Journal of
Mathematical
Inequalities
Volume 14, Number 3 (2020), 673–684 doi:10.7153/jmi-2020-14-43

SHARP WILKER AND HUYGENS TYPE INEQUALITIES FOR TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

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(*Communicated by E. Neuman*)

Abstract. In this paper, we prove Wilker and Huygens type inequalities for inverse trigonometric functions. This solves two conjectures proposed by Chao-Ping Chen. Also, we present new sharp Wilker and Huygens type inequalities for trigonometric functions.

1. Introduction

Wilker in [23] proposed two open problems:

(a) Prove that if $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1.1}$$

(b) Find the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x$$

for $0 < x < \pi/2$.

In [22], inequality (1.1) was proved, and the following inequality

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^3 \tan x \qquad \text{for } 0 < x < \frac{\pi}{2}, \tag{1.2}$$

where the constants $\left(\frac{2}{\pi}\right)^4$ and $\frac{8}{45}$ are the best possible, was also established.

Wilker type inequalities (1.1) and (1.2) have attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs and various generalizations and improvements (cf. [6, 7, 8, 9, 13, 14, 18, 15, 16, 17, 19, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] and the references cited therein). A brief survey

Mathematics subject classification (2010): 26D05.
Keywords and phrases: Inequalities, trigonometric functions.
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of some old and new inequalities associated with trigonometric functions can be found in [21]. These include (among other results) Wilker's inequality.

Another inequality which is of interest to us is Huygens inequality [10], which asserts that

$$2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}>3 \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}. \tag{1.3}$$

Neuman and Sándor [18] have pointed out that (1.3) implies (1.1).

It is known in the literature that

$$(\cos x)^{1/3}<\frac{\sin x}{x}<\frac{2+\cos x}{3} \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}. \tag{1.4}$$

The left-hand side inequality (1.4) first appeared in [12, p. 238]. The left-hand side of (1.4) can be rewritten as

$$\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}>1 \quad \left(\text{or} \quad \sqrt[3]{\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}}>1\right) \quad \text{for all} \quad 0<|x|<\frac{\pi}{2}. \tag{1.5}$$

Baricz and Sándor [4] have pointed out that inequality (1.5) implies (1.1) and (1.3), by using the arithmetic-geometric mean inequality.

The right-hand side inequality (1.4) was first mentioned by the German philosopher and theologian Nicolaus de Cusa (1401-1464), by a geometrical method. A rigorous proof of the right-hand side inequality (1.4) was given by Huygens [10], who used the right-hand side of (1.4) to estimate the number π. The right-hand side inequality (1.4) is now known as Cusa's inequality [13, 18, 32, 20]. Further interesting historical facts about Cusa's inequality can be found in [20].

Wu and Srivastava [26, Lemma 3] established another inequality

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2 \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}. \tag{1.6}$$

Neuman and Sándor [18, Theorem 2.3] proved that for $0<|x|<\pi/2$,

$$\frac{\sin x}{x}<\frac{2+\cos x}{3}<\frac{1}{2}\left(\frac{x}{\sin x}+\cos x\right). \tag{1.7}$$

By multiplying both sides of inequality (1.7) with $x/\sin x$, we obtain that for $0<|x|<\pi/2$,

$$\frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right]>\frac{2(x/\sin x)+x/\tan x}{3}>1. \tag{1.8}$$

The second inequality in (1.8) is equivalent to the second inequality in (1.4).

Chen and Sándor [8] proved the following inequality chain:

$$\begin{aligned}
&\frac{(\sin x/x)^2+\tan x/x}{2}>\left(\frac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right)>\frac{2\,(\sin x/x)+\tan x/x}{3}\\
&>\left(\frac{\sin x}{x}\right)^{2/3}\left(\frac{\tan x}{x}\right)^{1/3}>\frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right]>\frac{2(x/\sin x)+x/\tan x}{3}>1
\end{aligned} \tag{1.9}$$

for $0 < |x| < \pi/2$.

In 2012, Chen [5] proved that for $0 < x < 1$,

$$2 + \frac{17}{45}x^3 \arctan x < \left(\frac{\arcsin x}{x}\right)^2 + \frac{\arctan x}{x} \tag{1.10}$$

and

$$3 + \frac{7}{20}x^3 \arctan x < 2\left(\frac{\arcsin x}{x}\right) + \frac{\arctan x}{x}, \tag{1.11}$$

where the constants $\frac{17}{45}$ and $\frac{7}{20}$ are the best possible.

Also in [5], Chen proposed the following two conjectures.

CONJECTURE 1.1. For $0 < x < 1$, we have

$$\left(\frac{\arcsin x}{x}\right)^2 + \frac{\arctan x}{x} < 2 + \frac{\pi^2 + \pi - 8}{\pi}x^3 \arctan x. \tag{1.12}$$

The constant $\frac{\pi^2+\pi-8}{\pi}$ is the best possible.

CONJECTURE 1.2. For $0 < x < 1$, we have

$$2\left(\frac{\arcsin x}{x}\right) + \frac{\arctan x}{x} < 3 + \frac{5\pi - 12}{\pi}x^3 \arctan x. \tag{1.13}$$

The constant $\frac{5\pi-12}{\pi}$ is the best possible.

Conjectures 1.1 and 1.2 were proved in 2017 by Malešević et al. [11]. The proofs of these authors are rather complex. In this paper, we provide a simple proof of inequalities (1.10)-(1.13) (Theorems 2.1 and 2.2). Also, we present new sharp Wilker and Huygens type inequalities for trigonometric functions. More precisely, we prove the following inequality chain:

$$\begin{aligned} 2 + \frac{1}{9}x^3 \tan x < \frac{x}{\tan x} + \frac{\tan x}{x} &< 2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x \\ &< \left(\frac{2 + \cos x}{3}\right)^2 + \frac{\tan x}{x} < 2 + \frac{17}{90}x^3 \tan x \end{aligned}$$

for $0 < x < \pi/2$, where the constants $\frac{1}{9}$, $\left(\frac{2}{\pi}\right)^4$ and $\frac{17}{90}$ are the best possible (Theorems 2.3 and 2.4).

The following lemma will be useful in our present investigation.

LEMMA 1.1. (see [1,2,3]) *Let $-\infty < a < b < \infty$, and let f, $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b). Let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \text{ and } [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

The numerical values given have been calculated using the computer program MAPLE 13.

2. Results

THEOREM 2.1. *For* $0<x<1$,

$$2+\frac{17}{45}x^3\arctan x<\left(\frac{\arcsin x}{x}\right)^2+\frac{\arctan x}{x}<2+\frac{\pi^2+\pi-8}{\pi}x^3\arctan x, \tag{2.1}$$

where the constants $\frac{17}{45}$ *and* $\frac{\pi^2+\pi-8}{\pi}$ *are the best possible.*

Proof. For $0<x<1$, let

$$f(x)=\frac{\left(\frac{\arcsin x}{x}\right)^2+\frac{\arctan x}{x}-2}{x^3\arctan x}.$$

Elementary calculations reveal that

$$\lim_{x\to 0^+}f(x)=\frac{17}{45}=0.377\ldots \quad \text{and} \quad \lim_{x\to 1^-}f(x)=\frac{\pi^2+\pi-8}{\pi}=1.595\ldots.$$

In order to prove Theorem 2.1, it suffices to show that $f(x)$ is strictly increasing on $(0,1)$.

For $0\leqslant x<1$, let

$$f_1(x)=\frac{\left(\frac{\arcsin x}{x}\right)^2+\frac{\arctan x}{x}-2}{x^3}, \quad f_1(0)=\lim_{x\to 0^+}f_1(x)=0$$

and

$$f_2(x)=\arctan x.$$

Then, we have, for $0<x<1$,

$$f(x)=\frac{f_1(x)}{f_2(x)}=\frac{\frac{\left(\frac{\arcsin x}{x}\right)^2+\frac{\arctan x}{x}-2}{x^3}}{\arctan x}.$$

Elementary calculation reveals that

$$\frac{f_1'(x)}{f_2'(x)}=\frac{1+x^2}{x^3}\left\{\frac{2\arcsin x}{x^2\sqrt{1-x^2}}-\frac{5}{x}\left(\frac{\arcsin x}{x}\right)^2-\frac{4}{x}\left(\frac{\arctan x}{x}\right)+\frac{1}{x(1+x^2)}+\frac{6}{x}\right\}.$$

It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\arcsin x}{x}\right)^2=\frac{2\arcsin x}{x^2\sqrt{1-x^2}}-\frac{2}{x}\left(\frac{\arcsin x}{x}\right)^2.$$

Using the expansion [5, Lemma 2]

$$(\arcsin x)^2=\sum_{n=0}^{\infty}\frac{2^{2n+1}\cdot(n!)^2}{(2n+2)!}x^{2n+2}, \qquad 0<|x|<1, \tag{2.2}$$

we find

$$\frac{2\arcsin x}{x^2\sqrt{1-x^2}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\arcsin x}{x}\right)^2 + \frac{2}{x}\left(\frac{\arcsin x}{x}\right)^2 = \sum_{n=0}^{\infty}\frac{2^{2n+2}(n+1)\cdot(n!)^2}{(2n+2)!}x^{2n-1}.$$

We then obtain

$$\begin{aligned}\frac{f_1'(x)}{f_2'(x)} &= \frac{1+x^2}{x^3}\left\{\sum_{n=0}^{\infty}\frac{2^{2n+2}(n+1)\cdot(n!)^2}{(2n+2)!}x^{2n-1} - \sum_{n=0}^{\infty}\frac{5\cdot 2^{2n+1}\cdot(n!)^2}{(2n+2)!}x^{2n-1}\right.\\ &\qquad\left. - \sum_{n=0}^{\infty}\frac{(-1)^n 4}{2n+1}x^{2n-1} + \sum_{n=0}^{\infty}(-1)^n x^{2n-1} + \frac{6}{x}\right\}\\ &= \frac{17}{45} + 8\sum_{n=3}^{\infty}a_n x^{2n-4}\\ &= \frac{17}{45} + \frac{92}{315}x^2 + \frac{92}{105}x^4 + \frac{7864}{10395}x^6 + \frac{175064}{189189}x^8 + \dots,\end{aligned}$$

where

$$a_n = \frac{2^{2n-3}(8n^3-10n^2-13n-5)\Gamma(n)^2}{\Gamma(2n+3)} + \frac{(-1)^n}{4n^2-1}.$$

By induction on n, it is easy to show that for $n \geqslant 3$,

$$\frac{\Gamma(n)^2}{\Gamma(2n+3)} > \frac{1}{2^{2n-3}(8n^3-10n^2-13n-5)(4n^2-1)}$$

(we here omit the proof), which yields $a_n > 0$ for $n \geqslant 3$. We then obtain

$$\left(\frac{f_1'(x)}{f_2'(x)}\right)' > 0 \quad \text{for} \quad 0 < x < 1.$$

Therefore, the functions $f_1'(x)/f_2'(x)$ is strictly increasing on $(0,1)$. By Lemma 1.1, the function

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x)-f_1(0)}{f_2(x)-f_2(0)}$$

is strictly increasing on $(0,1)$. The proof of Theorem 2.1 is complete.

THEOREM 2.2. *For* $0 < x < 1$,

$$3 + \frac{7}{20}x^3\arctan x < 2\left(\frac{\arcsin x}{x}\right) + \frac{\arctan x}{x} < 3 + \frac{5\pi-12}{\pi}x^3\arctan x, \tag{2.3}$$

where the constants $\frac{7}{20}$ *and* $\frac{5\pi-12}{\pi}$ *are the best possible.*

Proof. For $0<x<1$, let

$$F(x)=\frac{2\left(\frac{\arcsin x}{x}\right)+\frac{\arctan x}{x}-3}{x^3\arctan x}.$$

Elementary calculations reveal that

$$\lim_{x\to0^+}F(x)=\frac{7}{20}=0.35 \quad \text{and} \quad \lim_{x\to1^-}F(x)=\frac{5\pi-12}{\pi}=1.18028\ldots.$$

In order to prove Theorem 2.2, it suffices to show that $F(x)$ is strictly increasing on $(0,1)$.

For $0\leqslant x<1$, let

$$F_1(x)=\frac{2\left(\frac{\arcsin x}{x}\right)+\frac{\arctan x}{x}-3}{x^3},\quad F_1(0)=\lim_{x\to0^+}F_1(x)=0$$

and

$$F_2(x)=\arctan x.$$

Then, we have, for $0<x<1$,

$$F(x)=\frac{F_1(x)}{F_2(x)}=\frac{\frac{2\left(\frac{\arcsin x}{x}\right)+\frac{\arctan x}{x}-3}{x^3}}{\arctan x}.$$

Elementary calculations reveal that

$$\begin{aligned}
\frac{F_1'(x)}{F_2'(x)}&=\frac{1+x^2}{x^4}\left\{\frac{2}{\sqrt{1-x^2}}+\frac{1}{1+x^2}-8\left(\frac{\arcsin x}{x}\right)-4\left(\frac{\arctan x}{x}\right)+9\right\}\\
&=\frac{1+x^2}{x^4}\left\{\sum_{n=0}^{\infty}\frac{2\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}x^{2n}+\sum_{n=0}^{\infty}(-1)^n x^{2n}\right.\\
&\qquad\left.-\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n-3}(n!)^2(2n+1)}x^{2n}-\sum_{n=0}^{\infty}\frac{(-1)^n 4}{2n+1}x^{2n}+9\right\}\\
&=\frac{1+x^2}{x^4}\sum_{n=2}^{\infty}\left\{\frac{2(2n-3)\Gamma(n+\frac{1}{2})}{(2n+1)\sqrt{\pi}\Gamma(n+1)}+(-1)^n\frac{2n-3}{2n+1}\right\}x^{2n}\\
&=\frac{7}{20}+\sum_{n=2}^{\infty}\left\{\frac{(16n^3+12n^2-20n-19)\Gamma(n+\frac{1}{2})}{(n+1)\sqrt{\pi}\Gamma(n+1)}-(-1)^n 8\right\}\frac{x^{2n-2}}{(2n+1)(2n+3)}\\
&=\frac{7}{20}+\frac{53}{280}x^2+\frac{313}{448}x^4+\frac{755}{1408}x^6+\frac{49897}{73216}x^8+\ldots.
\end{aligned}$$

By induction on n, it is easy to show that for $n\geqslant2$,

$$\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}>\frac{8(n+1)}{16n^3+12n^2-20n-19}$$

(we here omit the proof), which yields

$$\frac{(16n^3+12n^2-20n-19)\Gamma(n+\frac{1}{2})}{(n+1)\sqrt{\pi}\Gamma(n+1)}-(-1)^n 8>0 \quad \text{for} \quad n \geqslant 2.$$

We then obtain

$$\left(\frac{F_1'(x)}{F_2'(x)}\right)'>0 \quad \text{for} \quad 0<x<1.$$

Therefore, the functions $F_1'(x)/F_2'(x)$ is strictly increasing on $(0,1)$. By Lemma 1.1, the function

$$F(x)=\frac{F_1(x)}{F_2(x)}=\frac{F_1(x)-F_1(0)}{F_2(x)-F_2(0)}$$

is strictly increasing on $(0,1)$. The proof of Theorem 2.2 is complete.

THEOREM 2.3. *For* $0<x<\pi/2$,

$$2+\frac{1}{9}x^3\tan x<\frac{x}{\tan x}+\frac{\tan x}{x}<2+\left(\frac{2}{\pi}\right)^4 x^3\tan x, \tag{2.4}$$

where the constants $\frac{1}{9}$ *and* $\left(\frac{2}{\pi}\right)^4$ *are the best possible.*

Proof. For $0<x<\pi/2$, let

$$g(x)=\frac{\frac{x}{\tan x}+\frac{\tan x}{x}-2}{x^3\tan x}.$$

Elementary calculations give that

$$\lim_{x\to 0^+} g(x)=\frac{1}{9}=0.111111\ldots \quad \text{and} \quad \lim_{x\to\pi/2^-} g(x)=\left(\frac{2}{\pi}\right)^4=0.164255\ldots.$$

In order to Theorem 2.3, it suffices to show that $g(x)$ is strictly increasing for $0<x<\pi/2$.

Differentiating $g(x)$ and using the power series expansions of $\sin x$ and $\cos x$, we obtain

$$\begin{aligned}&\frac{x^5\sin^3 x}{2}g'(x)\\ =&(x^2-2)\sin^3 x-3x\cos^3 x+(3x-x^3)\cos x\\ =&(x^2-2)\left(\frac{3\sin x-\sin(3x)}{4}\right)-3x\left(\frac{3\cos x+\cos(3x)}{4}\right)+(3x-x^3)\cos x\\ =&\frac{2}{135}x^9-\frac{17}{4725}x^{11}+\frac{23}{56700}x^{13}-\frac{3713}{130977000}x^{15}+\sum_{n=8}^{\infty}(-1)^n u_n(x),\end{aligned} \tag{2.5}$$

where

$$u_n(x) = \frac{(4n^2-16n+9)9^n+96n^3-36n^2-24n-9}{12\cdot(2n+1)!}x^{2n+1}.$$

Elementary calculations show that for $0<x<\pi/2$ and $n\geqslant 8$,

$$\begin{aligned}\frac{u_{n+1}(x)}{u_n(x)} &= \frac{3x^2}{2}\frac{(12n^2-24n-9)9^n+9+64n+84n^2+32n^3}{(1+n)(2n+3)\Big((4n^2-16n+9)9^n+96n^3-36n^2-24n-9\Big)}\\ &< \frac{3(\pi/2)^2}{2}\frac{(12n^2-24n-9)9^n+9+64n+84n^2+32n^3}{(1+n)(2n+3)(4n^2-16n+9)9^n}\\ &< \frac{9}{2}\frac{12n^2-24n-9+x_n}{(1+n)(2n+3)(4n^2-16n+9)},\end{aligned}$$

where

$$x_n = \frac{9+64n+84n^2+32n^3}{9^n}.$$

Noting that the sequence $\{x_n\}$ is strictly decreasing for $n\geqslant 8$, we have

$$x_n \leqslant x_8 = \frac{7427}{14348907}.$$

We then obtain

$$\frac{u_{n+1}(x)}{u_n(x)} < \frac{9}{2}\frac{12n^2-24n-9+\frac{7427}{14348907}}{(1+n)(2n+3)(4n^2-16n+9)} < 1.$$

Hence, for every $x\in(0,\pi/2)$, the sequence $n\longmapsto u_n(x)$ is strictly decreasing for $n\geqslant 8$. Therefore, we obtain from (2.5) that

$$\frac{x^5\sin^3 x}{2}g'(x) > x^9\left(\frac{2}{135}-\frac{17}{4725}x^2+\frac{23}{56700}x^4-\frac{3713}{130977000}x^6\right) > 0.$$

Hence, the function $g(x)$ is strictly increasing for $x\in(0,\pi/2)$. The proof of Theorem 2.3 is complete.

THEOREM 2.4. *For* $0<x<\pi/2$,

$$2+\left(\frac{2}{\pi}\right)^4 x^3\tan x < \left(\frac{2+\cos x}{3}\right)^2+\frac{\tan x}{x} < 2+\frac{17}{90}x^3\tan x, \tag{2.6}$$

where the constants $\left(\frac{2}{\pi}\right)^4$ *and* $\frac{17}{90}$ *are the best possible.*

Proof. For $0<x<\pi/2$, let

$$h(x)=\frac{\left(\frac{2+\cos x}{3}\right)^2+\frac{\tan x}{x}-2}{x^3\tan x}.$$

Elementary calculations give that

$$\lim_{x\to 0^+}h(x)=\frac{17}{90}=0.1888888\ldots \quad\text{and}\quad \lim_{x\to\pi/2^-}h(x)=\left(\frac{2}{\pi}\right)^4=0.164255\ldots.$$

In order to Theorem 2.4, it suffices to show that $h(x)$ is strictly decreasing for $0<x<\pi/2$.

Differentiating $h(x)$ and using the power series expansions of $\sin x$ and $\cos x$, we obtain

$$\begin{aligned}
&-9x^5\sin^2 x h'(x)\\
=&3x\sin x\cos^3 x+12x\sin x\cos^2 x-42x\sin x\cos x-2x^2\cos^4 x-4x^2\cos^3 x\\
&+3x^2\cos^2 x+8x^2\cos x+36\sin^2 x-14x^2\\
=&-\frac{53}{4}x^2+18+3x\sin x-\frac{81}{4}x\sin(2x)+3x\sin(3x)+\frac{3}{8}x\sin(4x)\\
&+5x^2\cos x-18\cos(2x)+\frac{1}{2}x^2\cos(2x)-x^2\cos(3x)-\frac{1}{4}x^2\cos(4x)\\
=&\frac{11}{42}x^8-\frac{1961}{12600}x^{10}+\frac{60769}{1663200}x^{12}-\frac{115223}{22702680}x^{14}+\sum_{n=8}^{\infty}(-1)^n v_n(x),
\end{aligned}\tag{2.7}$$

where

$$v_n(x)=\frac{(2n^2-7n)16^n-(16n^2-656n+576)4^n+\left(\frac{128}{9}n^2-\frac{640}{9}n\right)9^n-640n^2+128n}{32\cdot(2n)!}x^{2n}.$$

Elementary calculations show that for $0<x<\pi/2$ and $n\geqslant 8$,

$$\frac{v_{n+1}(x)}{v_n(x)}=\frac{72x^2}{(2n+1)(n+1)}\frac{P(n)}{Q(n)}<\frac{72(\pi/2)^2}{(2n+1)(n+1)}\frac{P(n)}{Q(n)}<\frac{178P(n)}{(2n+1)(n+1)Q(n)},\tag{2.8}$$

where

$$\begin{aligned}P(n)=&(2n^2-3n-5)16^n+(8n^2-24n-32)9^n\\ &-(4n^2-156n-16)4^n-40n^2-72n-32\end{aligned}$$

and

$$\begin{aligned}Q(n)=&(18n^2-63n)16^n+(128n^2-640n)9^{n-1}\\ &-(144n^2-5904n+5184)4^n-5760n^2+1152n.\end{aligned}$$

We claim that for $n \geqslant 8$,

$$R(n) > 0, \tag{2.9}$$

where

$$\begin{aligned} R(n) &= (2n+1)(n+1)Q(n) - 178P(n) \\ &= (36n^4 - 72n^3 - 527n^2 + 471n + 890)16^n - (1296n^2 - 3632n - 5696)9^n \\ &\quad + \left(3200 + 22440(n-39) + 568(n-39)^2\right)4^n + 1360n^2 + 13968n + 5696. \end{aligned}$$

By direct computation, we find that (2.9) holds for $n = 8, 5, \ldots, 39$. We prove now (2.9) for $n \geqslant 40$, it suffices to show that

$$\left(\frac{16}{9}\right)^n > \frac{1296n^2 - 3632n - 5696}{36n^4 - 72n^3 - 527n^2 + 471n + 890}, \qquad n \geqslant 40. \tag{2.10}$$

For $n = 40$, elementary calculations show that

$$\begin{aligned} &\left[\left(\frac{16}{9}\right)^n - \frac{1296n^2 - 3632n - 5696}{36n^4 - 72n^3 - 527n^2 + 471n + 890}\right]_{n=40} \\ &= \frac{21125648099669692144278649784718245365463382958383913376}{2136540416021163806782770363146770002914542755} > 0, \end{aligned}$$

which shows that the inequality (2.10) holds true for $n = 40$.

We assume now that the inequality (2.10) holds true for a fixed positive integer $n \geqslant 40$, we try to obtain it for $n+1$. By inductive assumption, we have

$$\begin{aligned} &\left(\frac{16}{9}\right)^{n+1} - \frac{1296(n+1)^2 - 3632(n+1) - 5696}{36(n+1)^4 - 72(n+1)^3 - 527(n+1)^2 + 471(n+1) + 890} \\ &> \left(\frac{16}{9}\right) \frac{1296n^2 - 3632n - 5696}{36n^4 - 72n^3 - 527n^2 + 471n + 890} \\ &\quad - \frac{1296(n+1)^2 - 3632(n+1) - 5696}{36(n+1)^4 - 72(n+1)^3 - 527(n+1)^2 + 471(n+1) + 890} = \frac{16S(n)}{9T(n)}, \end{aligned}$$

where

$$\begin{aligned} S(n) &= 85639613888892 + 12836987226692(n-40) + 800704481739(n-40)^2 \\ &\quad + 26601591562(n-40)^3 + 496464759(n-40)^4 + 4934988(n-40)^5 \\ &\quad + 20412(n-40)^6 \end{aligned}$$

and

$$\begin{aligned} T(n) = {} & 8317213816424940 + 1672218300452028(n-40) \\ & + 146979332535359(n-40)^2 + 7376539810632(n-40)^3 \\ & + 231207222769(n-40)^4 + 4634506656(n-40)^5 \\ & + 58017672(n-40)^6 + 414720(n-40)^7 + 1296(n-40)^8. \end{aligned}$$

We then obtain

$$\left(\frac{16}{9}\right)^{n+1} > \frac{1296(n+1)^2-3632(n+1)-5696}{36(n+1)^4-72(n+1)^3-527(n+1)^2+471(n+1)+890}.$$

The proof of the inequality (2.10) is thus completed by means of the principle of mathematical induction on n.

Hence, the claim (2.9) holds for $n \geqslant 8$. We then obtain from (2.8) that

$$\frac{v_{n+1}(x)}{v_n(x)} < 1$$

for $0 < x < \pi/2$ and $n \geqslant 8$. Hence, for every $x \in (0,\pi/2)$, the sequence $n \longmapsto v_n(x)$ is strictly decreasing for $n \geqslant 8$. Therefore, we obtain from (2.7) that

$$-9x^5\sin^2 xh'(x) > x^8\left(\frac{11}{42}-\frac{1961}{12600}x^2+\frac{60769}{1663200}x^4-\frac{115223}{22702680}x^6\right) > 0$$

for $0 < x < \pi/2$, which implies $h'(x) < 0$ for $0 < x < \pi/2$. Hence, the function $h(x)$ is strictly decreasing on $(0,\pi/2)$. The proof of Theorem 2.4 is complete.

REFERENCES

[1] G. D. ANDERSON, S.-L. QIU, M. K. VAMANAMURTHY AND M. VUORINEN, *Generalized elliptic integral and modular equations*, Pacific J. Math. **192** (2000), 1–37.

[2] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, New York, 1997.

[3] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Monotonicity of Some Functions in Calculus*, https://www.math.auckland.ac.nz/Research/Reports/Series/538.pdf.

[4] A. BARICZ AND J. SÁNDOR, *Extensions of generalized Wilker inequality to Bessel functions*, J. Math. Inequal. **2** (2008), 397–406.

[5] C.-P. CHEN, *Sharp Wilker- and Huygens-type inequalities for inverse trigonometric and inverse hyperbolic functions*, Integral Transforms Spec. Funct. **23**, 12 (2012), 865–873.

[6] C. P. CHEN AND W.S. CHEUNG, *Sharpness of Wilker and Huygens type inequalities*, J. Inequal. Appl. **2012**, 72, http://www.journalofinequalitiesandapplications.com/content/2012/1/72.

[7] C. P. CHEN AND W. S. CHEUNG, *Inequalities and solution to Oppenheim's problem*, Integral Transforms Spec. Funct. **23**, 5 (2012), 325–336.

[8] C. P. CHEN AND J. SÁNDOR, *Inequality chains for Wilker, Huygens and Lazarević type inequalities*, J. Math. Inequal. **8**, 1 (2014), 55–67.

[9] B. N. GUO, B. M. QIAO, F. QI AND W. LI, *On new proofs of Wilker inequalities involving trigonometric functions*, Math. Inequal. Appl. **6** (2003), 19–22.

[10] C. HUYGENS, *Oeuvres Completes 1888-1940*, Société Hollondaise des Science, Haga.

[11] B. J. MALEŠEVIĆ, B. BANJAC AND I. JOVOVIĆ, *A proof of two conjectures of Chao-Ping Chen for inverse trigonometric functions*, J. Math. Inequal. **11**, 1 (2017), 151–162.

[12] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[13] C. MORTICI, *The natural approach of Wilker-Cusa-Huygens inequalities*, Math. Inequal. Appl. **14** (2011), 535–541.

[14] C. MORTICI, *A subtly analysis of Wilker inequality*, Appl. Math. Comput. **231** (2014), 516–520.

[15] E. NEUMAN, *One- and two-sided inequalities for Jacobian elliptic functions and related results*, Integral Transforms Spec. Funct. **21** (2010), 399–407.

[16] E. NEUMAN, *On Wilker and Huygens type inequalities*, Math. Inequal. Appl. **15** (2012), 271–279.

[17] E. NEUMAN, *Wilker and Huygens-type inequalities for the generalized trigonometric and for the generalized hyperbolic functions*, Appl. Math. Comput. **230** (2014), 211–217.
[18] E. NEUMAN AND J. SÁNDOR, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities*, Math. Inequal. Appl. **13** (2010), 715–723.
[19] I. PINELIS, *L'Hospital rules of monotonicity and Wilker-Anglesio inequality*, Amer. Math. Monthly, **111** (2004), 905–909.
[20] J. SÁNDOR AND M. BENCZE, *On Huygens' trigonometric inequality*, RGMIA Res. Rep. Collection **8**, 3 (2005), Article 14.
[21] R. SRIVASTAVA, *Some families of integral, trigonometric and other related inequalities*, Appl. Math. Inform. Sci. **5** (2011), 342–360.
[22] J. S. SUMNER, A. A. JAGERS, M. VOWE AND J. ANGLESIO, *Inequalities involving trigonometric functions*, Amer. Math. Monthly **98** (1991), 264–267.
[23] J. B. WILKER, *Problem E 3306*, Amer. Math. Monthly **96** (1989), 55.
[24] S. H. WU, *On extension and refinement of Wilker's inequality*, Rocky Mountain J. Math. **39** (2009), 683–687.
[25] S. H. WU AND A. BARICZ, *Generalizations of Mitrinović, Adamović and Lazarevic's inequalities and their applications*, Publ. Math. Debrecen **75** (2009), 447–458.
[26] S. H. WU AND H. M. SRIVASTAVA, *A weighted and exponential generalization of Wilker's inequality and its applications*, Integral Transforms and Spec. Funct. **18** (2007), 529–535.
[27] S. H. WU AND H. M. SRIVASTAVA, *A further refinement of Wilker's inequality*, Integral Transforms and Spec. Funct. **19** (2008), 757–765.
[28] L. ZHANG AND L. ZHU, *A new elementary proof of Wilker's inequalities*, Math. Inequal. Appl. **11** (2008), 149–151.
[29] L. ZHU, *A new simple proof of Wilker's inequality*, Math. Inequal. Appl. **8** (2005), 749–750.
[30] L. ZHU, *On Wilker-type inequalities*, Math. Inequal. Appl. **10** (2007), 727–731.
[31] L. ZHU, *Some new Wilker-type inequalities for circular and hyperbolic functions*, Abstr. Appl. Anal. **2009**, Article ID 485842.
[32] L. ZHU, *A source of inequalities for circular functions*, Comput. Math. Appl. **58** (2009), 1998–2004.
[33] L. ZHU, *Inequalities for Hyperbolic functions and their Applications*, J. Inequal. Appl. **2010**, Article ID 130821.

(Received August 4, 2019)

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