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NEW UPPER BOUNDS FOR THE INFINITY NORM OF NEKRASOV MATRICES

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Abstract. Some new upper bounds for the infinity norm of the inverse of Nekrasov matrices are presented. It is shown that the new bounds are better than those given by Kolotilina (2013) and Zhu, Li (2017). Numerical examples are given to illustrate the effectiveness of the derived results.

1. Introduction

A matrix $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ is called an H-matrix if its comparison matrix $<A>=[m_{ij}]$ defined by

$$<A>=[m_{ij}]\in\mathbb{R}^{n\times n}, m_{ij}=\begin{cases}|a_{ii}|, & i=j,\\ -|a_{ij}|, & i\neq j,\end{cases}$$

is a nonsingular M-matrix, that is, $<A>^{-1}\geqslant 0$ [1, 2, 3]. H-matrices are widely used in many disciplines, like scientific computing, economics, dynamical system theory, etc [15]. An important topic among them concerns infinity norm bounds for the inverse of H-matrices, since these bounds can often be used in the convergence analysis of iterative algorithms, see [1, 4, 6, 7, 9], or can be used to estimate error bounds for linear complementary problems, see [10, 11, 12, 17]. Until now, infinity norm bounds for the inverse of H-matrices have been studied extensively, see [3, 4, 13, 16]. It is worth noting that when the involved matrix in $||A^{-1}||_\infty$ is an *SDD* matrix as one of the most important subclasses of H-matrices, an elegant so-called Varah's bound is provided by Varah in [16]. Here a matrix $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ is said to be a strictly diagonally dominant matrix (*SDD*) if for each $i\in N=\{1,2,\cdots,n\}$,

$$|a_{ii}|>r_i(A),$$

where $r_i(A)=\sum\limits_{j\neq i}|a_{ij}|$.

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THEOREM 1. [16] *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be SDD. Then*

$$||A^{-1}||_\infty \leqslant \frac{1}{\min\limits_{i\in N}(|a_{ii}| - r_i(A))}. \tag{1}$$

Note that the Varah's bound works only for *SDD* matrices. Then, several new upper bounds for a wider class of H-matrices which sometimes are tighter in the SDD case were derived, see [3, 4, 13] and references therein.

In [8], Kolotilina provided the following upper bound for $||A^{-1}||_\infty$ when A is a Nekrasov matrix as a subclass of H-matrices, which only depends on the entries of A and improves the bounds proposed by Varah in [16] and Cvetković et al. in [4].

DEFINITION 1. [4, 14] A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called a Nekrasov matrix if for each $i \in N$,

$$|a_{ii}| > h_i(A),$$

where $h_1(A) = \sum\limits_{j\neq 1} |a_{1j}|$, and

$$h_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j(A) + \sum_{j=i+1}^{n} |a_{ij}|, i = 2,3,\ldots,n. \tag{2}$$

THEOREM 2. [8, Theorem 2.2] *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be a Nekrasov matrix. Then*

$$||A^{-1}||_\infty \leqslant \max_{i\in N} \frac{z_i(A)}{|a_{ii}| - h_i(A)}, \tag{3}$$

where $z_1(A) = 1$ and

$$z_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A) + 1, i = 2,3,\ldots,n. \tag{4}$$

Observe from Theorem 2 that the estimate by using bound (3) may be inaccurate when the value of $\min\limits_{i\in N}\{|a_{ii}| - h_i(A)\}$ is very small, implying that the scope of applications of this bound is limited. To overcome this drawback, Zhu and Li in [18] gave the following new upper bounds for the infinity norm of the inverse of Nekrasov matrices, which involve an adjustable parameter ε.

THEOREM 3. [18, Theorem 5] *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be a Nekrasov matrix such that, for each $i = 1,2,\ldots,n-1$, $a_{ij} \neq 0$ for some $j > i$. Then*

$$||A^{-1}||_\infty \leqslant \max_{i\in N}\{w_i\} \max\left\{ \max_{i\neq n} \frac{1}{\sum\limits_{j=i+1}^{n} (1-w_j)|a_{ij}|}, \frac{1}{\varepsilon} \right\}, \tag{5}$$

where $w_i = \frac{h_i(A)}{|a_{ii}|}$ for $i = 1,2,\ldots,n-1$, and $w_n = \frac{h_n(A)+\varepsilon}{|a_{nn}|}$ with $\varepsilon \in (0, |a_{nn}| - h_n(A))$.

THEOREM 4. [18, Theorem 6] *Let* $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ *be a Nekrasov matrix with* $h_i(A)\neq 0$ *for all* $i\in N$. *Then*

$$||A^{-1}||_\infty \leqslant \max_{i\in N}\{w_i\}\max\left\{\max_{i\neq n}\frac{1}{\sum\limits_{j=i+1}^{n}(1-w_j)|a_{ij}|},\frac{1}{h_n(A)(\varepsilon-1)}\right\}, \tag{6}$$

where $w_i=\frac{h_i(A)}{|a_{ii}|}$ *for* $i=1,2,\ldots,n-1$, *and* $w_n=\frac{h_n(A)}{|a_{nn}|}\varepsilon$ *with* $\varepsilon\in\left(1,\frac{|a_{nn}|}{h_n(A)}\right)$.

It is not difficult to see that in some cases the bounds in Theorems 3 and 4 are not always effective to estimate $||A^{-1}||_\infty$ because they can be arbitrarily large when $\varepsilon\to 0$ or $\varepsilon\to 1$. Hence it is interesting to find alternative bounds for $||A^{-1}||_\infty$ to overcome this drawback. In this paper, we give new upper bounds for $||A^{-1}||_\infty$ when A is a Nekrasov matrix, and then prove that the new bounds are better than those in Theorems 2 (Theorem 2.2 in [8]), 3, and 4 (Theorems 5 and 6 in [18]). Numerical examples are presented to demonstrate their usefulness.

2. Main results

In this section, two upper bounds for $||A^{-1}||_\infty$ are provided when A is a Nekrasov matrix. First, some lemmas and notations which will be used later are given as follows.

Given a matrix $A=[a_{ij}]$, by $A=D-L-U$ we denote the standard splitting of A into its diagonal (D), strictly lower $(-L)$ and strictly upper $(-U)$ triangular parts. And we denote $|A|=[|a_{ij}|]$.

LEMMA 1. [5, 18] *Let* $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ *be a Nekrasov matrix such that, for each* $i=1,2,\ldots,n-1$, $a_{ij}\neq 0$ *for some* $j>i$. *Then the matrix* $W=diag(w_i)$, *where* $w_i=\frac{h_i(A)}{|a_{ii}|}$ *for* $i=1,2,\ldots,n-1$, *and* $w_n=\frac{h_n(A)+\varepsilon}{|a_{nn}|},\varepsilon\in(0,|a_{nn}|-h_n(A))$, *has positive diagonal entries and it satisfies that* $B=AW$ *is SDD.*

LEMMA 2. [18] *Let* $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ *be a Nekrasov matrix with* $h_i(A)\neq 0$ *for all* $i\in N$. *Then the matrix* $W=diag(w_i)$, *where* $w_i=\frac{h_i(A)}{|a_{ii}|}$ *for* $i=1,2,\ldots,n-1$, *and* $w_n=\frac{h_n(A)}{|a_{nn}|}\varepsilon,\varepsilon\in\left(1,\frac{|a_{nn}|}{h_n(A)}\right)$, *has positive diagonal entries and it satisfies that* $B=AW$ *is SDD.*

To get the desired upper bounds, we next give the two technical lemmas.

LEMMA 3. *Suppose that* $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ *is a matrix with* $a_{ii}\neq 0$ *for all* $i\in N$. *Let* $B=AW=[b_{ij}]$, *where* $W=diag(w_i)$ *with* w_i *defined in Lemma 1. Then*

$$z_i(B)=z_i(A) \text{ and } h_i(A)-h_i(B)=\eta_i(A), \tag{7}$$

where $h_i(B), z_i(B)$ are defined as in (2) and (4), respectively, and

$$\eta_1(A) = \sum_{j=2}^{n}(1-w_j)|a_{1j}|, \eta_i(A) = \sum_{j=1}^{i-1}\frac{|a_{ij}|}{|a_{jj}|}\eta_j(A) + \sum_{j=i+1}^{n}(1-w_j)|a_{ij}|, i = 2,3,\cdots,n.$$

Proof. We prove (7) by induction on i. Note that $z_1(B) = z_1(A) = 1$. Then for $i = 2$, we have

$$z_2(B) = \frac{|b_{21}|}{|b_{11}|}z_1(B) + 1 = \frac{|w_1 \cdot a_{21}|}{|w_1 \cdot a_{11}|}z_1(A) + 1 = z_2(A).$$

Now suppose that $z_i(B) = z_i(A)$ holds for $i = 3,\ldots,k$ and $k < n$. Since

$$z_{k+1}(B) = \sum_{j=1}^{k}\frac{|b_{k+1,j}|}{|b_{jj}|}z_j(B) + 1 = \sum_{j=1}^{k}\frac{w_j \cdot |a_{k+1,j}|}{w_j \cdot |a_{jj}|}z_j(A) + 1 = z_{k+1}(A),$$

by mathematical induction we have that for each $i \in N$, $z_i(B) = z_i(A)$ holds.

We are now in a position to prove that $h_i(A) - h_i(B) = \eta_i(A)$ holds for each $i \in N$. Suppose that $W = diag(w_i)$ is a diagonal matrix with w_i defined in Lemma 1. Then for $i = 1$, we have

$$\begin{aligned}
h_1(A) - h_1(B) &= r_1(A) - r_1(B) \\
&= \sum_{j=2}^{n}|a_{1j}| - \left(\sum_{j=2}^{n-1}|a_{1j}|\frac{h_j(A)}{|a_{jj}|} + |a_{1n}|\frac{h_n(A)+\varepsilon}{|a_{nn}|}\right) \\
&= \sum_{j=2}^{n-1}\left(1 - \frac{h_j(A)}{|a_{jj}|}\right)|a_{1j}| + \left(1 - \frac{h_n(A)+\varepsilon}{|a_{nn}|}\right)|a_{1n}| \\
&= \sum_{j=2}^{n}(1-w_j)|a_{1j}| = \eta_1(A).
\end{aligned}$$

Now suppose that $h_i(A) - h_i(B) = \eta_i(A)$ holds for $i = 2,\ldots,k$ and $k < n$. Since

$$\begin{aligned}
h_{k+1}(A) - h_{k+1}(B) &= \sum_{j=1}^{k}\frac{|a_{k+1,j}|}{|a_{jj}|}h_j(A) + \sum_{j=k+2}^{n}|a_{k+1,j}| - \left[\sum_{j=1}^{k}\frac{|b_{k+1,j}|}{|b_{jj}|}h_j(B)\right. \\
&\qquad \left. + \sum_{j=k+2}^{n}|b_{k+1,j}|\right] \\
&= \sum_{j=1}^{k}\frac{|a_{k+1,j}|}{|a_{jj}|}\left[h_j(A) - h_j(B)\right] + \sum_{j=k+2}^{n}\left(|a_{k+1,j}| - |b_{k+1,j}|\right) \\
&= \sum_{j=1}^{k}\frac{|a_{k+1,j}|}{|a_{jj}|}\eta_j(B) + \sum_{j=k+2}^{n}(1-w_j)|a_{k+1,j}| \\
&= \eta_{k+1}(A),
\end{aligned}$$

by mathematical induction we have that for each $i \in N$, $h_i(A) - h_i(B) = \eta_i(A)$ holds. This completes the proof.

Similarly, we can easily obtain the following result.

LEMMA 4. *Suppose that $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a Nekrasov matrix with $a_{ii} \neq 0$ for all $i \in N$. Let $B = AW = [b_{ij}]$, where $W = diag(w_i)$ with w_i defined in Lemma 2. Then*

$$z_i(B) = z_i(A), h_i(A) - h_i(B) = \eta_i(A). \tag{8}$$

where $h_i(B), z_i(B), \eta_i(A)$ are defined as in (2), (4), and Lemma 2, respectively.

By Lemmas 1, 2, 3 and 4, we give the following bounds for $||A^{-1}||_\infty$ when A is a Nekrasov matrix.

THEOREM 5. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be a Nekrasov matrix such that, for each $i = 1,2,\dots,n-1$, $a_{ij} \neq 0$ for some $j > i$. Then*

$$||A^{-1}||_\infty \leqslant \max_{i\in N}\{w_i\} \max\left\{\max_{i\neq n} \frac{z_i(A)}{\eta_i(A)}, \frac{z_n(A)}{\eta_n(A)+\varepsilon}\right\}, \tag{9}$$

where $w_i = \frac{h_i(A)}{|a_{ii}|}$ for $i = 1,2,\dots,n-1$, and $w_n = \frac{h_n(A)+\varepsilon}{|a_{nn}|}$ with $\varepsilon \in (0, |a_{nn}| - h_n(A))$.

Proof. Since A is a Nekrasov matrix, it follows from Lemma 1 that there exists a positive diagonal matrix $W = diag(w_i)$ such that $B = AW$ is SDD. Then $A^{-1} = WB^{-1}$, which together with Theorem 2 imply that

$$||A^{-1}||_\infty = ||WB^{-1}||_\infty \leqslant ||W||_\infty \cdot ||B^{-1}||_\infty \leqslant \max_{i\in N}\{w_i\} \cdot \max_{i\in N} \frac{z_i(B)}{|b_{ii}| - h_i(B)}. \tag{10}$$

By Lemma 3, it holds that for each $i \in N$, $z_i(B) = z_i(A)$, for $i = 1,2,\dots,n-1$,

$$|b_{ii}| - h_i(B) = w_i|a_{ii}| - h_i(B) = \frac{h_i(A)}{|a_{ii}|}|a_{ii}| - h_i(B) = h_i(A) - h_i(B) = \eta_i(A),$$

and for $i = n$,

$$\begin{aligned} |b_{nn}| - h_n(B) = w_n|a_{nn}| - h_n(B) &= \frac{h_n(A)+\varepsilon}{|a_{nn}|}|a_{nn}| - h_n(B) \\ &= h_n(A) + \varepsilon - h_n(B) \\ &= \eta_n(A) + \varepsilon. \end{aligned}$$

Hence,

$$\max_{i\in N} \frac{z_i(B)}{|b_{ii}| - h_i(B)} = \max\left\{\max_{i\neq n} \frac{z_i(A)}{\eta_i(A)}, \frac{z_n(A)}{\eta_n(A)+\varepsilon}\right\}. \tag{11}$$

The conclusion follows from (10) and (11).

The following Theorem can be obtained in a similar way.

THEOREM 6. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be a Nekrasov matrix with $h_i(A) \neq 0$ for all $i \in N$. Then*

$$||A^{-1}||_\infty \leqslant \max_{i\in N}\{w_i\} \max\left\{\max_{i\neq n} \frac{z_i(A)}{\eta_i(A)}, \frac{z_n(A)}{\eta_n(A)+(\varepsilon-1)h_n(A)}\right\}, \tag{12}$$

where $w_i = \frac{h_i(A)}{|a_{ii}|}$ for $i = 1,2,\ldots,n-1$, and $w_n = \frac{h_n(A)}{|a_{nn}|}\varepsilon, \varepsilon \in \left(1, \frac{|a_{nn}|}{h_n(A)}\right)$.

REMARK 1. Obverse from Theorem 5 that when $\varepsilon \to 0$ bound (9) converges to a constant, i.e.,

$$\max_{i\in N}\{w_i\} \max\left\{\max_{i\neq n} \frac{z_i(A)}{\eta_i(A)}, \frac{z_n(A)}{\eta_n(A)+\varepsilon}\right\} \to \max_{i\in N}\left\{\frac{h_i(A)}{|a_{ii}|}\right\} \max\left\{\max_{i\neq n} \frac{z_i(A)}{\tilde{\eta}_i(A)}, \frac{z_n(A)}{\tilde{\eta}_n(A)}\right\}$$

when $\varepsilon \to 0$, where $\tilde{\eta}_1(A) = \sum_{j=2}^{n}(1-\frac{h_i(A)}{|a_{ii}|})|a_{1j}|$ and

$$\tilde{\eta}_i(A) = \sum_{j=1}^{i-1}\frac{|a_{ij}|}{|a_{jj}|}\eta_j(A) + \sum_{j=i+1}^{n}(1-\frac{h_i(A)}{|a_{ii}|})|a_{ij}|, i = 2,3,\cdots,n.$$

In contrast, bound (5) in Theorem 3 is

$$\max_{i\in N}\{w_i\} \max\left\{\max_{i\neq n} \frac{1}{\sum\limits_{j=i+1}^{n}(1-w_j)|a_{ij}|}, \frac{1}{\varepsilon}\right\},$$

and it can be arbitrarily large when $\varepsilon \to 0$. Obviously, bound (6) in Theorem 4 can also be arbitrarily large when $\varepsilon \to 1$, while bound (12) converges to a constant in the same setting.

The comparisons of the bounds in Theorems 3, 4, 5 and 6 are established as follows.

THEOREM 7. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be a Nekrasov matrix such that, for each $i = 1,2,\ldots,n-1$, $a_{ij} \neq 0$ for some $j > i$. Then*

$$\max\left\{\max_{i\neq n} \frac{z_i(A)}{\eta_i(A)}, \frac{z_n(A)}{\eta_n(A)+\varepsilon}\right\} \leqslant \max\left\{\max_{i\neq n} \frac{1}{\sum\limits_{j=i+1}^{n}(1-w_j)|a_{ij}|}, \frac{1}{\varepsilon}\right\},$$

where $w_i = \frac{h_i(A)}{|a_{ii}|}$ for $i = 1,2,\ldots,n-1$, and $w_n = \frac{h_n(A)+\varepsilon}{|a_{nn}|}$ with $\varepsilon \in (0, |a_{nn}| - h_n(A))$.

Proof. It follows from the proof of Theorem 5 that

$$\max\left\{\max_{i\neq n} \frac{z_i(A)}{\eta_i(A)}, \frac{z_n(A)}{\eta_n(A)+\varepsilon}\right\} = \max_{i\in N} \frac{z_i(B)}{|b_{ii}| - h_i(B)}.$$

Then, Theorem 2.4 of [8] can be applied to obtain that

$$\max_{i\in N} \frac{z_i(B)}{|b_{ii}|-h_i(B)} \leqslant \max_{i\in N} \frac{1}{|b_{ii}|-r_i(B)}.$$

By Lemma 3, we know that for $i=1,2,\dots,n-1$,

$$\begin{aligned}
|b_{ii}|-r_i(B) &= w_i|a_{ii}|-r_i(B)\\
&= \frac{h_i(A)}{|a_{ii}|}|a_{ii}|-r_i(B)\\
&= h_i(A)-r_i(B)\\
&= \sum_{j=1}^{i-1}\frac{|a_{ij}|}{|a_{jj}|}h_j(A)+\sum_{j=i+1}^{n}|a_{ij}|-\left[\sum_{j=1}^{i-1}|a_{ij}|w_j+\sum_{j=i+1}^{n}|a_{ij}|w_j\right]\\
&= \sum_{j=i+1}^{n}(1-w_j)|a_{ij}|,
\end{aligned}$$

and that for $i=n$,

$$\begin{aligned}
|b_{nn}|-r_n(B) = w_n|a_{nn}|-r_n(B) &= \frac{h_n(A)+\varepsilon}{|a_{nn}|}|a_{nn}|-r_n(B)\\
&= h_n(A)+\varepsilon-r_n(B)\\
&= \varepsilon.
\end{aligned}$$

Hence,

$$\max_{i\in N}\frac{1}{|b_{ii}|-r_i(B)} = \max\left\{\max_{i\neq n}\frac{1}{\sum\limits_{j=i+1}^{n}(1-w_j)|a_{ij}|}, \frac{1}{\varepsilon}\right\}.$$

The proof is completed.

Using the same technique, we can easily obtain the following result.

THEOREM 8. *Let $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ be a Nekrasov matrix with $h_i(A)\neq 0$ for all $i\in N$. Then*

$$\begin{aligned}
&\max\left\{\max_{i\neq n}\frac{z_i(A)}{\eta_i(A)}, \frac{z_n(A)}{\eta_n(A)+(\varepsilon-1)h_n(A)}\right\}\\
&\leqslant \max\left\{\max_{i\neq n}\frac{1}{\sum\limits_{j=i+1}^{n}(1-w_j)|a_{ij}|}, \frac{1}{h_n(A)(\varepsilon-1)}\right\},
\end{aligned}$$

where $w_i=\frac{h_i(A)}{|a_{ii}|}$ for $i=1,2,\dots,n-1$, and $w_n=\frac{h_n(A)}{|a_{nn}|}\varepsilon, \varepsilon\in\left(1,\frac{|a_{nn}|}{h_n(A)}\right)$.

We finally give comparisons of the bounds in Theorems 2, 4 and 5 for Nekrasov matrices.

THEOREM 9. *Let $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ be a Nekrasov matrix such that, for each $i=1,2,\dots,n-1$, $a_{ij}\neq 0$ for some $j>i$. If $\eta_i(A)\geqslant |a_{ii}|-h_i(A)$ holds for all $i\in N$, then*

$$\max_{i\in N}\{w_i\}\cdot\max\left\{\max_{i\neq n}\frac{z_i(A)}{\eta_i(A)},\frac{z_n(A)}{\eta_n(A)+\varepsilon}\right\}\leqslant\max_{i\in N}\frac{z_i(A)}{|a_{ii}|-h_i(A)},$$

where $w_i=\frac{h_i(A)}{|a_{ii}|}$ for $i=1,2,\dots,n-1$, $w_n=\frac{h_n(A)+\varepsilon}{|a_{nn}|}$, and $\eta_i(A)$ is defined in Lemma 3 with $\varepsilon\in(0,|a_{nn}|-h_n(A))$.

Proof. Since A is a Nekrasov matrix and $\varepsilon\in(0,|a_{nn}|-h_n(A))$, it follows that $w_i=\frac{h_i(A)}{|a_{ii}|}<1$ for all $i\in N$. Furthermore, it follows from $\eta_i(A)\geqslant|a_{ii}|-h_i(A)$ that for each $i\in N$,

$$\frac{z_i(A)}{\eta_i(A)}\leqslant\frac{z_i(A)}{|a_{ii}|-h_i(A)},$$

which together with $\varepsilon>0$ yield that

$$\max\left\{\max_{i\neq n}\frac{z_i(A)}{\eta_i(A)},\frac{z_n(A)}{\eta_n(A)+\varepsilon}\right\}\leqslant\max_{i\in N}\frac{z_i(A)}{|a_{ii}|-h_i(A)}.$$

This completes the proof.

Similarly, we can obtain the following result.

THEOREM 10. *Let $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ be a Nekrasov matrix with $h_i(A)\neq 0$ for all $i\in N$. If $\eta_i(A)\geqslant|a_{ii}|-h_i(A)$ holds for all $i\in N$, then*

$$\max_{i\in N}\{w_i\}\cdot\max\left\{\max_{i\neq n}\frac{z_i(A)}{\eta_i(A)},\frac{z_n(A)}{\eta_n(A)+(\varepsilon-1)h_n(A)}\right\}\leqslant\max_{i\in N}\frac{z_i(A)}{|a_{ii}|-h_i(A)},$$

where $w_i=\frac{h_i(A)}{|a_{ii}|}$ for $i=1,2,\dots,n-1$, $w_n=\frac{h_n(A)}{|a_{nn}|}\varepsilon$, and $\eta_i(A)$ is defined in Lemma 4 with $\varepsilon\in\left(1,\frac{|a_{nn}|}{h_n(A)}\right)$.

3. Numerical examples

In this section, we give some examples to show the sharpness of the proposed bounds.

EXAMPLE 1. Consider the following five Nekrasov matrices in [4, 5, 13]:

$$A_1=\begin{bmatrix}21&-9.1&-4.2&-2.1\\-0.7&9.1&-4.2&-2.1\\-0.7&-0.7&4.9&-2.1\\-0.7&-0.7&-0.7&2.8\end{bmatrix},\ A_2=\begin{bmatrix}5&1&0.2&2\\1&21&1&-3\\2&0.5&6.4&-2\\0.5&-1&1&9\end{bmatrix},$$

$$A_3 = \begin{bmatrix} 60 & -15 & -15 & -15 \\ -75 & 105 & -45 & 0 \\ -60 & -60 & 120 & -15 \\ -15 & -15 & -15 & 45 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 5 & -1/5 & -2/5 & -1/2 \\ -1/10 & 2 & -1/2 & -1/10 \\ -1/2 & -1/10 & 1.5 & -1/10 \\ -2/5 & -2/5 & -4/5 & 1.2 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 6 & -3 & -2 \\ -1 & 11 & -8 \\ -7 & -3 & 10 \end{bmatrix}.$$

Obviously, A_1 and A_2 are *SDD* matrices, and all matrices satisfy the conditions in Theorems 5 and 6. We compute by Matlab 12.0 the upper bounds for the infinity norm of the inverse of A_i, $i = 1,\ldots,5$, which are shown in Table 1. It is easy to see from Table 1 that this example illustrates Theorems 7 and 8.

Table 1. The upper bounds for $||A_i^{-1}||_\infty$, $i = 1,2,\ldots,5$.

Matrix	A_1	A_2	A_3	A_4	A_5
Exact $\|\|A^{-1}\|\|_\infty$	0.8759	0.2707	0.6843	1.9864	1.1519
Varah (1)	1.4286	0.5556	–	–	–
The bound (3)	0.9676	0.5556	1.5333	2.2094	1.4138
The bound (4)	1.3298	0.4445	1.2222	10.3667	1.1567
ε	0.5630	1.4500	1.3000	0.0300	0.7800
The bound (5)	1.3297	0.4442	1.2222	10.3643	1.1555
ε	1.3790	2.3500	1.0320	1.0880	1.0948
Theorem 5	**0.9477**	**0.3998**	**0.6985**	**2.0316**	**1.1528**
ε	0.5630	1.4500	1.3000	0.0300	0.7800
Theorem 6	**0.9477**	**0.4006**	**0.6980**	**2.0316**	**1.1544**
ε	1.3790	2.3500	1.0320	1.0880	1.0948

EXAMPLE 2. Consider the following matrix:

$$A_6 = \begin{bmatrix} 8 & -0.5 & -0.5 & -0.5 \\ -9 & 16 & -5 & -5 \\ -6 & -4 & 15 & -3 \\ -4.9 & -0.9 & -0.9 & 2 \end{bmatrix}.$$

By computations, we have

$$h_1(A_6) = 1.5, h_2(A_6) = 11.6875, h_3(A_6) = 7.0469, h_4(A_6) = 1.9990.$$

Obviously, A_6 is a Nekrasov matrix. Hence, by Theorems 3 and 5, we can get the bounds (5) and (9) involved with $\varepsilon \in (0, 0.001)$ for $||A_6^{-1}||_\infty$, which is drawn in Figure 1a. And by Theorems 4 and 6, we can get the bounds (6) and (12) involved with

$\varepsilon \in (1, 1.0005)$ for $||A_6^{-1}||_\infty$, which is drawn in Figure 1b. Furthermore, by the bound (3) of Theorem 2 we have

$$\max_{i \in N} \frac{z_i(A_6)}{|a_{ii}| - h_i(A_6)} = 1840.3.$$

In fact, $||A_6^{-1}||_\infty = 3.2592$.

It is easy to see from Figure 1 that the bounds in Theorem 5 and 6 are considerably smaller than those in Theorems 2 (Theorem 2.2 in [8]), 3, and 4 (Theorems 5 and 6 in [18]).

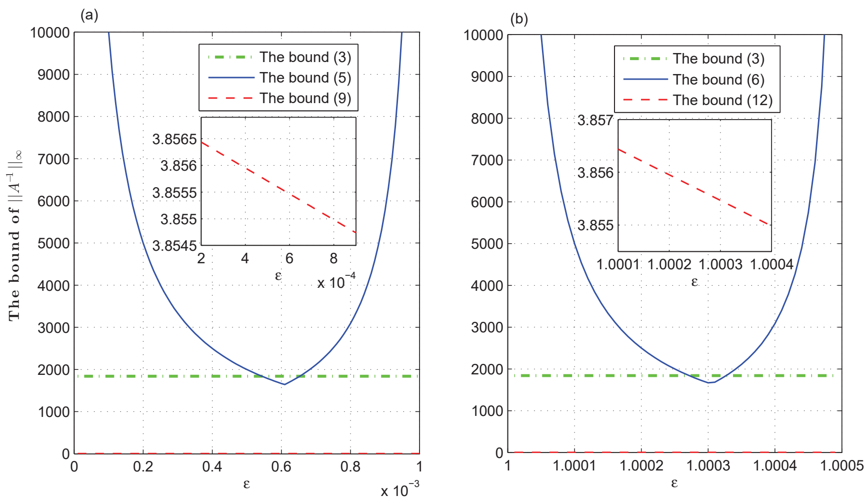

Figure 1: The bounds (3), (5), (6), (9) and (12).

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