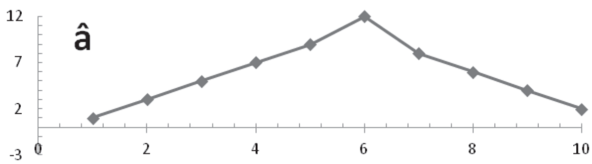

Figure 1: Symmetrical Decreasing Order According to (1.4) -10 Terms

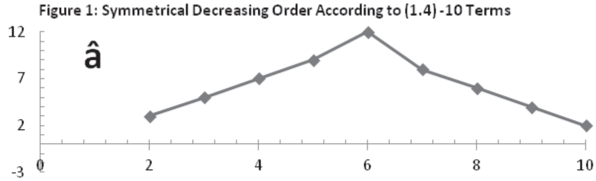

Figure 2: Symmetrical Decreasing Order According to (1.5) -9 Terms

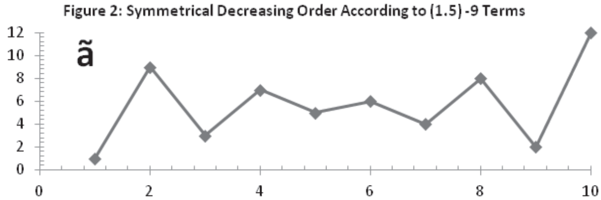

Figure 3: Alternating Order According to (1.6) - 10 Terms

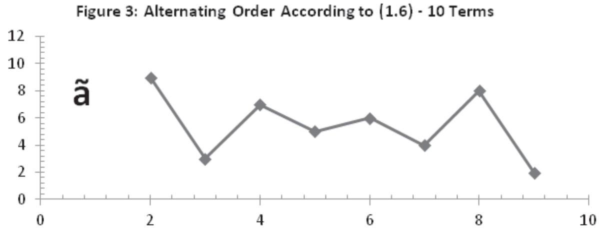

Figure 4: Alternating Order According to (1.7) - 8 Terms

DEFINITION 1. (See [5] and the illustrations in Figure 1 and Figure 2). An ordered set $(\mathbf{x}) = (x_1, \ldots, x_n)$ of $n$ real numbers is arranged in **symmetrical decreasing**

**order** if

$$x_1 \leqslant x_n \leqslant x_2 \leqslant x_{n-1} \leqslant \dots \leqslant x_{[(n+2)/2]} \tag{1.4}$$

or if

$$x_n \leqslant x_1 \leqslant x_{n-1} \leqslant x_2 \leqslant \dots \leqslant x_{[(n+1)/2]}. \tag{1.5}$$

REMARK 1. When the ordered set $(\mathbf{x}) = (x_1, \dots, x_n)$ of $n$ terms satisfies (1.4), then the ordered set $(x_2, \dots, x_n)$ of $n-1$ terms satisfies (1.5).

DEFINITION 2. (See [5]) A **circular rearrangement** of an ordered set $(\mathbf{x})$ is a cyclic rearrangement of $(\mathbf{x})$ or a cyclic rearrangement followed by inversion; For example, the circular rearrangements of the ordered set $(1,2,3,4)$ are the sets

$$(1,2,3,4), (2,3,4,1), (3,4,1,2), (4,1,2,3),$$
$$(4,3,2,1), (1,4,3,2), (2,1,4,3), (3,2,1,4).$$

DEFINITION 3. (See [5]) A set $(\mathbf{x})$ is arranged in **circular symmetrical order** if one of its circular rearrangements is symmetrically decreasing.

DEFINITION 4. (See [8] and the illustrations in Figure 3 and Figure 4). An ordered set $(\mathbf{x}) = (x_1, \dots, x_n)$ of n real numbers is arranged in **alternating order** if

$$x_1 \leqslant x_{n-1} \leqslant x_3 \leqslant x_{n-3} \leqslant x_5 \leqslant \dots \leqslant x_{\left[\frac{n+1}{2}\right]} \leqslant \dots \leqslant x_{n-4} \leqslant x_4 \leqslant x_{n-2} \leqslant x_2 \leqslant x_n, \tag{1.6}$$

or if

$$x_n \leqslant x_2 \leqslant x_{n-2} \leqslant x_4 \leqslant x_{n-4} \leqslant \dots \leqslant x_{\left[\frac{n+1}{2}\right]} \leqslant \dots \leqslant x_5 \leqslant x_{n-3} \leqslant x_3 \leqslant x_{n-1} \leqslant x_1. \tag{1.7}$$

REMARK 2. When the ordered set $(\mathbf{x}) = (x_1, \dots, x_n)$ of $n$ terms satisfies (1.6), then the ordered set $(x_2, \dots, x_{n-1})$ of $n-2$ terms satisfies (1.7).

DEFINITION 5. A set $(\mathbf{x})$ is arranged in **circular alternating order** if one of its circular rearrangements is arranged in an alternating order.

DEFINITION 6. (See [8]). Given the set

$$(\mathbf{x}) = \left(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n\right),$$

we call its permutation

$$(\mathbf{z}) = \left(x_1, \dots, x_{i-1}, x_j, x_{j-1}, x_{j-2}, \dots, x_{i+2}, x_{i+1}, x_i, x_{j+1}, \dots, x_n\right)$$

**a turnover of** $(\mathbf{x})$, for some $1 \leqslant i < j \leqslant n$, in which the order of

$$\left(x_i, x_{i+1}, x_{i+2}, \dots, x_{j-2}, x_{j-1}, x_j\right)$$

in $(\mathbf{x})$ is reversed in $(\mathbf{z})$.

One more main result proved in Section 2 is the following:

THEOREM 2. *Let* $F = F(u,v)$ *be a differentiable real function defined on* $\alpha \leqslant u,v \leqslant \beta$ *which is symmetric in* $u$ *and* $v$*. Then* $\sum_{i=1}^{n} F(x_i, x_{i+1})$*, where* $x_{n+1} = x_1$*, is maximal if* $(\mathbf{x})$ *is arranged in circular symmetrical order and minimal if* $(\mathbf{x})$ *is arranged in circular alternating order in each of the following cases:*

*Case 2(a)* $F(x,y) = f(x+y)$*, where* $f$ *is convex on* $\mathbb{R}_+$*,*

*Case 2(b)* $F(x,y) = f(|x-y|)$*, where* $f$ *is concave and decreasing on* $\mathbb{R}_+$*.*

*Case 2(c)* $F(x,y) = f\left(\frac{x+y}{2}\right) + f\left(\left|\frac{x-y}{2}\right|\right)$*, where* $f'$ *is convex and differentiable on* $\mathbb{R}_+$ *and* $f'(0) = 0$*,*

*Case 2(d)* $F(x,y) = f\left(\frac{x+y}{2}\right) + C \times \left(\frac{x-y}{2}\right)^2$*, where the constant* $C$ *satisfies* $C \leqslant \varphi'\left(\frac{x_m + x_j}{2}\right)$*, with* $0 \leqslant x_m \leqslant x_j \leqslant x_i,\ i \neq m, j,\ i = 1, \ldots, n,\ f(x) = x\varphi(x)$*, and* $\varphi$ *is twice differentiable and convex function on* $0 \leqslant x \leqslant b$*.*

*Case 2(e)* $F(x,y) = f\left(\frac{x+y}{2}\right) + \varphi'\left(\frac{x+y}{2}\right)\left(\frac{x-y}{2}\right)^2$*, where* $\varphi$ *and* $\varphi'$ *are twice differentiable and convex on* $\mathbb{R}_+$ *and* $f(x) = x\varphi(x)$*.*

Next we recall (as mentioned in the introduction) that the maximum value of $\sum_{i=1}^{n} F(x_i, x_{i+1}),\ x_{n+1} = x_1$, stated in the following theorem, is proved in [1, Theorem 1]:

THEOREM A. *Let* $F(u,v)$ *be a symmetric function which is defined on* $\alpha < u, v < \beta,\ -\infty \leqslant \alpha < \beta \leqslant \infty$*. Let the set* $(\mathbf{x}) = (x_1, \ldots, x_n),\ \alpha < x_i < \beta,\ i = 1, \ldots, n$*, be given except its arrangement. If*

$$F(u,v) + F(u,w) - F(w,v), \qquad \alpha < u, v, w < \beta$$

*is decreasing in* $v$ *and* $w$ *for* $u \leqslant \min(v,w)$*, then*

$$\sum_{i=1}^{n} F(x_i, x_{i+1}), \quad x_{n+1} = x_1$$

*is maximal if* $(\mathbf{x}) = (x_1, \ldots, x_n)$*, is arranged in circular symmetrical order.*

REMARK 3. We get a special case of Theorem A when $F(u,v)$ is differentiable, symmetric in $u$ and $v$ and (1.1) and (1.2) hold.

Moreover, the case $F(u,v) = -(x-y)^2$ leads to the maximum of $\sum_{i=1}^{n} x_i x_{i+1}$, $x_{n+1} = x_1$ (see [1]).

We also mention that the maximum of $\sum_{i=1}^{n}\left(f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\left|\frac{x_i - x_{i+1}}{2}\right|\right)\right)$ appeared as Theorem 2.1 in [2]:

THEOREM B. *Let* $F(x,y) = f\left(\frac{x+y}{2}\right) + f\left(\left|\frac{x-y}{2}\right|\right)$. *If* $f'$ *is convex,* $x \geqslant 0$, *and* $f'(0) = 0$, *then*

$$\sum_{i=1}^{n}\left(f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\left|\frac{x_i - x_{i+1}}{2}\right|\right)\right)$$

*attains its maximum value when* $(\mathbf{x}) = (x_1, ..., x_n)$ *is arranged in circular symmetrical order.*

The remaining part of the paper is organized as follows: In Section 2 we prove Theorems 1 and 2. For these proofs we need to prove some lemmas of independent interest, since they may be regarded as new independent results on rearrangements (see Lemmas 1, 2, 3 and c.f. also Corollary 1).

With the help of these results we will formulate and prove in Section 3 our new Jensen type inequalities (see Theorems 3, 4 and 5).

## 2. Proofs of Theorems 1 and 2: New results on rearrangements

The case for which $\sum_{i=1}^{n} F(x_i, x_{i+1})$, $x_{n+1} = x_1$ gets its maximal value under rearrangements of $(\mathbf{x})$ was proved in [1, Theorem 1] as quoted in Theorem A. Therefore it is enough to discuss now the case for which $\sum_{i=1}^{n} F(x_i, x_{i+1})$, $x_{n+1} = x_1$, gets its minimal value under rearrangements of $(\mathbf{x})$ and we implement these results on inequalities related to convex, superquadratic, stronly convex and $1$-quasiconvex functions in Theorem 2.

An outline of the proofs of the following three lemmas, a corollary and Theorem 1 is as follows:

We denote a given set of $n$ real numbers according to their increasing order $(\mathbf{a}) = (a_1, a_2, ..., a_{n-1}, a_n)$, where $a_1 \leqslant a_2 \leqslant ... \leqslant a_{n-1} \leqslant a_n$. We start with an arbitrary permutation of $(\mathbf{a})$ called $(\mathbf{b}) = (b_1, ..., b_n)$. Since our $F(u,v)$ is symmetric and we are interested in $\sum_{i=1}^{n} F(b_i, b_{i+1})$, $b_{n+1} = b_1$, which is clearly invariant under all circular rearrangements, we can assume that $b_1 = a_1$. Now we go through three permutations which bring us from $(\mathbf{b}) \to (\mathbf{c}) \to (\mathbf{d}) \to (\mathbf{e})$ in which $\sum_{i=1}^{n} F(b_i, b_{i+1}) \geqslant \sum_{i=1}^{n} F(c_i, c_{i+1}) \geqslant \sum_{i=1}^{n} F(d_i, d_{i+1}) \geqslant \sum_{i=1}^{n} F(e_i, e_{i+1})$, and we make sure that the two first and two last numbers in $(\mathbf{e})$ are $e_1 = a_1$, $e_2 = a_{n-1}$, $e_{n-1} = a_2$, $e_n = a_n$, which are already the two first and the two last in the rearrangements of the alternating order of type (1.6) (see Figure 1). We realize also that when we check $(e_2, e_3, ..., e_{n-2}, e_{n-1})$ we already have that $e_2$ and $e_{n-1}$ are the largest and the smallest numbers, respectively, in $(e_2, e_3, ..., e_{n-2}, e_{n-1})$.

Now we use the induction procedure: We assume the validity of the Theorem 1 for the set of $n-2$ numbers and show that this implies its validity for the set of $n$ numbers. More specifically, the $n-2$ numbers if rearranged in alternating order of (1.7) gives, according to the induction assumption, the smallest value of $\sum_{i=2}^{n-2} F(e_i, e_{i+1})$ and in the same time we get that $(e_1, ..., e_n)$ is arranged in alternating order too, this time according to (1.6) and therefore the proof by induction for $n$ numbers is obtained.

In Lemma 1 we perform a permutation on $(\mathbf{b})$ so that in the resulting $(\mathbf{c})$, the minimum of $c_i$ and the maximum of $c_i$, $i = 1, ..., n$, are neighbouring terms.

LEMMA 1. *Let* $(\mathbf{a})$ *be the ordered set* $(\mathbf{a}) = (a_1, \ldots, a_n)$*, where* $a_1 \leqslant a_2 \leqslant, \ldots, \leqslant a_{n-1} \leqslant a_n$ *and its given permutation* $(\mathbf{b}) = (b_1, \ldots, b_n)$*, where* $b_1 = a_1,\ b_j = a_n$*. Let* $(\mathbf{c}) = (c_1, \ldots, c_n) = \left(b_1, b_2, \ldots, b_{j-1}, b_n, b_{n-1}, b_{n-2}, \ldots, b_{j+1}, b_j\right)$*, which is the turnover of* $(\mathbf{b})$ *where* $c_1 = b_1 = a_1,\ c_n = b_j = a_n$*. Let the symmetric and differentiable function* $F(x,y)$ *be defined on* $[a_1, a_n] \times [a_1, a_n]$*. Then,*

$$\sum_{i=1}^{n} F(b_i, b_{i+1}) - \sum_{i=1}^{n} F(c_i, c_{i+1}) \geqslant 0$$

*when* $\frac{\partial(F(x,z) - F(x,y))}{\partial x} \geqslant 0,\ \ y \leqslant \min(z, x)$.

*Proof.* Since $(\mathbf{c})$ is the turnover of $(\mathbf{b})$ that satisfies $(\mathbf{c}) = \left(a_1, b_2, \ldots, b_{j-1}, b_n, b_{n-1}, b_{n-2}, \ldots, b_{j+1}, b_j\right)$, we get that

$$\begin{aligned}
&\sum_{i=1}^{n} F(c_i, c_{i+1}) \\
&= \left(\sum_{i=1}^{j-2} F(c_i, c_{i+1})\right) + F\left(c_{j-1}, c_j\right) + \left(\sum_{i=j}^{n-1} F(c_i, c_{i+1})\right) + F(c_n, c_1) \\
&= \left(\sum_{i=1}^{j-2} F(c_i, c_{i+1})\right) + F\left(c_{j-1}, c_j\right) + \left(\sum_{k=0}^{n-j-1} F(c_{n-k-1}, c_{n-k})\right) + F(c_n, c_1) \\
&= \left(\sum_{i=2}^{i-2} F(b_i, b_{i+1})\right) + F\left(b_{j-1}, b_n\right) + \left(\sum_{k=0}^{n-j-1} F(b_{n-k-1}, b_{n-k})\right) + F(b_j, b_1).
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
&\sum_{i=1}^{n} F(b_i, b_{i+1}) \\
&= \left(\sum_{i=2}^{i-2} F(b_i, b_{i+1})\right) + F\left(b_{j-1}, b_j\right) + \left(\sum_{k=0}^{n-j-1} F(b_{n-k-1}, b_{n-k})\right) + F(b_n, b_1).
\end{aligned}$$

Therefore, using the symmetry of $F(u,v)$ the inequality

$$\begin{aligned}
&\sum_{i=1}^{n} F(b_i, b_{i+1}) - \sum_{i=1}^{n} F(c_i, c_{i+1}) \\
&= F\left(b_{j-1}, b_j\right) + F(b_n, b_1) - \left(F\left(b_{j-1}, b_n\right) + F(b_j, b_1)\right) \\
&= F\left(b_{j-1}, a_n\right) + F(b_n, a_1) - \left(F\left(b_{j-1}, b_n\right) + F(a_n, a_1)\right) \\
&= \left[F\left(a_n, b_{j-1}\right) - F(a_n, a_1)\right] - \left[F\left(b_n, b_{j-1}\right) - F(b_n, a_1)\right] \geqslant 0,
\end{aligned}$$

holds when $\frac{\partial(F(x,z) - F(x,y))}{\partial x} \geqslant 0,\ \ y \leqslant \min(z, x)$, because $a_1 \leqslant b_i,\ i = 2, \ldots, n,\ a_n \geqslant b_n$ and $b_1 = a_1,\ b_j = a_n$. The proof is complete.

In Lemma 2 we make a turnover from $(\mathbf{c})$ to $(\mathbf{d})$ such that in $(\mathbf{d})$ $d_2 = a_{n-1}$.

LEMMA 2. *Let* $(\mathbf{a})$ *be the ordered set* $(\mathbf{a}) = (a_1,...,a_n)$*, where* $a_1 \leqslant a_2 \leqslant ... \leqslant a_{n-1} \leqslant a_n$*, let* $(\mathbf{c}) = (c_1,...,c_n)$ *be such that* $c_1 = a_1$*,* $c_n = a_n$*,* $c_k = a_{n-1}$*. Let* $(\mathbf{d})$ *be a turnover of* $(\mathbf{c})$ *such that* $(\mathbf{d}) = (c_1,c_k,c_{k-1},c_{k-2},...,c_3,c_2,c_{k+1},c_{k+2},...,c_{n-1},c_n)$*, let the symmetric and continuouosly differentiable function* $F(x,y)$ *be defined on* $[a_{1,}a_n] \times [a_{1,}a_n]$*. Then,*

$$\sum_{i=1}^{n} F(c_i,c_{i+1}) - \sum_{i=1}^{n} F(d_i,d_{i+1}) \geqslant 0$$

*when* $\frac{\partial(F(x,z)-F(x,y))}{\partial x} \geqslant 0, \ \ y \leqslant \min(z,x)$.

*Proof.* By the turnover of $(\mathbf{c})$ we get that

$$\begin{aligned}
&\sum_{i=1}^{n} F(d_i,d_{i+1}) \\
&= F(d_1,d_2) + \left(\sum_{i=2}^{k-1} F(d_i,d_{i+1})\right) + F(d_k,d_{k+1}) + \left(\sum_{i=k+1}^{n} F(d_i,d_{i+1})\right) \\
&= F(d_1,d_2) + \left(\sum_{i=0}^{k-3} F(d_{k-i},d_{k-i-1})\right) + F(d_k,d_{k+1}) + \left(\sum_{i=k+1}^{n} F(d_i,d_{i+1})\right) \\
&= F(c_1,c_k) + \left(\sum_{j=0}^{k-3} F\left(c_{k-j},c_{k-j-1}\right)\right) + F(c_2,c_{k+1}) + \left(\sum_{i=k+1}^{n} F(c_i,c_{i+1})\right),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n} F(c_i,c_{i+1}) \\
&= F(c_1,c_2) + \left(\sum_{j=0}^{k-3} F\left(c_{k-j},c_{k-j-1}\right)\right) + F(c_k,c_{k+1}) + \left(\sum_{i=k+1}^{n} F(c_i,c_{i+1})\right).
\end{aligned}$$

Therefore

$$\sum_{i=1}^{n} F(c_i,c_{i+1}) - \sum_{i=1}^{n} F(d_i,d_{i+1}) = F(c_1,c_2) + F(c_k,c_{k+1}) - (F(c_1,c_k) + F(c_2,c_{k+1})),$$

and

$$\begin{aligned}
&\sum_{i=1}^{n} F(c_i,c_{i+1}) - \sum_{i=1}^{n} F(d_i,d_{i+1}) \\
&= F(a_{n-1},c_{k+1}) - F(a_{n-1},a_1) - [F(c_2,c_{k+1}) - F(c_2,a_1)] \geqslant 0
\end{aligned}$$

when $\frac{\partial(F(x,z)-F(x,y))}{\partial x} \geqslant 0, \ \ y \leqslant \min(z,x)$, because $a_1 \leqslant c_2, c_{k+1}, a_{n-1}$ and $a_{n-1} \geqslant c_2$, and $F(\cdot,\cdot)$ is symmetric and continuously differentiable. The proof is complete.

In Lemma 3 we make a turnover from $(\mathbf{d}\ )$ to $(\mathbf{e})$ such that $e_{n-1} = a_2$.

LEMMA 3. *Let* $(\mathbf{a})$ *be as before and let* $(\mathbf{d})$ *be such that* $d_1 = a_1, d_2 = a_{n-1}$ $d_n = a_n$ *and* $d_m = a_2$*. By the turnover of* $(\mathbf{d})$ *we get*

$$(\mathbf{e}) = (e_1, e_2, e_3, ..., e_{n-2}, e_{n-1}, e_n) = (d_1, d_2, ..., d_{m-1}, d_{n-1}, d_{n-2}, ..., d_m, d_n),$$

*where* $e_1 = a_1 = d_1,\ e_2 = a_{n-1} = d_2,\ e_n = a_n = d_n$ *and* $e_{n-1} = a_2 = d_m$*. Let the symmetric and continuously differentiable function* $F(x,y)$ *be defined on* $[a_{1,}a_n] \times [a_{1,}a_n]$*. Then,*

$$\sum_{i=1}^{n} F(d_i, d_{i+1}) - \sum_{i=1}^{n} F(e_i, e_{i+1}) \geqslant 0 \tag{2.1}$$

*when* $\frac{\partial(F(x,z) - F(x,y))}{\partial x} \geqslant 0,\ \ y \leqslant \min(z,x)$.

*Proof.* Using the same idea as in the proof of the former lemmas, we get that

$$\begin{aligned}
&\sum_{i=1}^{n} F(d_i, d_{i+1}) - \sum_{i=1}^{n} F(e_i, e_{i+1}) \\
&= F(d_{m-1}, d_m) + F(d_{n-1}, d_n) - [F(d_{m-1}, d_{n-1}) + F(d_m, d_n)] \\
&= F(a_n, d_{n-1}) - F(a_n, a_2) - [F(d_{m-1}, d_{n-1}) - F(d_{m-1}, a_2)].
\end{aligned}$$

For inequality (2.1) to hold, we see that it is sufficient that

$$F(a_n, d_{n-1}) - F(a_n, a_2) \geqslant [F(d_{m-1}, d_{n-1}) - F(d_{m-1}, a_2)]$$

holds and this occurs because $\frac{\partial(F(x,z) - F(x,y))}{\partial x} \geqslant 0.\ \ y \leqslant \min(z,x)$ and $d_{n-1},\ d_{m-1} \geqslant a_2$, $a_n \geqslant d_{m-1}$ are satisfied and $F(\cdot,\cdot)$ is symmetric and continuously differentiable. The proof is complete.

COROLLARY 1. *By the three turnovers, from* $(\mathbf{b})$ *to* $(\mathbf{c})$ *in Lemma 1, from* $(\mathbf{c})$ *to* $(\mathbf{d})$ *in Lemma 2 and from* $(\mathbf{d})$ *to* $(\mathbf{e})$ *in Lemma 3 it is clear that starting with a given arrangement* $(\mathbf{b})$ *we get an arrangement* $(\mathbf{e})$ *for which* $e_1 = a_1,\ e_2 = a_{n-1},\ e_{n-1} = a_2$ *and* $e_n = a_n$ *that leads to*

$$\sum_{i=1}^{n} F(b_i, b_{i+1}) \geqslant \sum_{i=1}^{n} F(e_i, e_{i+1}).$$

*Moreover, we realize that the rearrangement* $(\mathbf{e})$ *is a step in rearranging the original* $(\mathbf{b})$ *toward its alternating order (see also Figure 3 and Figure 4).*

With help of the inequalities obtained in lemmas 1, 2, 3 and Corollary 1, we are ready to present and complete the

*Proof of Theorem 1.* First we verify the statement in the theorem for $n = 3$ and $n = 4$ by just computing all the different rearrangements. We prove now that $\sum_{i=1}^{n} F(b_i, b_{i+1})$, $b_{n+1} = b_1$ gets its minimal value when $(\mathbf{b})$ is arranged in alternating order. We can assume that $b_1 = a_1,\ b_2 = a_{n-1},\ b_n = a_n$ and $b_{n-1} = a_2$ because otherwise the arrangement does not give the minimal sum of $\sum_{i=1}^{n} F(b_i, b_{i+1})$ as shown in Lemmas

1, 2, 3 and Corollary 1. Then, the set $(b_2, b_3, ..., b_{n-1})$, $b_2 = a_{n-1}$, $b_{n-1} = a_2$, satisfies $b_2 \leqslant b_l \leqslant b_{n-1}$ $l = 3, 4, ..., n-2$. According to the induction assumption when $(b_2, b_3, ..., b_{n-1})$ is arranged in the alternating order (1.7), the sum $\sum_{i=2}^{n-1} F(b_i, b_{i+1})$, $b_n = b_2$, attains its minimum value which we denote $(\widetilde{b}_2, ..., \widetilde{b}_{n-1})$. Because already $b_1 = a_1$, $b_n = a_n$, $b_2 = \widetilde{b}_2 = a_{n-1}$ and $b_{n-1} = \widetilde{b}_{n-1} = a_2$, we get

$$\begin{aligned}
&\sum_{i=1}^{n} F(b_i, b_{i+1}) \\
&= F(b_1, b_2) + \left[\sum_{i=2}^{n-2} F(b_i, b_{i+1}) + F(b_{n-1}, b_2)\right] - F(b_{n-1}, b_2) + F(b_{n-1}, b_n) + F(b_n, b_1) \\
&= F(a_1, a_{n-1}) + \left[F(a_{n-1}, b_3) + \sum_{i=3}^{n-2} F(b_i, b_{i+1}) + F(a_2, a_{n-1})\right] \\
&\quad - F(a_2, a_{n-1}) + F(a_2, a_n) + F(a_n, a_1) \\
&\geqslant F(a_1, a_{n-1}) + \left[F\left(a_{n-1}, \widetilde{b}_3\right) + \sum_{i=3}^{n-2} F\left(\widetilde{b}_i, \widetilde{b}_{i+1}\right) + F(a_2, a_{n-1})\right] \\
&\quad - F(a_2, a_{n-1}) + F(a_2, a_n) + F(a_n, a_1) \\
&= F(a_1, a_{n-1}) + F\left(a_{n-1}, \widetilde{b}_3\right) + \sum_{i=3}^{n-2} F\left(\widetilde{b}_i, \widetilde{b}_{i+1}\right) + F(a_2, a_n) + F(a_n, a_1) \\
&= \sum_{i=1}^{n} F\left(\widetilde{b}_i, \widetilde{b}_{i+1}\right).
\end{aligned}$$

Indeed, using the induction assumption on $(\widetilde{b}_2, ..., \widetilde{b}_{n-1})$ we obtain $\sum_{i=2}^{n-1} F(b_i, b_{i+1}) \geqslant \sum_{i=2}^{n-1} F\left(\widetilde{b}_i, \widetilde{b}_{i+1}\right)$ as simultaneously both $(\widetilde{b}_2, ..., \widetilde{b}_{n-1})$ and $(\widetilde{b}_1, \widetilde{b}_2, ..., \widetilde{b}_{n-1}, \widetilde{b}_n)$, according to (1.7) and (1.6) respectively, are arranged in alternating order as is emphasized in Remark 2, the minimum of $\sum_{i=1}^{n} F(b_i, b_{i+1})$ and $\sum_{i=2}^{n-1} F(b_i, b_{i+1})$ are obtained simultaneously when $b_1 = a_1$, $b_2 = a_{n-1}$, $b_{n-1} = a_2$ and $b_n = a_n$. The proof of Theorem 1 is complete.

*Proof of Theorem* 2. Case 2(a) is trivial to verify and therefore its proof is omitted.

Case 2(b) that states that (1.1) and (1.2) hold in this case was proved in [1, Theorem1] (see Theorem A).

Case 2(c): In Theorem B it is proved that the maximum of $\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\left|\frac{x_i - x_{i+1}}{2}\right|\right)$, $x_{n+1} = x_1$ is obtained when $(\mathbf{x})$ is arranged in circular symmetrical order. To complete the proof of Case 2(c), we prove now that:

The minimum of $\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\left|\frac{x_i - x_{i+1}}{2}\right|\right)$, $x_{n+1} = x_1$, is obtained when $(\mathbf{x})$ is arranged in alternating order and $f'$ is convex on $\mathbb{R}_+$ and $f'(0) = 0$ for which, according to Theorem 1, we have to show that

$$\frac{\partial F(u, v)}{\partial v} - \frac{\partial F(w, v)}{\partial v} \leqslant 0,$$

which means in our case that

$$\frac{\partial f(u+v)}{\partial v}+\frac{\partial f(|v-u|)}{\partial v}\leqslant\frac{\partial f(w+v)}{\partial v}+\frac{\partial f(|v-w|)}{\partial v}.$$

has to be satisfied. Therefore when $u\leqslant w\leqslant v$ we want to prove that the inequality

$$\frac{\partial f(u+v)}{\partial v}+\frac{\partial f(v-u)}{\partial v}\leqslant\frac{\partial f(w+v)}{\partial v}+\frac{\partial f(v-w)}{\partial v}$$

which is the same as

$$\frac{\partial f(v-u)}{\partial v}-\frac{\partial f(v-w)}{\partial v}\leqslant\frac{\partial f(w+v)}{\partial v}-\frac{\partial f(u+v)}{\partial v}$$

holds. The last inequality is satisfied because $f'$ is convex.

Now we deal with the case that $u\leqslant v\leqslant w$ and we have to show that

$$H_1(u,v):=\frac{\partial f(u+v)}{\partial v}+\frac{\partial f(v-u)}{\partial v}\leqslant\frac{\partial f(w+v)}{\partial v}-\frac{\partial f(w-v)}{\partial v}=:H_2(w,v).$$

It is obvious that when $u=v=w$ there is equality in the last inequality because $f'(0)=0$ which means that $H_1(v,v)=H_2(v,v)=\frac{\partial f(2v)}{\partial v}$. Moreover,

$$\frac{\partial H_1(u,v)}{\partial u}=\frac{\partial^2 f(u+v)}{\partial v\partial u}-\frac{\partial^2 f(v-u)}{\partial v\partial u}=f''(x)_{/x=u+v}-f''(y)_{/y=v-u}\geqslant 0$$

because $f''(x)$ is increasing as a result of the convexity of $f'(x)$ when $x\geqslant 0$. Therefore $H_1(u,v)\leqslant H_1(v,v)$ because $u\leqslant v$. Similarly we get that also $H_2(w,v)$ is increasing with $w$, which leads to $H_2(v,v)\leqslant H_2(w,v)$. Summing up, we get that

$$H_1(u,v)\leqslant H_1(v,v)=H_2(v,v)\leqslant H_2(w,v)$$

holds. The proof of the Case 2(c) is complete.

We prove now Case 2(d): We have to show that

$$\begin{aligned}&\sum_{i=1}^{n}f\left(\frac{\widetilde{x}_i+\widetilde{x}_{i+1}}{2}\right)+C\times\left(\frac{\widetilde{x}_i-\widetilde{x}_{i+1}}{2}\right)^2\\ \leqslant&\sum_{i=1}^{n}f\left(\frac{x_i+x_{i+1}}{2}\right)+C\times\left(\frac{x_i-x_{i+1}}{2}\right)^2\leqslant\sum_{i=1}^{n}f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+C\times\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2,\end{aligned}$$

where $(\widehat{\mathbf{x}})=(\widehat{x}_1,\dots,\widehat{x}_n)$ is the circular symmetrical order of $(\mathbf{x})$ and $(\widetilde{\mathbf{x}})$ is the circular alternating order of $(\mathbf{x})$ when $x_i\geqslant 0$, $i=1,\dots,n$.

In our case, according to Theorem 1, we have to prove that

$$\frac{1}{2}f'\left(\frac{u+v}{2}\right)+C\times\left(\frac{v-u}{2}\right)\leqslant\frac{1}{2}f'\left(\frac{w+v}{2}\right)+C\times\left(\frac{v-w}{2}\right),$$

which is the same as to prove that

$$\frac{1}{2}f'\left(\frac{u+v}{2}\right)-C\frac{u}{2}\leqslant\frac{1}{2}f'\left(\frac{w+v}{2}\right)-C\frac{w}{2}.$$

It is obvious that for $u=v=w$ there is equality in the last inequality.

We show now that the function $H(u,v)=\frac{1}{2}f'\left(\frac{u+v}{2}\right)-C\frac{u}{2}$ is increasing with $u$, that is $\frac{\partial H(u,v)}{\partial u}=\frac{1}{4}f''\left(\frac{u+v}{2}\right)-\frac{C}{2}\geqslant 0$. This follows because $C\leqslant\varphi'\left(\frac{x_m+x_j}{2}\right)$, where $x_m\leqslant x_j\leqslant x_i,\ \ i\neq m,j,\ \ i=1,\ldots,n$, and because $f''(x)=x\varphi''(x)+2\varphi'(x)$ and hence $\frac{\partial H(u,v)}{\partial u}$ can be rewritten as

$$\frac{\partial H(u,v)}{\partial u}=\frac{1}{4}\left(\frac{u+v}{2}\right)\varphi''\left(\frac{u+v}{2}\right)+\frac{2}{4}\varphi'\left(\frac{u+v}{2}\right)-\frac{1}{2}C$$

and because $\varphi'$ is increasing and

$$\varphi'\left(\frac{u+v}{2}\right)\geqslant\varphi'\left(\frac{x_m+x_j}{2}\right)\geqslant C$$

we get that $\frac{\partial H(u,v)}{\partial u}\geqslant 0$ and therefore when $0\leqslant u\leqslant w,v$, we conclude that $H(u,v)\leqslant H(w,v)$ and the proof of Case 2(d) is complete.

We prove finally Case 2(e). We have to show that

$$\begin{aligned}&\sum_{i=1}^{n}\left(f\left(\frac{\widetilde{x}_i+\widetilde{x}_{i+1}}{2}\right)+\varphi'\left(\frac{\widetilde{x}_i+\widetilde{x}_{i+1}}{2}\right)\left(\frac{\widetilde{x}_i-\widetilde{x}_{i+1}}{2}\right)^2\right)\\ \leqslant&\sum_{i=1}^{n}\left(f\left(\frac{x_i+x_{i+1}}{2}\right)+\varphi'\left(\frac{x_i+x_{i+1}}{2}\right)\left(\frac{x_i-x_{i+1}}{2}\right)^2\right)\\ \leqslant&\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+\varphi'\left(\frac{x_i+x_{i+1}}{2}\right)\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\right)\end{aligned}\tag{2.2}$$

holds, where $(\widehat{\mathbf{x}})$ is the circular symmetrical arrangement of $(\mathbf{x})$ and $(\widetilde{\mathbf{x}})$ is the circular and alternating order of $(\mathbf{x})$.

As before we have to prove that under our conditions

$$\frac{\partial F(v,u)}{\partial v}-\frac{\partial F(v,w)}{\partial v}\leqslant 0$$

when $0\leqslant u\leqslant v,w$, which means in our case that $F(v,u)=f\left(\frac{u+v}{2}\right)+\varphi'\left(\frac{u+v}{2}\right)\left(\frac{u-v}{2}\right)^2$ and $f(x)=x\varphi(x)$, $\varphi$ and $\varphi'$ are convex, we must prove that

$$\begin{aligned}&\varphi''(u+v)(v-u)^2+\varphi'(u+v)(3v-u)+\varphi(u+v)\\&-\left[\varphi''(u+v)(v-w)^2+\varphi'(w+v)(3v-w)+\varphi(w+v)\right]\leqslant 0.\end{aligned}$$

We show now that for every $v \geqslant 0$

$$H(u,v) = \varphi''(u+v)(v-u)^2 + \varphi'(u+v)(3v-u) + \varphi(u+v)$$

is increasing with $u$. In other words, we must show that

$$\frac{\partial H(u,v)}{\partial u} = \varphi'''(u+v)(v-u)^2 + \varphi''(u+v)(u+v) \geqslant 0,$$

and this holds because it is given that $\varphi$ and $\varphi'$ are convex. Hence (2.2) is proved. The proof of Theorem 2 is complete.

REMARK 4. The function $f(x) = -x^2$ is an example of functions where Theorem 2 Case 2(b) can be applied.

The functions $f(x) = x^p$ $p \geqslant 2$, $x \geqslant 0$ and $f(x) = x^2 \ln x$, $x \geqslant 0$ are immediate examples of functions where we can apply Theorem 2 Case 2(c).

Theorem 2 Case 2(d) is satisfied for example for the 1-quasiconvex function $f(x) = x^4$, and the 1-quasiconvex function $f(x) = x^2 \ln x$, $x > 0$, $f(0) = 0$, for $x_i > 0$, $i = 1, \ldots, n$.

The assumptions in Theorem 2 Case 2(e) are satisfied e.g., by $f(x) = x^p$, $x \geqslant 0$, $p > 3$.

## 3. Jensen type inequalities and rearrangements

We start with quoting some definitions and lemmas that we need for the proof of the theorems presented in the sequel:

DEFINITION 7. A function $\varphi : [0,B) \to \mathbb{R}$ is superquadratic provided that for all $x \in [0,B)$ there exists a constant $C_\varphi(x) \in \mathbb{R}$ such that the inequality

$$\varphi(y) \geqslant \varphi(x) + C_\varphi(x)(y-x) + \varphi(|y-x|)$$

holds for all $y \in [0,B)$, (see [3, Definition 2.1], but there $[0,\infty)$ instead $[0,B)$).

LEMMA 4. *([3, Inequality 1.2]) The inequality*

$$\int \varphi(f(s))\, d\mu(s) \geqslant \varphi\left(\int f d\mu\right) + \int \varphi\left(\left|f(s) - \int f d\mu\right|\right) d\mu(s)$$

*holds for all probability measures $\mu$ and all non-negative, $\mu-$integrable functions $f$ if and only if $\varphi$ is superquadratic.*

LEMMA 5. *([3, Lemma 2.1]) Let $\varphi$ be a superquadratic function with $C_\varphi(x)$ as in Definition 7.*

*(i) Then $\varphi(0) \leqslant 0$.*

*(ii) If* $\varphi(0) = \varphi'(0) = 0$*, then* $C_\varphi(x) = \varphi'(x)$ *whenever* $\varphi$ *is differentiable on* $[0,B)$.

*(iii) If* $\varphi \geqslant 0$*, then* $\varphi$ *is convex and* $\varphi(0) = \varphi'(0) = 0$.

COROLLARY 2. *Suppose that* $f$ *is superquadratic. Let* $n \in \mathbb{N}$, $0 \leqslant x_i < B$, $i = 1,\dots,n$ *and let* $\overline{x} = \sum_{i=1}^n a_i x_i$*, where* $a_i \geqslant 0$, $i = 1,\dots,n$ *and* $\sum_{i=1}^n a_i = 1$*. Then*

$$\sum_{i=1}^n a_i f(x_i) - f(\overline{x}) \geqslant \sum_{i=1}^n a_i f(|x_i - \overline{x}|). \tag{3.1}$$

*If* $f$ *is non-negative, then it is also convex and the inequality refines Jensen's inequality. In particular, the functions* $f(x) = x^r$, $x \geqslant 0$, $r \geqslant 2$ *are superquadratic and convex, and equality holds in inequality (3.1) when* $r = 2$.

LEMMA 6. *(*[3, Lemma 3.1]*) Suppose* $\varphi : [0,B) \to \mathbb{R}$ *is continuously differentiable and* $\varphi(0) \leqslant 0$*. If* $\varphi'$ *is superadditive or* $\varphi'(x)/x$ *is non-decreasing, then* $\varphi$ *is superquadratic and* $C_\varphi(x) = \varphi'(x)$ *with* $C_\varphi(x)$ *as in Definition 7.*

The following is a definition of the Jensen inequality for strongly convex functions with modulus $C$.

DEFINITION 8. ([6] and [7]) Let $n \in \mathbb{N}$. *The function* $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$ is strongly convex with modulus $C$ if for all $x_i \in I$, and all $a_i \geqslant 0$, $i = 1,\dots,n$ such that $\sum_{i=1}^n a_i = 1$, the inequality

$$\sum_{i=1}^n a_i f(x_i) - f(\overline{x}) \geqslant C \sum_{i=1}^n a_i (x_i - \overline{x})^2,$$

where $C \geqslant 0$ and $\overline{x} = \sum_{i=1}^n a_i x_i$, holds.

DEFINITION 9. [4] Let $N \in \mathbb{N}$. A real-valued function $\psi_N$ defined on an interval $[a,b)$ with $0 \leqslant a < b \leqslant \infty$ is called $N$-quasiconvex if it can be represented as the product of a convex function $\varphi$ and the function $p(x) = x^N$. For $N = 0$, $\psi_0 = \varphi$ and for $N = 1$ the function $\psi_1(x) = x\varphi(x)$ is called $1$-quasiconvex function and for all $n \in \mathbb{N}$ satisfies the inequalities

$$\begin{aligned}\sum_{i=1}^n a_i \psi_1(x_i) &\geqslant \psi_1(\overline{x}) + \varphi'(\overline{x}) \sum_{i=1}^n a_i (x_i - \overline{x})^2 \\ &\geqslant \psi_1(\overline{x}) + C \sum_{i=1}^n a_i (x_i - \overline{x})^2,\end{aligned} \tag{3.2}$$

where $C \leqslant \min \varphi'(x_i)$, $a_i \geqslant 0$, $i = 1,\dots,n$, $\sum_{i=1}^n a_i = 1$ and $\overline{x} = \sum_{i=1}^n a_i x_i$.

In the following theorems we introduce new types of Jensen inequalities which involve rearrangement of $(\mathbf{x}) = (x_1, x_2, \dots, x_n)$. As in the former section, $\widehat{x}_i$ and $\widetilde{x}_i$ are the $i-$th numbers in $(\widehat{\mathbf{x}})$, the circular symmetrical decreasing order and the circular alternating order $(\widetilde{\mathbf{x}})$, respectively.

The new Jensen type inequalities are derived from the cases 2(c), 2(d) and 2(e) of Theorem 2.

We start with a Jensen type inequality for superquadratic functions. This is a result of Theorem B together with Theorem 2 Case 2(c), when $F(x,y) = f\left(\frac{x+y}{2}\right) + f\left(\left|\frac{x-y}{2}\right|\right)$, and it shows the use of rearrangements to get a refinement of Jensen type inequality:

THEOREM 3. *Let $x_i$, $i = 1,\dots,n$, $n \in \mathbb{N}$ be a sequence of real non-negative numbers and let $f$ be differentiable on $\mathbb{R}_+$, $f'$ is convex on $\mathbb{R}_+$, $f'(0) = 0$ and $f(0) \leqslant 0$. Then, for the superquadratic function $f$, the following Jensen type inequalities hold:*

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i) - (n-1) f\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) - (n-1)\sum_{i=1}^{n}\frac{1}{n} f\left(\left|x_i - \frac{\sum_{j=1}^{n} x_j}{n}\right|\right) \\
&\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i + \widehat{x}_{i+1}}{2}\right) + f\left(\left|\frac{\widehat{x}_i - \widehat{x}_{i+1}}{2}\right|\right)\right) \\
&\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\left|\frac{x_i - x_{i+1}}{2}\right|\right)\right) \\
&\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widetilde{x}_i + \widetilde{x}_{i+1}}{2}\right) + f\left(\left|\frac{\widetilde{x}_i - \widetilde{x}_{i+1}}{2}\right|\right)\right)
\end{aligned} \tag{3.3}$$

*and*

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i) - n f\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) - \sum_{i=1}^{n} f\left(\left|x_i - \frac{\sum_{j=1}^{n} x_j}{n}\right|\right) \\
&\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i + \widehat{x}_{i+1}}{2}\right) + f\left(\left|\frac{\widehat{x}_i - \widehat{x}_{i+1}}{2}\right|\right)\right) \\
&\quad - \left(f\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) + \sum_{i=1}^{n}\frac{1}{n} f\left(\left|x_i - \frac{\sum_{j=1}^{n} x_j}{n}\right|\right)\right). \\
&\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\left|\frac{x_i - x_{i+1}}{2}\right|\right)\right) \\
&\quad - \left(f\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) + \sum_{i=1}^{n}\frac{1}{n} f\left(\left|x_i - \frac{\sum_{j=1}^{n} x_j}{n}\right|\right)\right) \\
&\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widetilde{x}_i + \widetilde{x}_{i+1}}{2}\right) + f\left(\left|\frac{\widetilde{x}_i - \widetilde{x}_{i+1}}{2}\right|\right)\right) \\
&\quad - \left(f\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) + \sum_{i=1}^{n}\frac{1}{n} f\left(\left|x_i - \frac{\sum_{j=1}^{n} x_j}{n}\right|\right)\right).
\end{aligned} \tag{3.4}$$

*If, in addition*

$$\frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i + \widehat{x}_{i+1}}{2}\right) + f\left(\left|\frac{\widehat{x}_i - \widehat{x}_{i+1}}{2}\right|\right)\right) \tag{3.5}$$

$$\geqslant \left( f\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) + \sum_{i=1}^{n} \frac{1}{n} f\left(\left|x_i - \frac{\sum_{j=1}^{n} x_j}{n}\right|\right)\right)$$

*then (3.4) refines (3.1) when* $a_i = \frac{1}{n}$, $i = 1, \ldots, n$.

*Proof.* According to (3.1), a superquadratic function $f$ satisfies

$$\sum_{i=1}^{n} f(x_i) - nf\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) - \sum_{i=1}^{n} f\left(\left|x_i - \frac{\sum_{j=1}^{n} x_j}{n}\right|\right) \geqslant 0. \tag{3.6}$$

Inequality (3.6) is not affected by any rearrangement of $(\mathbf{x})$.
Moreover, from (3.1) it follows that

$$\begin{aligned}\sum_{i=1}^{n} f(x_i) &= \sum_{i=1}^{n} \frac{f(x_i) + f(x_{i+1})}{2} \\ &\geqslant \sum_{i=1}^{n} \left( f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\left|\frac{x_i - x_{i+1}}{2}\right|\right)\right)\end{aligned} \tag{3.7}$$

holds.

Therefore, by using (3.6) and (3.7) we get the inequalities

$$\begin{aligned}&\sum_{i=1}^{n} f(x_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} f(x_i) + (n-1)\sum_{i=1}^{n} \frac{f(x_i)}{n} \\ &\geqslant \frac{1}{n}\sum_{i=1}^{n} \left( f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\left|\frac{x_i - x_{i+1}}{2}\right|\right)\right) + (n-1)\sum_{i=1}^{n} \frac{f(x_i)}{n} \\ &\geqslant \frac{1}{n}\sum_{i=1}^{n} \left( f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\left|\frac{x_i - x_{i+1}}{2}\right|\right)\right) + (n-1) f\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) \\ &\quad + (n-1)\sum_{i=1}^{n} \frac{1}{n} f\left(\left|x_i - \sum_{j=1}^{n} \frac{x_j}{n}\right|\right).\end{aligned} \tag{3.8}$$

By combining the inequalities (3.8) together with Theorem 2 Case 2(c), regarding rearrangement we find that (3.3) holds.

If (3.5) holds it is clear that (3.4) refines (3.1) for $a_i = \frac{1}{n}$, $i = 1, \ldots, n$. The proof is complete.

EXAMPLE 1. When $n = 3$, $x_i = i$, $i = 1, 2, 3$, and $f(x) = x^4$, (3.4) is a refinement of (3.6). In this specific case (3.4) reads:

$$\sum_{i=1}^{3} x_i^4 - 3\left(\frac{\sum_{j=1}^{3} x_j}{3}\right)^4 - \sum_{i=1}^{3}\left(\left|x_i - \frac{\sum_{j=1}^{3} x_j}{3}\right|\right)^4$$

$$
\begin{aligned}
&> \frac{1}{3}\sum_{i=1}^{3}\left(\left(\frac{x_i+x_{i+1}}{2}\right)^4+\left(\left|\frac{x_i-x_{i+1}}{2}\right|\right)^4\right)\\
&-\left(\left(\frac{\sum_{j=1}^{3}x_j}{3}\right)^4+\sum_{i=1}^{3}\frac{1}{3}\left(\left|x_i-\frac{\sum_{j=1}^{3}x_j}{3}\right|\right)^4\right)>0,
\end{aligned}
$$

whereas from the basic inequality (3.1) satisfied by superquadratic functions we get the weaker result

$$
\sum_{i=1}^{3}x_i^4-3\left(\frac{\sum_{j=1}^{3}x_j}{3}\right)^4-\sum_{i=1}^{3}\left(\left|x_i-\frac{\sum_{j=1}^{3}x_j}{3}\right|\right)^4\geqslant 0,
$$

and from the convexity of $f(x)=x^4$ we get only that:

$$
\sum_{i=1}^{3}x_i^4-3\left(\frac{\sum_{j=1}^{3}x_j}{3}\right)^4>0.
$$

The three inequalities show that we get in this case a refinement of Jensen type inequality for the superquadratic and the convex function $f(x)=x^4$.

The following theorem shows the use of rearrangements for refinements of Jensen type inequality for $1$-quasiconvex functions by using Case 2(d) of Theorem 2.

THEOREM 4. *Let $x_i,\ i=1,\dots,n,\ n\in\mathbb{N}$ be a sequence of real non-negative numbers and let $\varphi$ be convex on $x\geqslant 0$ and $f(x)=x\varphi(x)$. Let $C\leqslant \min\varphi'(x_i),\ i=1,\dots,n$. Then the $1$-quasiconvex function $f$ (which is strongly convex when $C\geqslant 0$) satisfies*

$$
\begin{aligned}
\sum_{i=1}^{n}f(x_i) &= \frac{1}{2}\sum_{i=1}^{n}\left(f(x_i)+f(x_{i+1})\right) \qquad (3.9)\\
&\geqslant \sum_{i=1}^{n}f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+\varphi'\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\\
&\geqslant \sum_{i=1}^{n}f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+C\times\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\\
&\geqslant \sum_{i=1}^{n}f\left(\frac{x_i+x_{i+1}}{2}\right)+C\times\left(\frac{x_i-x_{i+1}}{2}\right)^2\\
&\geqslant \sum_{i=1}^{n}f\left(\frac{\widetilde{x}_i+\widetilde{x}_{i+1}}{2}\right)+C\times\left(\frac{\widetilde{x}_i-\widetilde{x}_{i+1}}{2}\right)^2,
\end{aligned}
$$

*and*

$$
\sum_{i=1}^{n}f(x_i)-(n-1)f\left(\frac{\sum_{j=1}^{n}x_j}{n}\right)-\frac{n-1}{n}\varphi'\left(\frac{\sum_{j=1}^{n}x_j}{n}\right)\sum_{i=1}^{n}\left(x_i-\frac{\sum_{j=1}^{n}x_j}{n}\right)^2 \qquad (3.10)
$$

$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+C\times\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\right)$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_i+x_{i+1}}{2}\right)+C\times\left(\frac{x_i-x_{i+1}}{2}\right)^2\right)$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widetilde{x}_i+\widetilde{x}_{i+1}}{2}\right)+C\times\left(\frac{\widetilde{x}_i-\widetilde{x}_{i+1}}{2}\right)^2\right).$$

*If, in addition,*

$$\frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+C\times\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\right) - \left(f\left(\frac{\sum_{j=1}^{n}x_j}{n}\right)+C\sum_{i=1}^{n}\frac{1}{n}\left(x_i-\frac{\sum_{j=1}^{n}x_j}{n}\right)^2\right)\geqslant 0 \tag{3.11}$$

*then*

$$\sum_{i=1}^{n}f(x_i)-nf\left(\frac{\sum_{j=1}^{n}x_j}{n}\right)-C\times\sum_{i=1}^{n}\left(x_i-\frac{\sum_{j=1}^{n}x_j}{n}\right)^2 \geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+C\times\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\right) - \left(f\left(\frac{\sum_{j=1}^{n}x_j}{n}\right)+C\sum_{i=1}^{n}\frac{1}{n}\left(x_i-\frac{\sum_{j=1}^{n}x_j}{n}\right)^2\right). \tag{3.12}$$

*Proof.* Inequality (3.9) is an immediate consequence of (3.2) for $n=2$

$$\frac{f(x_i)+f(x_{i+1})}{2} \geqslant f\left(\frac{x_i+x_{i+1}}{2}\right)+\varphi'\left(\frac{x_i+x_{i+1}}{2}\right)\left(\frac{x_i-x_{i+1}}{2}\right)^2\geqslant f\left(\frac{x_i+x_{i+1}}{2}\right)+C\times\left(\frac{x_i-x_{i+1}}{2}\right)^2 \tag{3.13}$$

together with the inequalities in Theorem 2, Case 2(d), concerning rearrangement. We prove now (3.10):

$$\sum_{i=1}^{n}f(x_i) = \frac{1}{n}\sum_{i=1}^{n}f(x_i)+\frac{n-1}{n}\sum_{i=1}^{n}f(x_i) \geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_i+x_{i+1}}{2}\right)+\varphi'\left(\frac{x_i+x_{i+1}}{2}\right)\left(\frac{x_i-x_{i+1}}{2}\right)^2\right)+\frac{n-1}{n}\sum_{i=1}^{n}f(x_i) \tag{3.14}$$

$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_i+x_{i+1}}{2}\right)+C\times\left(\frac{x_i-x_{i+1}}{2}\right)^2\right)+\frac{n-1}{n}\sum_{i=1}^{n}f\left(x_i\right)$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_i+x_{i+1}}{2}\right)+C\times\left(\frac{x_i-x_{i+1}}{2}\right)^2\right)+(n-1)f\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)$$
$$+(n-1)\varphi'\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)\sum_{i=1}^{n}\frac{1}{n}\left(x_i-\frac{\sum_{j=1}^{n}x_j}{n}\right)^2.$$

Indeed, the first two inequalities in (3.14) follow from (3.13). The third inequality follows from (3.2). From the first and the last lines in (3.14) we get

$$\sum_{i=1}^{n}f\left(x_i\right)-(n-1)f\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)-\frac{(n-1)}{n}\varphi'\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)\sum_{i=1}^{n}\left(x_i-\frac{\sum_{j=1}^{n}x_j}{n}\right)^2$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_i+x_{i+1}}{2}\right)+C\times\left(\frac{x_i-x_{i+1}}{2}\right)^2\right).$$

Therefore, also

$$\sum_{i=1}^{n}f\left(\widehat{x}_i\right)-(n-1)f\left(\frac{\sum_{i=1}^{n}\widehat{x}_i}{n}\right)-\frac{(n-1)}{n}\varphi'\left(\frac{\sum_{i=1}^{n}\widehat{x}_i}{n}\right)\sum_{i=1}^{n}\left(\widehat{x}_i-\frac{\sum_{j=1}^{n}\widehat{x}_j}{n}\right)^2 \quad (3.15)$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+C\times\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\right).$$

Since $\sum_{i=1}^{n}f\left(x_i\right)-(n-1)f\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)-\frac{n-1}{n}\varphi'\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)\sum_{i=1}^{n}\left(x_i-\frac{\sum_{j=1}^{n}x_j}{n}\right)^2$ is not dependent on the arrangement of $(\mathbf{x})$ we get from (3.15) that

$$\sum_{i=1}^{n}f\left(x_i\right)-(n-1)f\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)-\frac{n-1}{n}\varphi'\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)\sum_{i=1}^{n}\left(x_i-\frac{\sum_{j=1}^{n}x_j}{n}\right)^2 \quad (3.16)$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+C\times\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\right)$$

holds. Inequality (3.10) is derived from Inequality (3.16) and Theorem 2 Case 2(d) (which implies that the two last inequalities in (3.10) hold). Inequality (3.12) is a rewriting of (3.10). If (3.11) holds then it is obvious that (3.12) is a refinement of (3.2) for $a_i=\frac{1}{n}$, $i=1,\dots,n$. The proof is complete.

We finish by stating a result which follows from Theorem 2, Case 2(e). Its proof is omitted because it is similar to the proofs of Theorems 3 and 4.

THEOREM 5. *Let $x_i$, $i=1,\dots,n$, $n\in\mathbb{N}$ be a sequence of real non-negative numbers, and let $\varphi$ and $\varphi'$ be convex on $\mathbb{R}_+$ and $f(x)=x\varphi(x)$, $x\in\mathbb{R}_+$. Then denoting $\overline{x}=\frac{1}{n}\sum_{j=1}^{n}x_j$, it yields that*

$$\sum_{i=1}^{n}f\left(x_i\right)-(n-1)f\left(\overline{x}\right)-(n-1)\varphi'\left(\overline{x}\right)\sum_{i=1}^{n}\frac{1}{n}\left(x_i-\overline{x}\right)^2$$

$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+\varphi'\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\right)$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_i+x_{i+1}}{2}\right)+\varphi'\left(\frac{x_i+x_{i+1}}{2}\right)\left(\frac{x_i-x_{i+1}}{2}\right)^2\right)$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widetilde{x}_i+\widetilde{x}_{i+1}}{2}\right)+\varphi'\left(\frac{\widetilde{x}_i+\widetilde{x}_{i+1}}{2}\right)\left(\frac{\widetilde{x}_i-\widetilde{x}_{i+1}}{2}\right)^2\right).$$

*If also*

$$\frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+\varphi'\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\right)\geqslant f(\overline{x})+\frac{1}{n}\varphi'(\overline{x})\sum_{i=1}^{n}(x_i-\overline{x})^2,$$

*then*

$$\sum_{i=1}^{n}f(x_i)-nf(\overline{x})-\varphi'(\overline{x})\sum_{i=1}^{n}(x_i-\overline{x})^2$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+\varphi'\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)\left(\frac{\widehat{x}_i-\widehat{x}_{i+1}}{2}\right)^2\right)-f(\overline{x})-\varphi'(\overline{x})\sum_{i=1}^{n}\frac{1}{n}(x_i-\overline{x})^2\geqslant 0.$$

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