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BOUNDEDNESS FOR A CLASS OF FRACTIONAL CARLESON TYPE MAXIMAL OPERATOR

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Abstract. In this paper, the authors study the fractional Carleson type maximal operators $\mathscr{T}_\beta^*$ which is defined by

$$\mathscr{T}_\beta^* f(x) = \sup_\lambda \left| \int_{\mathbb{R}^n} e^{iP_\lambda(y)} \frac{\Omega(y)}{|y|^{n-\beta}} f(x-y)dy \right|,$$

where $0<\beta<n$ and Ω satisfies the L^q-Dini conditions with $1<q<\infty$. The authors prove the $L^p \to L^p$ boundedness of $\mathscr{T}_\beta^*$ under certain conditions.

1. Introduction

In 1966, Carleson [2] studied the following Carleson type maximal operator $\mathscr{C}^*$ as

$$\mathscr{C}^* f(x) = \sup_{\lambda\in\mathbb{R}} \left| \int_{-\pi}^{\pi} \frac{e^{-i\lambda t} f(t)}{x-t} dt \right|, \tag{1.1}$$

where $f \in L^2([-\pi,\pi])$ and $x \in [-\pi,\pi]$. Carleson [2] proved the almost everywhere convergence of the Fourier series of the functions in $L^2([-\pi,\pi])$ by using the weak type (2,2) of $\mathscr{C}^*$. Later, Hunt [9] improved Carleson's results to $L^p([-\pi,\pi])$ with $1<p<\infty$.

In 1970, Sjölin [12] studied another type of following Carleson type operator $\mathscr{J}^*$ on $\mathbb{R}^n$, that is

$$\mathscr{J}^*(f)(x) = \sup_{\lambda\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-i\lambda\cdot y} K(x-y) f(y) dy \right|, \tag{1.2}$$

where $\lambda = (\lambda_1,\cdots,\lambda_n) \in R^n$ and K is an appropriate Calderón-Zygmund kernel. Sjölin [12] proved the following theorem.

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THEOREM A. *([12]) If K satisfies the following conditions:*

1. $K(tx) = t^{-n}K(x), \quad \text{for} \quad t > 0;$
2. $\displaystyle\int_{\mathbb{S}^{n-1}} K(x')d\sigma(x') = 0;$
3. $K \in C^{n+1}(\mathbb{R}^n \setminus \{0\}).$

Then $\|\mathscr{J}^*(f)\|_{L^p} \leqslant C_p\|f\|_{L^p}$ *for* $1 < p < \infty$.

In 2001, Stein and Wainger [13] extended Theorem A to a broader context. That is, the authors in [13] replace the linear phase $\lambda \cdot y$ in the definition of $\mathscr{J}^*$ by a more general phase with a fixed degree. Now, let us state the main results of [13].

Define

$$T_\lambda(f)(x) = \int_{\mathbb{R}^n} e^{iP_\lambda(y)}K(y)f(x-y)dy,$$

where $P_\lambda(x) = \sum_{2\leqslant|\alpha|\leqslant d} \lambda_\alpha x^\alpha$ is the polynomial in $\mathbb{R}^n$ with real coefficients $\lambda := (\lambda_\alpha)_{1\leqslant|\alpha|\leqslant d}$.

Then, the definition of the Carleson type maximal operator $\mathscr{T}^*$ is

$$\mathscr{T}^* f(x) = \sup_\lambda |T_\lambda(f)(x)|, \tag{1.1}$$

where the supremum is taken over all the real coefficients λ of P_λ. Stein and Wainger proved the following result.

THEOREM B. *([13]) Suppose that $P_\lambda(x) = \sum_{2\leqslant|\alpha|\leqslant d} \lambda_\alpha x^\alpha$ and K satisfies the following conditions:*

1. *K is a tempered distribution and agrees with a C^1 function $K(x)$ for $x \neq 0$;*
2. $\widehat{K} \in L^\infty$;
3. $|\partial_x^\gamma K(x)| \leqslant A|x|^{-n-|\gamma|}$ *for* $0 \leqslant |\gamma| \leqslant 1$.

Then $\|\mathscr{T}^*(f)\|_{L^p} \leqslant C_p\|f\|_{L^p}$ *for* $1 < p < \infty$.

Obviously, Theorem B is a essential extension of Theorem A. Recently, Ding and Liu [5] gave a weighted variant version of Theorem B under weak conditions. Before giving the main results of [5], we introduce some definitions.

Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ $(n \geqslant 2)$, equipped with the usual Lebesgue measure $d\sigma$. Suppose that Ω is a homogeneous of degree zero and measurable function on $\mathbb{R}^n \setminus \{0\}$. Furthermore, we assume that Ω satisfies the following conditions:

$$\Omega \in L^1(\mathbb{S}^{n-1}), \quad \int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{1.3}$$

DEFINITION 1.1. ([1]) Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $1 \leqslant q \leqslant \infty$. Then a function Ω is said to satisfy the L^q-Dini condition if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty,$$

where $\omega_q(\delta)\,(0<\delta\leqslant 1)$ is called the integral continuous modulus of Ω of degree q, which is defined by

$$\omega_q(\delta)=\sup_{\|\rho\|<\delta}\left(\int_{\mathbb{S}^{n-1}}\left|\Omega(\rho x')-\Omega(x')\right|^q d\sigma(x')\right)^{1/q}\quad \text{for}\quad 1\leqslant q<\infty$$

and

$$\omega_\infty(\delta)=\sup_{\|\rho\|<\delta}|\Omega(\rho x')-\Omega(x')|,$$

where ρ is a rotation in $\mathbb{R}^n$ and $\|\rho\|=\sup\left\{|\rho x'-x'|:x'\in\mathbb{S}^{n-1}\right\}$.
Then the Carleson type maximal operator with a rough kernel on $\mathbb{R}^n$ studied by Ding and Liu in [5] can be written as

$$\mathscr{T}^*f(x):=\sup_\lambda|T_\lambda f(x)|=\sup_\lambda\left|\int_{\mathbb{R}^n}e^{iP_\lambda(y)}K(y)f(x-y)dy\right|,$$

where $K(y)=\frac{\Omega(y)}{|y|^n}$. In [5], Ding and Liu proved the following theorem.

THEOREM C. ([5]) *Suppose that* $P_\lambda(x)=\sum_{2\leqslant|\alpha|\leqslant d}\lambda_\alpha x^\alpha$ *and* $K(x)=\Omega(x)|x|^{-n}$, *where* Ω *satisfies (1.3). If* Ω *satisfies the* L^q*-Dini condition for some* $1<q\leqslant\infty$, *then for* $1\leqslant q'<p<\infty$ *and* $w\in A_{p/q'}$, *the Carleson type maximal operator* $\mathscr{T}^*$ *is a bounded operator on the weighted space* $L^p(w)$. *That is, there exists a constant* $C>0$ *such that for all* $f\in L^p(w)$

$$\left\|\mathscr{T}^*f\right\|_{L^p(w)}\leqslant C\|f\|_{L^p(w)}, \tag{1.4}$$

where $A_{p/q'}$ *denotes the classical Muckenhoupt class (see* [8] *or* [10]*).*

By the way, we would like to point out that Ding and Liu [4] also proved that if $\Omega\in H^1(\mathbb{S}^{n-1})$, then $\mathscr{T}^*$ is bounded on L^p for $1<p<\infty$. Here $H^1(\mathbb{S}^{n-1})$ denotes the Hardy space on the unit sphere $\mathbb{S}^{n-1}$ and one may see [3] for more details. Noting the following fact

$$C^1(\mathbb{S}^{n-1})\subset \mathrm{Lip}_1(\mathbb{S}^{n-1})\subset L^q(\mathbb{S}^{n-1})\,(1<q\leqslant\infty)\subset H^1(\mathbb{S}^{n-1})\subset L^1(\mathbb{S}^{n-1}),$$

we find that Ding and Liu's results in [4, 5] are improvements of the main results of [13].

On the other hand, the fractional integral was also studied a lot by many authors. Especially in [6], Ding and Lu studied the fractional integral with a rough kernel defined by

$$T_{\Omega,\beta}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\beta}}f(y)dy,$$

where $0<\beta<n$ and $\Omega\in L^s(\mathbb{S}^{n-1})$. Ding and Lu [6] proved the following results.

THEOREM D. *([6]) Let $0<\beta<n, s'<p<n/\alpha$ and $1/q=1/p-\beta/n$. If $\Omega\in L^s(\mathbb{S}^{n-1})$ and $\omega(x)^{s'}\in A(p/s',q/s')$, then there exists a constant C independent of f, such that*

$$\left(\int_{\mathbb{R}^n}[T_{\Omega,\beta}f(x)\omega(x)]^q dx\right)^{1/q}\leqslant C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p dx\right)^{1/p}$$

where $A(p/s',q/s')$ denotes the fractional type Muckenhoupt-Wheeden class (see [11]).

In this paper, we will study the following fractional Carleson type maximal operators $\mathscr{T}_\beta^*$ with the following definition,

$$\mathscr{T}_\beta^* f(x):=\sup_\lambda\left|T_{\lambda,\beta}f(x)\right|=\sup_\lambda\left|\int_{\mathbb{R}^n}e^{iP_\lambda(y)}K_\beta(y)f(x-y)dy\right|, \tag{1.5}$$

where $K_\beta(y)=\frac{\Omega(y)}{|y|^{n-\beta}}$ with $0<\beta<n$ and $\Omega\in L^q(\mathbb{S}^{n-1})$ for some $q>1$.

Furthermore, for any $c>0$ and vector $(\lambda_\alpha)_{2\leqslant|\alpha|\leqslant d}$, the set Λ is defined by

$$\Lambda=\left\{\lambda=(\lambda_\alpha)_{2\leqslant|\alpha|\leqslant d}:|\lambda|=\sum_{2\leqslant|\alpha|\leqslant d}|\lambda_\alpha|\geqslant c\right\}.$$

Our results can be stated as follows.

THEOREM 1.2. *Suppose that $P_\lambda(x)=\sum_{2\leqslant|\alpha|\leqslant d}\lambda_\alpha x^\alpha$ with $\lambda\in\Lambda$ and Ω satisfies the L^q-Dini condition for some $1<q\leqslant\infty$. Then for $q'<p<\infty$ and $0<\beta<\frac{\delta\gamma(p)}{n-1+\gamma(p)}$ with $\gamma(p)=\min\{1/p,1/p'\}$ and $\delta=\frac{1}{6d}$, we have*

$$\|\mathscr{T}_\beta^* f\|_{L^p}\leqslant C\|f\|_{L^p}, \tag{1.6}$$

where the constant C is dependent on $1/c$ and α but independent of f.

REMARK 1.3. By a simple computation, we have $\mathscr{T}_\beta^*(f)(x)\leqslant T_{|\Omega|,\beta}(|f|)(x)$ with $0<\beta<n$. Thus by Theorem D, we can easily get the $L^p_{\omega^p}\to L^q_{\omega^q}$ boundedness of $\mathscr{T}_\beta^*$ under the conditions of Theorem D.

2. Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2. Some basic ideas and techniques of this proof comes from [4, 5].

First, we will introduce a variant version of the Hardy-Littlewood maximal function which will be very useful in the proof of Theorem 1.2 (See [13]).

Let $B_3=\{x\in\mathbb{R}^n:\ |x|\leqslant 3\}$. For a measurable set $E\subset B_3$, χ_E denotes the characteristic function of E. Then for any $\varepsilon>0$, the maximal operator $\mathscr{M}_\varepsilon$ is defined by

$$\mathscr{M}_\varepsilon(f)(x)=\sup_{\substack{E\subset B_3,|E|\leqslant\varepsilon\\ a>0}}|f|*(\chi_E)_a(x),$$

where $(\chi_E)_a(x)=a^{-n}\chi_E(x/a)$.

LEMMA 2.1. *([13]) For $1 < p < \infty$, there exists a constant $C > 0$, independent of ε, such that*

$$\|\mathscr{M}_\varepsilon(f)\|_{L^p} \leqslant C\varepsilon^{1-1/p}\|f\|_{L^p}.$$

Proof of Theorem 1.2. As $\lambda \in \Lambda$, we have $|\lambda| = \sum_{2\leqslant|\alpha|\leqslant d} |\lambda_\alpha| \geqslant c$. Thus, it suffices to consider the case for $\mathscr{T}_\beta^* f(x) := \sup_{\lambda>0} \left|T_{\lambda,\beta} f(x)\right|$.

Let $\lambda(x)$ be the nonzero vector $\left(\lambda_\alpha(x)\right)_{2\leqslant|\alpha|\leqslant d}$ satisfying

$$\left|T_{\lambda(x),\beta}(f)(x)\right| \geqslant \frac{1}{2}\sup_\lambda \left|T_{\lambda,\beta}(f)(x)\right|,$$

where $f \in L^p$ and $x \in \mathbb{R}^n$. Thus by the above estimates, to prove Theorem 1.2, it suffices to show that there exists a constant C, such that

$$\|T_{\lambda(\cdot),\beta}(f)\|_{L^p} \leqslant C\|f\|_{L^p},$$

where the constant C is independent of the choice of $\lambda(\cdot)$.

For any $x \in \mathbb{R}^n$, we define $N(\lambda(x)) = \sum_{2\leqslant|\alpha|\leqslant d} |\lambda_\alpha(x)|^{\frac{1}{|\alpha|}}$. By the definition of $\lambda(\cdot)$ and the condition $\lambda \in \Lambda$, we have $\sum_{2\leqslant|\alpha|\leqslant d} |\lambda_\alpha(x)| \geqslant c$. As $\frac{1}{|\alpha|} < 1$, we obtain

$$N(\lambda(x)) = \sum_{2\leqslant|\alpha|\leqslant d} |\lambda_\alpha(x)|^{\frac{1}{|\alpha|}} \geqslant \left(\sum_{2\leqslant|\alpha|\leqslant d} |\lambda_\alpha(x)|\right)^{\frac{1}{|\alpha|}} \geqslant C_\alpha, \tag{2.1}$$

where the constant C_α is only dependent on c and α but independent of x.

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function with $\mathrm{supp}(\psi) \subseteq \{\frac{1}{4} < |y| \leqslant 1\}$. Furthermore, we assume that ψ satisfies

$$\sum_{j=-\infty}^{\infty} \psi_j(y) = 1 \quad \text{for} \quad y \neq 0,$$

where $\psi_j(y) = \psi(2^{-j}y)$. Then, it is easy to get that

$$\sum_{j=-\infty}^{\infty} \psi_j\big(N(\lambda(x))y\big) = 1 \quad \text{for} \quad y \neq 0.$$

Now we may decompose K_β as

$$K_\beta(y) = \sum_{j=0}^{\infty} K_{\beta,j}(y),$$

where $K_{\beta,0}(y) = \sum_{j=-\infty}^{0} \psi_{j,\lambda}(y) K_\beta(y)$, $K_{\beta,j}(y) = \psi_{j,\lambda}(y) K_\beta(y)$ and $\psi_{j,\lambda}(y) = \psi_j\big(N(\lambda(x))y\big)$ for $j \geqslant 1$ and $x \in \mathbb{R}^n$. Thus, we have

$$\begin{aligned} \left|T_{\lambda(x),\beta}(f)(x)\right| &\leqslant \left|\int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} K_{\beta,0}(y) f(x-y)dy\right| \\ &\quad + \sum_{j=1}^{\infty} \left|\int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} K_{\beta,j}(y) f(x-y)dy\right| \\ &:= T^0_{\lambda(x),\beta}(f)(x) + \sum_{j=1}^{\infty} T^j_{\lambda(x),\beta}(f)(x). \end{aligned} \tag{2.2}$$

The estimates of $T^0_{\lambda(\cdot),\beta}$

By the fact $\operatorname{supp}(K_{\beta,0}) \subset \{|y| \leqslant \frac{1}{N(\lambda(x))}\}$ and (2.1), we have

$$\begin{aligned} \left|T^0_{\lambda(x),\beta} f(x)\right| &= \left|\int_{\mathbb{R}^n} e^{iP_\lambda(y)} K_{\beta,0} f(x-y)dy\right| \\ &= \left|\sum_{j=-\infty}^{0} \int_{\mathbb{R}^n} e^{iP_\lambda(y)} K_\beta(y) \psi_{j,\lambda}(y) f(x-y)dy\right| \\ &\leqslant \sum_{j=-\infty}^{0} \int_{\frac{2^{j-2}}{N(\lambda(x))} \leqslant |y| \leqslant \frac{2^j}{N(\lambda(x))}} \frac{|\Omega(y)|}{|y|^{n-\beta}} |f(x-y)|dy \\ &\leqslant C \sum_{j=-\infty}^{0} \left(\frac{2^{j-2}}{N(\lambda(x))}\right)^{\beta-n} \int_{|y| \leqslant \frac{2^j}{N(\lambda(x))}} |\Omega(y) f(x-y)|dy \\ &\leqslant C M_\Omega f(x), \end{aligned}$$

where M_Ω is the usual Hardy-Littlewood maximal function with a rough kernel(see [7]). Thus, by the L^p boundedness of M_Ω (see [7]), we have

$$\|T^0_{\lambda(\cdot),\beta} f\|_{L^p} \leqslant C \|M_\Omega f\|_{L^p} \leqslant C \|f\|_{L^p}. \tag{2.3}$$

The estimates of $T^j_{\lambda(\cdot),\beta}$

We use some basic ideas from [5]. Choose a nonnegative function $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp}(\phi) \subseteq \{|y| \leqslant 2^{-6}\}$ and $\|\phi\|_{L^1} = 1$. For any $a > 0$, we denote $\phi_a(x) = a^{-n}\phi(x/a)$.

For a positive real number σ which satisfies $\beta < \sigma$ and will be chosen later, we may denote

$$L_{j,\lambda(x),\beta}(y) = K_{\beta,j} * \phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(y) \quad \text{and} \quad R_{j,\lambda(x),\beta}(y) = K_{\beta,j}(y) - L_{j,\lambda(x),\beta}(y), j \in \mathbb{N}.$$

Thus, it is easy to see

$$\left|T^j_{\lambda(x),\beta}(f)(x)\right| \leqslant \left|\mathscr{L}^j_{\lambda(x),\beta}(f)(x)\right| + \left|\mathscr{R}^j_{\lambda(x),\beta}(f)(x)\right|,$$

where $\mathscr{L}^j_{\lambda(\cdot),\beta}$ and $\mathscr{R}^j_{\lambda(\cdot),\beta}$ are defined by

$$\mathscr{L}^j_{\lambda(x),\beta}(f)(x)=\int_{\mathbb{R}^n}e^{iP_{\lambda(x)}(y)}L_{j,\lambda(x),\beta}(y)f(x-y)dy$$

and

$$\mathscr{R}^j_{\lambda(x),\beta}(f)(x)=\int_{\mathbb{R}^n}e^{iP_{\lambda(x)}(y)}R_{j,\lambda(x),\beta}(y)f(x-y)dy.$$

Next, we will give the estimates of $\mathscr{L}^j_{\lambda(\cdot),\beta}$ and $\mathscr{R}^j_{\lambda(\cdot),\beta}$ respectively.

The estimates of $\mathscr{L}^j_{\lambda(\cdot),\beta}$

First, we give the estimate of $L_{j,\lambda(x),\beta}$. By the definition of $L_{j,\lambda(x),\beta}$, we have

$$\text{supp }(L_{j,\lambda(\cdot),\beta})\subseteq\left\{\tfrac{2^{j-3}}{N(\lambda(x))}\leqslant|y|\leqslant\tfrac{2^{j+1}}{N(\lambda(x))}\right\}. \tag{2.4}$$

Define

$$L_{j,\beta}(y)=\int_{\mathbb{R}^n}\psi(y-z)K_\beta(y-z)2^{jn\sigma}\phi(2^{j\sigma}z)dz.$$

Then, it is easy to see that

$$L_{j,\lambda(x),\beta}\left(\tfrac{2^jy}{N(\lambda(x))}\right)=\left(\tfrac{2^j}{N(\lambda(x))}\right)^{\beta-n}L_{j,\beta}(y). \tag{2.5}$$

By the definition of $L_{j,\lambda(\cdot),\beta}$ and the Hölder inequality , we have the following estimates for $L_{j,\lambda(\cdot),\beta}$.

$$\begin{aligned}
\left|L_{j,\lambda(x),\beta}(y)\right|&=(2^{-j}N(\lambda(x)))^n2^{jn\sigma}\left|\int_{\mathbb{R}^n}K_{\beta,j}(y)\phi\left(\frac{|x-y|}{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}\right)dy\right|\\
&\leqslant(2^{-j}N(\lambda(x)))^n2^{jn\sigma}\int_{\frac14\leqslant|2^{-j}N(\lambda(x))y|\leqslant1}|\psi_{j,\lambda}(y)|\frac{|\Omega(y)|}{|y|^{n-\beta}}dy\\
&\leqslant(2^{-j}N(\lambda(x)))^n2^{jn\sigma}\left(\int_{\frac14\leqslant|2^{-j}N(\lambda(x))y|\leqslant1}|\Omega(y)|^qdy\right)^{1/q}\left(\int_{\frac14\leqslant|2^{-j}N(\lambda(x))y|\leqslant1}|y|^{q'(\beta-n)}dy\right)^{1/q'}\\
&\leqslant C(2^{-j}N(\lambda(x)))^n2^{jn\sigma}\left(\frac{2^j}{N(\lambda(x))}\right)^{n/q}\left(\frac{2^j}{N(\lambda(x))}\right)^{\beta-n}\left(\frac{2^j}{N(\lambda(x))}\right)^{n/q'}\\
&=C(2^{-j}N(\lambda(x)))^n2^{jn\sigma}\frac{2^{j\beta}}{(N(\lambda(x)))^\beta}.
\end{aligned}$$

From (2.1), we get

$$|L_{j,\lambda(x),\beta}(y)|\leqslant C(2^{-j}N(\lambda(x)))^n2^{jn\sigma}\tfrac{2^{j\beta}}{(N(\lambda(x)))^\beta}\leqslant C(2^{-j}N(\lambda(x)))^n2^{jn\sigma}2^{j\beta}, \tag{2.6}$$

which implies

$$\left|\mathscr{L}^{j}_{\lambda(x),\beta}f(x)\right| \leqslant C\int_{\frac{2^{j-1}}{N(\lambda)}\leqslant|y|\leqslant\frac{2^{j+1}}{N(\lambda)}}(2^{-j}N(\lambda(x)))^{n}2^{jn\sigma}2^{j\beta}|f(x-y)|dy \leqslant C2^{jn\sigma+j\beta}Mf(x). \tag{2.7}$$

Next, we adopt some notations from [5]. Recall that $\lambda(x)=\big(\lambda_\alpha(x)\big)_{2\leqslant|\alpha|\leqslant d}$. For any $j\in\mathbb{Z}^+$, we denote

$$A_{j,\lambda}\circ\lambda = \left(\left(\frac{2^j}{N(\lambda(x))}\right)^{|\alpha|}\lambda_\alpha(x)\right)_{2\leqslant|\alpha|\leqslant d}$$

for convenience. From [5, p.2744], there is

$$\begin{aligned} P_{\lambda(x)}(y) &= \sum_{2\leqslant|\alpha|\leqslant d}\lambda_\alpha(x)y^\alpha = \sum_{2\leqslant|\alpha|\leqslant d}\lambda_\alpha(x)\left(\frac{2^j}{N(\lambda(x))}\right)^{|\alpha|}\left(2^{-j}N(\lambda(x))y\right)^{\alpha} \\ &= P_{A_{j,\lambda}\circ\lambda}\left(2^{-j}N(\lambda(x))y\right). \end{aligned} \tag{2.8}$$

Thus, by (2.5), we obtain

$$\mathscr{L}^{j}_{\lambda(x),\beta}(f)(x) = \int_{\mathbb{R}^n} e^{iP_{A_{j,\lambda}\circ\lambda}\left(2^{-j}N(\lambda(x))y\right)}\left(2^{-j}N(\lambda(x))\right)^{n-\beta}L_{j,\beta}\left(2^{-j}N(\lambda(x))y\right)f(x-y)dy.$$

From now on, we denote $\mathscr{L}^{j}_{\lambda(x),\beta}(f)(x)$ by $\mathscr{L}_{j,\beta}(f)(x)$ for simplicity.

From [5, p.2745], we know that there exists a constant $c_0>0$ such that $N(v)\leqslant c_0|v|$ for any vector v satisfying $N(v)\geqslant 1$. Moreover, there is

$$\left|A_{j,\lambda}\circ\lambda\right| \geqslant 2^j/c_0 \quad \text{for all} \quad \lambda(x),\ x\in\mathbb{R}^n. \tag{2.9}$$

For $r\geqslant 2^j/c_0$, we define

$$U_{j,r} = \left\{x : r\leqslant|A_{j,\lambda}\circ\lambda|<2r\right\}$$

and

$$\mathscr{L}_{j,r,\beta}(f)(x) = \mathscr{L}_{j,\beta}(f)(x)\chi_{U_{j,r}}(x).$$

Let $\mathscr{L}^*_{j,r,\beta}$ be the adjoint operator of $\mathscr{L}_{j,r,\beta}$. Then, it is easy to check that

$$\mathscr{L}^*_{j,r,\beta}(g)(y) = \int_{\mathbb{R}^n} e^{-iP_{\lambda(x)}(x-y)}L_{j,\lambda(x),\beta}(x-y)g(x)\chi_{U_{j,r}}(x)dx$$

and

$$\left(\mathscr{L}_{j,r,\beta}\mathscr{L}^*_{j,r,\beta}\right)(f)(x) = \int_{\mathbb{R}^n}\mathscr{K}_\beta(x,z)f(z)dz,$$

where

$$\mathscr{K}_\beta(x,z) = \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)}e^{-iP_{\lambda(z)}(z-x+y)}L_{j,\lambda(x),\beta}(y)L_{j,\lambda(z),\beta}(z-x+y)dy\chi_{U_{j,r}}(x)\chi_{U_{j,r}}(z)$$

$$= \left(e^{iP_{\lambda(x)}(\cdot)}L_{j,\lambda(x),\beta}(\cdot)\right) * \left(e^{-iP_{\lambda(z)}(-\cdot)}L_{j,\lambda(z),\beta}(-\cdot)\right)(x-z)\chi_{U_{j,r}}(x)\chi_{U_{j,r}}(z).$$

Following [5], we are going to prove that for $r \geqslant 2^j/c_0$ and fixed $x,z \in U_{j,r}$, the following inequality holds.

$$\begin{aligned}\left|\mathscr{K}_\beta(x,z)\right| \leqslant\ & C\left(2^{-j}N(\lambda(z))\right)^n 2^{2jn\sigma+4j\beta}\left[r^{-2\delta}\chi_{B_3}(2^{-j}N(\lambda(z))(x-z))\right.\\ &\left.+\chi_{E^j_{\lambda(z)}}\left(2^{-j}N(\lambda(z))(x-z)\right)\right]\\ &+C\left(2^{-j}N(\lambda(x))\right)^n 2^{2jn\sigma+4j\beta}\left[r^{-2\delta}\chi_{B_3}(2^{-j}N(\lambda(x))(x-z))\right.\\ &\left.+\chi_{E^j_{\lambda(x)}}\left(2^{-j}N(\lambda(x))(x-z)\right)\right],\end{aligned} \tag{2.10}$$

where the sets $E^j_{\lambda(x)}, E^j_{\lambda(z)} \subset B_3 = \{|y| \leqslant 3\}$ satisfying $|E^j_{\lambda(x)}|, |E^j_{\lambda(z)}| \leqslant r^{-4\delta}$ with $\delta = (6d)^{-1}$.

Now, we give the proof of (2.10) according to [5, p.2747-p.2748].

Define $\mathscr{F}^{\mu,\nu}_{j,\beta}$ by

$$\mathscr{F}^{\mu,\nu}_{j,\beta}(u) = \left(e^{iP_\nu(\cdot)}L_{j,\nu,\beta}(\cdot)\right) * \left(e^{-iP_\mu(-\cdot)}L_{j,\mu,\beta}(-\cdot)\right)(u),$$

where $\nu = (\nu_\alpha)_{2\leqslant|\alpha|\leqslant d}$ and $\mu = (\mu_\alpha)_{2\leqslant|\alpha|\leqslant d}$ to satisfy

$$r \leqslant \left|A_{j,\nu}\circ\nu\right|,\ \left|A_{j,\mu}\circ\mu\right| < 2r.$$

Let $h = \frac{N(\mu)}{N(\nu)}$ and we may assume that $h \leqslant 1$. Hence, by (2.5) and (2.8), we have

$$\begin{aligned}\mathscr{F}^{\mu,\nu}_{j,\beta}\left(\tfrac{2^j u}{N(\mu)}\right) = &\int_{\mathbb{R}^n} e^{i\left[P_{A_{j,\nu}\circ\nu}(y)-P_{A_{j,\mu}\circ\mu}(-u+hy)\right]}\\ &\times L_{j,\beta}(y)\left(\tfrac{2^j}{N(\mu)}\right)^{-n+\beta}\left(\tfrac{2^j}{N(\nu)}\right)^{\beta}L_{j,\beta}(hy-u)dy.\end{aligned} \tag{2.11}$$

To estimate $\mathscr{F}^{\mu,\nu}_{j,\beta}\left(\frac{2^j u}{N(\mu)}\right)$, we adopt some basic ideas and estimates from [5, 13]. Moreover, we can divide it into two cases: h is near the origin and away from the origin.

Case 1. $0 < h \leqslant \eta \ll 1$, where η will be chosen later. Note that

$$\operatorname{supp}(L_{j,\beta}) \subseteq \left\{1/8 < |y| \leqslant 3/2\right\} \quad \text{and} \quad |u| \leqslant |hy-u| + h|y| \leqslant 3.$$

First, we have the following estimates.

$$\begin{aligned}|L_{j,\beta}(y)| &\leqslant C\int_{2^{j-2}\leqslant|y-z|\leqslant 2^j}\frac{|\Omega(y-z)|}{|y-z|^{n-\beta}}2^{jn\sigma}dz\\ &\leqslant C2^{jn\sigma}\left(\int_{2^{j-2}\leqslant|y|\leqslant 2^j}|\Omega(y)|^q dy\right)^{1/q}\left(\int_{2^{j-2}\leqslant|y|\leqslant 2^j}|y|^{(\beta-n)q'}dy\right)^{1/q'}\\ &\leqslant C2^{jn\sigma}2^{j\beta}.\end{aligned} \tag{2.13}$$

Similarly, there is

$$|\nabla L_{j,\beta}(y)| \leqslant C2^{jn\sigma}2^{j\beta}. \tag{2.14}$$

From [5, p.2747], we know that if we choose η small enough, then

$$\sum_{2\leqslant|\alpha|\leqslant d}\left|\left((A_{j,\nu}\circ\nu)_\alpha+O(h|A_{j,\mu}\circ\mu|)\right)\right|\geqslant\sum_{2\leqslant|\alpha|\leqslant d}\left|(A_{j,\nu}\circ\nu)_\alpha\right|-C\eta|A_{j,\mu}\circ\mu|\geqslant Cr.$$

Thus, using (2.13), (2.14) and the van der Corput lemma in n-dimensional (see Proposition 2.1 in [13, p. 791]), we obtain

$$\left|\mathscr{F}^{\mu,\nu}_{j,\beta}\left(\frac{2^j u}{N(\mu)}\right)\right|\leqslant C\left(2^{-j}N(\mu)\right)^n 2^{2jn\sigma+4j\beta}r^{-1/d}\chi_{B_3}(u). \tag{2.15}$$

Case 2. $\eta<h\leqslant 1$. From the assumption on polynomial, we know that there is no first order term in y of $P_{A_{j,\nu}\circ\nu}(y)$. Let $e_k=(0,\dots,1,0\dots)$ with 1 in the k^{th} component. From [5, p.2748], we know that the first order term in y in $P_{A_{j,\nu}\circ\nu}(y)-P_{A_{j,\mu}\circ\mu}(-u+hy)$ can be written as

$$-h\sum_{k=1}^n P^{(k)}_{A_{j,\mu}\circ\mu}(u)y_k=-h\sum_{k=1}^n\sum_{2\leqslant|\alpha|\leqslant d}\alpha_k\left(\frac{2^j}{N(\mu)}\right)^{|\alpha|}\mu_\alpha u^{\alpha-e_k}y_k.$$

Now applying (2.13), (2.14) and Proposition 2.1 in [13] again, we have

$$\left|\mathscr{F}^{\mu,\nu}_{j,\beta}\left(\frac{2^j u}{N(\mu)}\right)\right|\leqslant C\left(2^{-j}N(\mu)\right)^n 2^{2jn\sigma+4j\beta}\left(\sum_{k=1}^n\left|P^{(k)}_{A_{j,\mu}\circ\mu}(u)\right|\right)^{-1/d}\chi_{B_3}(u).$$

For $\rho>0$, we define $E^j_\mu=\left\{u\in B_3:\sum_{k=1}^n\left|P^{(k)}_{A_{j,\mu}\circ\mu}(u)\right|\leqslant\rho\right\}$. Thus, we get

$$\left|\mathscr{F}^{\mu,\nu}_{j,\beta}\left(\frac{2^j u}{N(\mu)}\right)\right|\leqslant C\left(2^{-j}N(\mu)\right)^n 2^{2jn\sigma+4j\beta}\rho^{-1/d}\chi_{B_3}(u), \tag{2.16}$$

for $u\in\left(E^j_\mu\right)^c$. From Proposition 2.2 in [13, p. 791], we have

$$\left|E^j_\mu\right|\leqslant C_{n,d}\left(\sum_{k=1}^n\sum_{2\leqslant|\alpha|\leqslant d}\alpha_k\left(\frac{2^j}{N(\mu)}\right)^{|\alpha|}|\mu_\alpha|\right)^{-1/d}\rho^{1/d}.$$

According to [5, p.2748], there is

$$\sum_{k=1}^n\sum_{2\leqslant|\alpha|\leqslant d}\alpha_k\left(\frac{2^j}{N(\mu)}\right)^{|\alpha|}|\mu_\alpha|\geqslant\sum_{2\leqslant|\alpha|\leqslant d}\left(\frac{2^j}{N(\mu)}\right)^{|\alpha|}|\mu_\alpha|=|A_{j,\mu}\circ\mu|\geqslant r.$$

Let $\rho=(C_{n,d})^{-d}r^{1/3}$ and $\delta=\frac{1}{6d}$. Then, for $u\in E^j_\mu$, we have

$$\left|\mathscr{F}^{\mu,\nu}_{j,\beta}\left(\frac{2^j u}{N(\mu)}\right)\right|\leqslant C\left(2^{-j}N(\mu)\right)^n 2^{2jn\sigma+4j\beta}\chi_{E^j_\mu}(u) \tag{2.17}$$

with $|E_\mu^j| \leqslant C_{n,d}(\rho/r)^{1/d} \leqslant r^{-4\delta}$. As $r \geqslant 2^j/c_0$, then using (2.15)-(2.17), we get

$$\left|\mathscr{F}_{j,\beta}^{\mu,\nu}\left(\frac{2^j u}{N(\mu)}\right)\right| \leqslant C\big(2^{-j}N(\mu)\big)^n 2^{2jn\sigma+4j\beta}\left[r^{-2\delta}\chi_{B_3}(u)+\chi_{E_\mu^j}(u)\right].$$

Now, we conclude that for $\mu,\ \nu$ with $r \leqslant \left|A_{j,\nu}\circ\nu\right| \leqslant 2r, r \leqslant \left|A_{j,\mu}\circ\mu\right| \leqslant 2r$ and $h \leqslant 1$, there is

$$\left|\mathscr{F}_{j,\beta}^{\mu,\nu}(u)\right| \leqslant C\big(2^{-j}N(\mu)\big)^n 2^{2jn\sigma+4j\beta}\left[r^{-2\delta}\chi_{B_3}(2^{-j}N(\mu)u)+\chi_{E_\mu^j}(2^{-j}N(\mu)u)\right]. \tag{2.18}$$

For fixed $x,z \in U_{j,r}$, let $\nu = \lambda(x)$, $\mu = \lambda(z)$, $u = x-z$. By the symmetry of μ, ν, and (2.18), we finish the proof of (2.10).

Now, we return to the estimates of $\mathscr{L}_{\lambda(\cdot),\beta}^{j}$. Recall the definition of $\mathscr{M}_\varepsilon(f)(x)$ and Lemma 2.1. Denoting $\varepsilon = r^{-4\delta}$ and using (2.10), we obtain

$$\begin{aligned}
&\left|\big(\mathscr{L}_{j,r,\beta}\mathscr{L}_{j,r,\beta}^{*}f,g\big)\right| \\
\leqslant&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\mathscr{K}_\beta(x,z)||f(z)||g(x)|dzdx \\
\leqslant& Cr^{-2\delta}2^{2jn\sigma+4j\beta}\int_{\mathbb{R}^n}|f(z)|\big(2^{-j}N(\lambda(z))\big)^n\int_{|x-z|\leqslant\frac{3\cdot 2^j}{N(\lambda(z))}}|g(x)|dxdz \\
&+C2^{2jn\sigma+4j\beta}\int_{\mathbb{R}^n}|f(z)|\big(2^{-j}N(\lambda(z))\big)^n\int_{\mathbb{R}^n}\chi_{E_{\lambda(z)}^j}\big(2^{-j}N(\lambda(z))(x-z)\big)|g(x)|dxdz \\
&+Cr^{-2\delta}2^{2jn\sigma+4j\beta}\int_{\mathbb{R}^n}|g(x)|\big(2^{-j}N(\lambda(x))\big)^n\int_{|x-z|\leqslant\frac{3\cdot 2^j}{N(\lambda(x))}}|f(z)|dzdx \\
&+C2^{2jn\sigma+4j\beta}\int_{\mathbb{R}^n}|g(x)|\big(2^{-j}N(\lambda(x))\big)^n\int_{\mathbb{R}^n}\chi_{E_{\lambda(x)}^j}\big(2^{-j}N(\lambda(x))(x-z)\big)|f(z)|dzdx \\
\leqslant& Cr^{-2\delta}2^{2jn\sigma+4j\beta}\int_{\mathbb{R}^n}|f(z)|M(g)(z)dz+C2^{2jn\sigma+4j\beta}\int_{\mathbb{R}^n}|f(z)|\mathscr{M}_\varepsilon(g)(z)dz \\
&+Cr^{-2\delta}2^{2jn\sigma+4j\beta}\int_{\mathbb{R}^n}|g(x)|M(f)(x)dx+C2^{2jn\sigma+4j\beta}\int_{\mathbb{R}^n}|g(x)|\mathscr{M}_\varepsilon(f)(x)dx.
\end{aligned}$$

Using the Hölder inequality, the L^2 boundedness of M (see [8]) and Lemma 2.1, we obtain that

$$\left|\big(\mathscr{L}_{j,r,\beta}\mathscr{L}_{j,r,\beta}^{*}f,g\big)\right| \leqslant Cr^{-2\delta}2^{2jn\sigma+4j\beta}\|f\|_{L^2}\|g\|_{L^2}. \tag{2.19}$$

As

$$\left\{x\in\mathbb{R}^n : \left|A_{j,\lambda}\circ\lambda\right| \geqslant \frac{2^j}{c_0}\right\} = \bigcup_{k=0}^{\infty}\left\{x : \frac{2^{j+k}}{c_0} \leqslant |A_{j,\lambda}\circ\lambda| < \frac{2^{j+k+1}}{c_0}\right\},$$

we may choose $r = 2^{j+k}/c_0$ for $k = 0,1,\cdots,$ and denote $\mathscr{L}_{j,\beta}^{(k)} := \mathscr{L}_{j,r,\beta}$. Thus, we obtain $\mathscr{L}_{j,\beta}(f)(x) = \sum_{k=0}^{\infty}\mathscr{L}_{j,\beta}^{(k)}(f)(x)$. Using (2.19), we get

$$\left\|\mathscr{L}_{j,\beta}\right\|_{L^2\to L^2} \leqslant \sum_{k=0}^{\infty}\left\|\mathscr{L}_{j,\beta}^{(k)}\right\|_{L^2\to L^2} \leqslant C2^{jn\sigma+2j\beta}\sum_{k=0}^{\infty}\big(2^{j+k}/c_0\big)^{-\delta} \leqslant C2^{jn\sigma+2j\beta}2^{-j\delta}. \tag{2.20}$$

By the fact $|\mathscr{L}_{j,\beta}f(x)| \leqslant C2^{jn\sigma+j\beta}Mf(x)$ (see (2.7)), we have

$$\left\|\mathscr{L}_{j,\beta}f\right\|_{L^s} \leqslant C2^{jn\sigma+j\beta}\|f\|_{L^s},$$

for $1<s<\infty$.

Thus, by the Riesz-Thörin interpolation theorem, we obtain

$$\left\|\mathscr{L}_{j,\beta}f\right\|_{L^p} \leqslant C2^{jn\sigma+j\beta}2^{-j(\delta-\beta)\gamma(p)}\|f\|_{L^p},$$

where $\gamma(p)=\min\{1/p,1/p'\}$ and $q'<p<\infty$. As $0<\beta<\frac{\delta\gamma(p)}{n-1+\gamma(p)}$, we may choose σ satisfying $\beta<\sigma<\frac{\beta+(\delta-\beta)\gamma(p)}{n}$. Moreover, we denote $\theta=(\delta-\beta)\gamma(p)-n\sigma-\beta>0$, which implies

$$\left\|\mathscr{L}^j_{\lambda(\cdot),\beta}f\right\|_{L^p} = \|\mathscr{L}_{j,\beta}f\|_{L^p} \leqslant C2^{-j\theta}\|f\|_{L^p}. \tag{2.21}$$

The estimates of $\mathscr{R}^j_{\lambda(\cdot),\beta}$

Before giving the estimates of $\mathscr{R}^j_{\lambda(\cdot),\beta}$, we recall the definition of $\mathscr{R}^j_{\lambda(\cdot),\beta}$ as

$$\mathscr{R}^j_{\lambda(x),\beta}(f)(x) = \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)}R_{j,\lambda,\beta}(y)f(x-y)dy,$$

where $R_{j,\lambda,\beta}(y)$ is defined by

$$R_{j,\lambda,\beta}(y) = \int_{\mathbb{R}^n}\left[K_\beta(y)\psi_{j,\lambda}(y)-K_\beta(y-z)\psi_{j,\lambda}(y-z)\right]\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz.$$

As

$$\text{supp }(R_{j,\lambda,\beta}) \subseteq \left\{\frac{2^{j-3}}{N(\lambda(x))} \leqslant |y| \leqslant \frac{2^{j+1}}{N(\lambda(x))}\right\} \tag{2.22}$$

and $|z| \leqslant \frac{2^{j(1-\sigma)-5}}{N(\lambda(x))}$, we get $|y-z|\sim|y|$. Thus, we have

$$\begin{aligned}
|R_{j,\lambda,\beta}(y)| \leqslant & \int_{\mathbb{R}^n}|\psi_{j,\lambda}(y-z)||K_\beta(y)-K_\beta(y-z)|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz\\
&+\int_{\mathbb{R}^n}|K_\beta(y)||\psi_{j,\lambda}(y)-\psi_{j,\lambda}(y-z)|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz\\
\leqslant & C\frac{|\Omega(y)|}{|y|^{n+1-\beta}}\int_{\mathbb{R}^n}|z|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz\\
&+\frac{C}{|y|^{n-\beta}}\int_{\mathbb{R}^n}|\Omega(y-z)-\Omega(y)|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz\\
&+C\frac{|\Omega(y)|}{|y|^{n-\beta}}\int_{\mathbb{R}^n}\left|2^{-j}N(\lambda(x))z\right|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz\\
\leqslant & C2^{-j\sigma+j\beta}\frac{|\Omega(y)|}{|y|^n}+\frac{C}{|y|^{n-\beta}}\int_{\mathbb{R}^n}|\Omega(y-z)-\Omega(y)|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz.
\end{aligned}$$

Now, we denote $\mathscr{R}^j_{\lambda(\cdot),\beta}(f)$ by $\mathscr{R}_{j,\beta}(f)$ for simplicity. Then, from (2.22) and the Hölder inequality, we have

$$\begin{aligned}
&|\mathscr{R}_{j,\beta}(f)(x)| \\
&\leqslant C2^{-j\sigma+j\beta}M_\Omega(f)(x)+\int_{\mathbb{R}^n}\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)\times\int_{\frac{2^{j-3}}{N(\lambda(x))}\leqslant|y|\leqslant\frac{2^{j+1}}{N(\lambda(x))}}\frac{|\Omega(y-z)-\Omega(y)|}{|y|^{n-\beta}}|f(x-y)|dydz \\
&\leqslant C2^{-j\sigma+j\beta}M_\Omega(f)(x)+\int_{\mathbb{R}^n}\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)\left(\int_{\frac{2^{j-3}}{N(\lambda(x))}\leqslant|y|\leqslant\frac{2^{j+1}}{N(\lambda(x))}}\frac{|\Omega(y-z)-\Omega(y)|^q}{|y|^{n-\beta}}dy\right)^{1/q} \\
&\quad\times\left(\int_{\frac{2^{j-3}}{N(\lambda(x))}\leqslant|y|\leqslant\frac{2^{j+1}}{N(\lambda(x))}}\frac{|f(x-y)|^{q'}}{|y|^{n-\beta}}dy\right)^{1/q'}dz.
\end{aligned}$$

From [5, p.2750], we know that for $\alpha=\frac{z}{|y|}$, there is

$$\int_{\mathbb{S}^{n-1}}\left|\Omega\left(\frac{y'-\alpha}{|y'-\alpha|}\right)-\Omega(y')\right|^q d\sigma(y')\leqslant C\omega_q^q(|\alpha|).$$

As $|z|\leqslant\frac{2^{j(1-\sigma)-5}}{N(\lambda(x))}$, we get

$$\left(\int_{\frac{2^{j-3}}{N(\lambda(x))}\leqslant|y|\leqslant\frac{2^{j+1}}{N(\lambda(x))}}\frac{|\Omega(y-z)-\Omega(y)|^q}{|y|^{n-\beta}}dy\right)^{1/q}\leqslant C2^{j\beta/q}\omega_q(2^{-j\sigma-2}).$$

Thus, we have

$$|\mathscr{R}_{j,\beta}(f)(x)|\leqslant C\left[2^{-j\sigma+j\beta}M_\Omega(f)(x)+2^{j\beta}\omega_q(2^{-j\sigma-2})M_{q'}(f)(x)\right],$$

where C is independent of the choice of $\lambda(\cdot)$. Since $p>q'$, using the L^p boundedness of M_Ω and $M_{q'}$ again, we obtain

$$\left\|\mathscr{R}^j_{\lambda(\cdot),\beta}(f)\right\|_{L^p}=\left\|\mathscr{R}_{j,\beta}(f)\right\|_{L^p}\leqslant C\left(2^{-j\sigma+j\beta}+2^{j\beta}\omega_q(2^{-j\sigma-2})\right)\|f\|_{L^p}. \tag{2.23}$$

Proof of Theorem 1.2

From (2.21) and (2.23), we have the following estimates.

$$\begin{aligned}
\left\|T^j_{\lambda(\cdot),\beta}(f)\right\|_{L^p}&\leqslant\left\|\mathscr{L}^j_{\lambda(\cdot),\beta}(f)\right\|_{L^p}+\left\|\mathscr{R}^j_{\lambda(\cdot),\beta}(f)\right\|_{L^p}\\
&\leqslant C\left(2^{-j\theta}\|f\|_{L^p}+2^{-j\sigma+j\beta}\|f\|_{L^p}+2^{j\beta}\omega_q(2^{-j\sigma-2})\|f\|_{L^p}\right).
\end{aligned}$$

By the fact that $\theta<0$ and $\beta<\sigma$, we can easily get

$$\sum_{j\geqslant1}2^{-j\theta}\|f\|_{L^p}\leqslant C\|f\|_{L^p}, \tag{2.24}$$

and

$$\sum_{j\geqslant 1} 2^{-j\sigma+j\beta}\|f\|_{L^p} \leqslant C\|f\|_{L^p}. \tag{2.25}$$

Moreover, we have

$$\begin{aligned}
&\sum_{j\geqslant 1} 2^{j\beta}\omega_q(2^{-j\sigma-2})\|f\|_{L^p}\\
&= (\ln 2)^{-1}\sum_{j\geqslant 1}\omega_q(2^{-j\sigma-2})2^{j\beta}\int_{2^{-j\sigma-2}}^{2^{-j\sigma-1}}\delta\frac{d\delta}{\delta}\|f\|_{L^p}\\
&\leqslant C\sum_{j\geqslant 1}2^{j\beta}\int_{2^{-j\sigma-2}}^{2^{-j\sigma-1}}\frac{\omega_q(\delta)}{\delta}\delta d\delta\|f\|_{L^p}\\
&\leqslant C\sum_{j\geqslant 1}2^{j\beta}2^{-j\sigma}\int_0^1\frac{\omega_q(\delta)}{\delta}d\delta\|f\|_{L^p}\leqslant C\|f\|_{L^p}.
\end{aligned} \tag{2.26}$$

Combining (2.2)-(2.3), (2.24)-(2.26), we finish the proof of Theorem 1.2.

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