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ASYMPTOTIC DISTRIBUTIONS AND BERRY–ESSEEN INEQUALITIES FOR LOTKA–NAGAEV ESTIMATOR OF A POISSON RANDOMLY INDEXED BRANCHING PROCESS

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Abstract. Consider a Galton–Watson process $\{Z_n\}$, the Lotka–Negaev estimator for offspring mean m is $R_n = Z_{n+1}/Z_n$. Let N_t be a Poisson process independent of $\{Z_n\}$, the continuous time process $\{Z_{N_t}\}$ is a Poisson randomly indexed branching process. We show the asymptotic distributions for $\{\mathbb{R}_t := R_{N_t}\}$.

1. Introduction

Consider a Galton–Watson process(GW) $\{Z_n\}$ with offspring distribution $\{p_i\}$. A basic task in statistical inference of GW is the estimation of the offspring mean $m := \sum_i ip_i$. We assume that $Z_0 = 1, p_0 = 0, 0 \leqslant p_i < 1, \forall i$ and GW is supercritical, that is $m > 1$. One of the most important estimator is Lotka-Negaev estimator defined as $R_n = Z_{n+1}/Z_n$, see [12]. Large deviations for R_n attracted the attention of several researchers in recent years, see [1, 2, 6, 10, 13], etc. [11], [14] and [9] extended these results to the Lotka-Negaev estimator of a branching process with immigration or random environment.

The model of Poisson randomly indexed branching process(PB) $\{Y_t := Z_{N_t}\}$ was introduced by Epps[5] to study the evolution of stock prices, where $\{N_t\}$ is a Poisson process which is independent of $\{Z_n\}$. The statistical investigation on various estimates and some parameters of the process were done in [3]. Particularly, $T_t := \log(Y_t)/(\lambda t)$ is used to estimate $\log m$, where λ is the density of underlying Poisson process. The asymptotic normality and Berry–Esseen type inequalities were given in [7].

In this paper, we concentrate on the Lotka–Negaev estimator $\{\mathbb{R}_t := R_{N_t}\}$ for offspring mean m. Wu[15] obtained the large deviations for $\mathbb{R}_t$. These results have been extended to the case that the random index is a renewal process, see [8] for details. In this manuscript, we focus on the asymptotic distribution of $\mathbb{R}_t$.

Note that $p_0 = 0$, there exists a positive random variable W such that $Y_t/C_t \to W$ *a.s.*, where $C_t = \exp(\lambda t(m-1))$, see [17]. Firstly, we consider the asymptotic distribution of normalized process $\sqrt{C_t}(\mathbb{R}_t - m)$ as $t \to \infty$.

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THEOREM 1. *Assume that $\mathbb{E}(Z_1^2) < \infty$, one has*

$$\lim_{t\to\infty} \mathbb{P}(\sigma^{-1}\sqrt{C_t}(\mathbb{R}_t - m) \leqslant x) = \int_0^\infty \Phi(xy^{1/2})dG(y),$$

where $\sigma^2 = \mathbb{V}ar(Z_1)$, $G(y)$ and $\Phi(x)$ are the cumulative distribution functions of W and the standard normal random variable respectively.

Theorem 1 shows that the typical asymptotic distribution of $\sqrt{C_t}(\mathbb{R}_t - m)$ as $t \to \infty$ is not a normal distribution. One naturally wonder whether the asymptotic distribution of $(\mathbb{R}_t - m)/\mathbb{V}\mathrm{ar}(\mathbb{R}_t)$ is normal distribution, so we consider the rate of $\mathbb{V}\mathrm{ar}(\mathbb{R}_t)$ as $t \to \infty$.

For a PB, we distinguish between the Schröder case and the Böttcher case depending on whether $p_0 + p_1 > 0$ or $p_0 + p_1 = 0$. Note that $p_0 = 0$ in this paper, the Schröder index is defined as $\alpha = -\log_m p_1 \in (0, +\infty]$. If $\alpha \in (0, +\infty)$, PB belongs to the Schröder case, else if $\alpha = \infty$, PB belongs to the Böttcher case.

Let $f_n(s)$ be the generating function of Z_n, if $1 > p_1 > 0$ (Shröder case), there exists a unique $Q(s)$ such that $f_n(s)/p_1^n \to Q(s)$ and (see [1])

$$Q(f(s)) = p_1 Q(s), \;\; Q(0) = 0, \;\; Q(s) > 0$$

for all $s \in (0,1)$, where $f(s) = f_1(s)$.

THEOREM 2. *Let $\phi(v)$ be the Laplace transformation of $V := \lim_{n\to\infty} Z_n/m^n$. Assume that $1 > p_1 > 0$ and $\mathbb{E}(Z_1^2) < \infty$. One has $\mathbb{V}ar(\mathbb{R}_t) \sim C(\alpha,t)$, where $f(t) \sim g(t)$ stands for $f(t)/g(t) \to 1$ as $t \to \infty$, α is the Schröder index and*

$$C(\alpha,t) = \begin{cases} \sigma^2 \exp(\lambda t(p_1 - 1)) \int_0^\infty Q(\exp(-v))dv, & \alpha < 1; \\ \sigma^2 p_1 \lambda t \exp(\lambda t(p_1 - 1)) \int_1^m Q(\phi(v))dv, & \alpha = 1; \\ \sigma^2 \exp(\lambda t(m^{-1} - 1)) \int_0^\infty \phi(v)dv, & \alpha > 1. \end{cases}$$

In Example 1, we choose $\lambda = 1, \alpha = 0.5, 1, 2$. The decay rates of $C(\alpha,t)$ are illustrated in the Figure 1 and Figure 2. From these figures, we know that the smaller the α, the faster the decay rate of $\mathbb{V}\mathrm{ar}(\mathbb{R}_t)$.

Note that $m > 1$, then $m + m^{-1} > 2$. When $\alpha \leqslant 1$, $p_1 \geqslant m^{-1}$, thus $C_t \mathbb{V}\mathrm{ar}(\mathbb{R}_t) \to \infty$ as $t \to \infty$. According to Slutsky's theorem,

$$\frac{\mathbb{R}_t - m}{\sqrt{\mathbb{V}\mathrm{ar}(\mathbb{R}_t)}} = \sqrt{C_t(\mathbb{R}_t - m)} \cdot \frac{1}{\sqrt{C_t \mathbb{V}\mathrm{ar}(\mathbb{R}_t)}} \xrightarrow{d} 0,$$

where $\xrightarrow{d}$ stands for convergence in distribution. That is $(\mathbb{R}_t - m)/\sqrt{\mathbb{V}\mathrm{ar}(\mathbb{R}_t)}$ has no proper asymptotic distribution. In order to balance the fluctuation, we consider the randomly normalized process $\sqrt{Y_t}(\mathbb{R}_t - m)$, where $Y_t = Z_{N_t}$.

THEOREM 3. *Assume that $\mathbb{E}(Z_1^2) < \infty$. We have*

$$\lim_{t\to\infty} \mathbb{P}(\sigma^{-1}\sqrt{Y_t}(\mathbb{R}_t - m) \leqslant x) = \Phi(x).$$

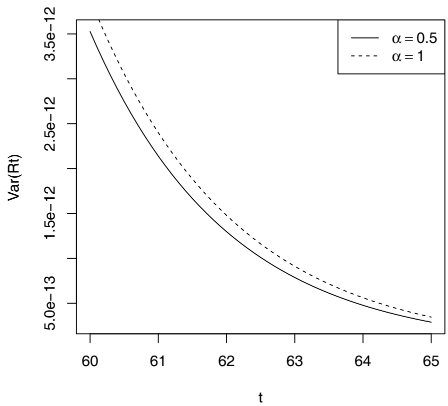

Figure 1: $\alpha = 0.5, 1$

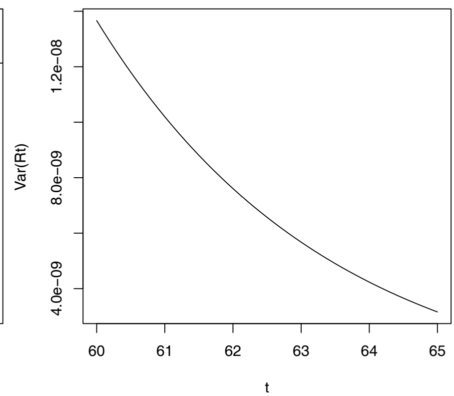

Figure 2: $\alpha = 2$

Assume that $\lambda = 1$ and the offspring distribution satisfies the following four cases respectively.

(a) $Z_1 - 1 \sim$ Geom(0.5): $p_k = 0.5^k$, (b) $Z_1 - 1 \sim$ Pois(1): $p_k = 1/(e(k-1)!)$,

(c) $Z_1 - 1 \sim$ Binom(2,0.5): $p_1 = p_3 = 1/4$, $p_2 = 1/2$,

(d) $Z_1 \sim$ Unif(1,2,3): $p_1 = p_2 = p_3 = 1/3$.

We conduct 10000 simulations for each case. Note that $\mathbb{E}(Z_1) = 2$ and $\mathbb{E}(Z_n) = 2^n$, we choose $t = 8(2^8 = 128)$ for relatively small sample and $t = 10(2^{10} = 1024)$ for relatively large sample. Compares for densities of $t = 8$, $t = 10$ and that of the standard normal distribution are given in Figure 3–6. From these figures, we know that for t large enough, Theorem 3 is efficient.

The rates of convergence in Theorem 3 can be characterized by Berry–Esseen type inequalities. Using the classical Berry–Esseen bound for sums of i.i.d. random sequence and the harmonic moments of $\{Z_n\}$ one can obtain Theorem 4. Define $G_t(x) = \mathbb{P}(\sigma^{-1}\sqrt{Y_t}(\mathbb{R}_t - m) \leqslant x)$, one has

THEOREM 4. *Assume that* $1 > p_1 > 0$, $\mathbb{E}(Z_1^3) < \infty$. *Then there exists constant C such that*

$$\sup_{x\in\mathbb{R}} |G_t(x) - \Phi(x)| \leqslant CH(\alpha,t),$$

where α *is the Schröder index,* $\mathbb{R} = (-\infty, +\infty)$ *and*

$$H(\alpha,t) = \begin{cases} \exp(\lambda t(p_1 - 1)), & \alpha < 0.5; \\ p_1 \lambda t \exp(\lambda t(p_1 - 1)), & \alpha = 0.5; \\ \exp(\lambda t(m^{-1/2} - 1)), & \alpha > 0.5. \end{cases}$$

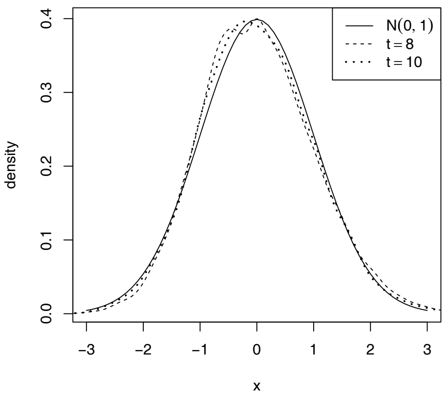

Figure 3: $Z_1 - 1 \sim$ Geom(0.5)

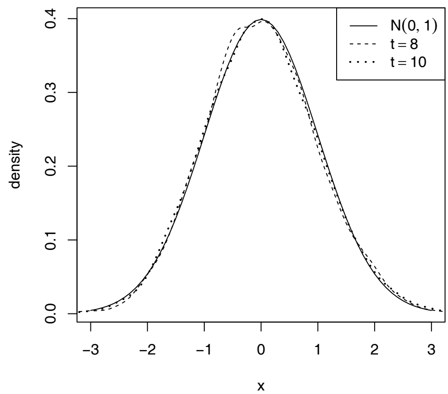

Figure 4: $Z_1 - 1 \sim$ Pois(1)

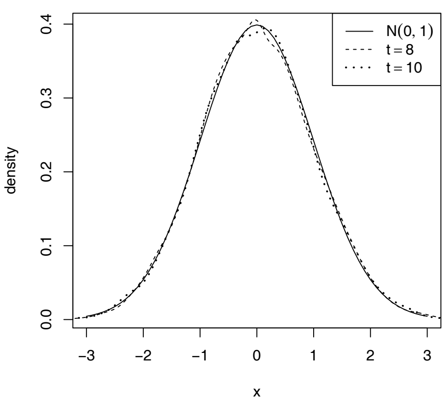

Figure 5: $Z_1 - 1 \sim$ Binom(2,0.5)

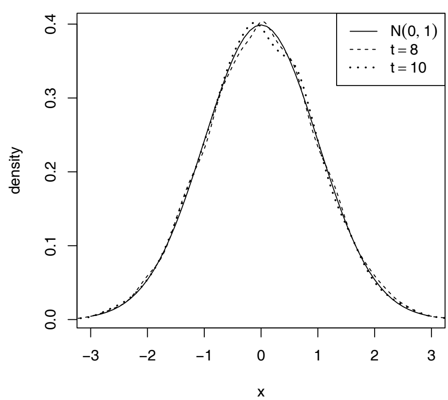

Figure 6: $Z_1 \sim$ Unif(1,2,3)

The rest of the paper is organized as follows. In Section 2, we obtain the asymptotic distribution for the normalized process $\sqrt{C_t}(\mathbb{R}_t - m)$. Section 3 is devoted to the decay rates of $\mathbb{V}\mathrm{ar}(\mathbb{R}_t)$. Asymptotic normality of the randomly normalized process $\sqrt{Y_t}(\mathbb{R}_t - m)$ is given in section 4.

In the rest of the paper, we denote by C an absolute and positive constant which may differ from line to line.

2. Asymptotic distribution of $\sqrt{C_t}(\mathbb{R}_t - m)$

Independent of Y_t, let $\{X_n\}$ be a sequence of i.i.d random variables with the same distribution as Z_1. Define $S_k = X_1 + \cdots + X_k$ for any $k \geqslant 1$ and

$$L_k(x) = \mathbb{P}\left(\frac{S_k - m}{\sqrt{k}\sigma} \leqslant x\right), \quad x \in \mathbb{R}, \quad \Delta_k = \sup_{x\in\mathbb{R}} |L_k(x) - \Phi(x)|,$$

where $\sigma^2 = \mathbb{V}\mathrm{ar}(Z_1) \in (0,\infty)$. Then

$$\Delta_k \to 0 \tag{1}$$

as $k \to \infty$, see [4]P105 for example. The proof of Theorem 1 depends on the convergence $W_t := Y_t/C_t \to W$ $a.s.$ and the independence between $\{N_t\}$ and the underlying $\mathtt{GW}$ $\{Z_n\}$.

The proof of Theorem 1.

Conditioning on Y_t,

$$\mathbb{P}\left(\frac{\sqrt{C_n}(\mathbb{R}_t - m)}{\sigma} \leqslant x\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(\frac{\sqrt{C_n}(S_k - m)}{k\sigma} \leqslant x\right)\mathbb{P}(Y_t = k). \tag{2}$$

For any $\varepsilon \in (0,1)$, we divide the right side of (2) into the following three parts.

$$J_1(\varepsilon,t) = \sum_{k<\varepsilon C_t} \mathbb{P}\left(\frac{\sqrt{C_n}(S_k - m)}{k\sigma} \leqslant x\right)\mathbb{P}(Y_t = k),$$
$$J_2(\varepsilon,t) = \sum_{\varepsilon C_t \leqslant k \leqslant \varepsilon^{-1} C_t} \mathbb{P}\left(\frac{\sqrt{C_n}(S_k - m)}{k\sigma} \leqslant x\right)\mathbb{P}(Y_t = k),$$
$$J_3(\varepsilon,t) = \sum_{k>\varepsilon^{-1} C_t} \mathbb{P}\left(\frac{\sqrt{C_n}(S_k - m)}{k\sigma} \leqslant x\right)\mathbb{P}(Y_t = k).$$

For $J_1(\varepsilon,t)$, when ε is a continuous point of $G(x) = \mathbb{P}(W \leqslant x)$, we have

$$J_1(\varepsilon,t) \leqslant \sum_{k<\varepsilon C_t} \mathbb{P}(Y_t = k) = \sum_{k<\varepsilon C_t} \mathbb{P}(W_t = k/C_t) = \mathbb{P}(W_t < \varepsilon) \to G(\varepsilon), \tag{3}$$

as $t \to \infty$. Similarly, when ε^{-1} is a continuous point of G, we obtain

$$J_3(\varepsilon,t) \leqslant \mathbb{P}(W_t > \varepsilon^{-1}) \to 1 - G(\varepsilon^{-1}). \tag{4}$$

Finally, for $J_2(\varepsilon,t)$, one has

$$\begin{aligned} J_2(\varepsilon,t) &= \sum_{\varepsilon C_t \leqslant k \leqslant \varepsilon^{-1} C_t} \mathbb{P}\left(\frac{S_k - m}{\sqrt{k}\sigma} \leqslant x\sqrt{\frac{k}{C_t}}\right)\mathbb{P}(Y_t = k) \\ &= \sum_{\varepsilon C_t \leqslant k \leqslant \varepsilon^{-1} C_t} \mathbb{P}\left(\frac{S_k - m}{\sqrt{k}\sigma} \leqslant x\sqrt{\frac{k}{C_t}}\right)\mathbb{P}(W_t = k/C_t) \\ &= \int_{\varepsilon}^{\varepsilon^{-1}} L_{yC_t}(x\sqrt{y})d\mathbb{P}(W_t \leqslant y) \\ &= \int_{\varepsilon}^{\varepsilon^{-1}} \Phi(x\sqrt{y})d\mathbb{P}(W_t \leqslant y) + o(1), \end{aligned} \tag{5}$$

as $t \to \infty$, where the last equality follows from formula (1) and $yC_t \geqslant \varepsilon C_t \to \infty$. Note that $\Phi(x\sqrt{y})$ is a bounded continuous function with respect to y and $W_t \to W$ *a.s.*, we obtain

$$J_2(\varepsilon,t) \to \int_{\varepsilon}^{\varepsilon^{-1}} \Phi(x\sqrt{y})dG(y), \tag{6}$$

as $t \to \infty$. Since ε is arbitrary, we complete the proof of Theorem 1 by (2)–(6).

3. Decay rates of $\mathbb{V}\mathrm{ar}(\mathbb{R}_t)$

Convergence rates for generating function $f_n(s)$ and harmonic moment $\mathbb{E}(Y_t^{-1})$ play important role in estimating the decay rates of $\mathbb{V}\mathrm{ar}(\mathbb{R}_t)$, so we need the following lemmas. Lemma 1 comes from [1].

LEMMA 1. *Assume that* $1 > p_1 > 0$, *then there exist constants* $0 \leqslant q_k < \infty$ *such that*

$$\lim_{n\to\infty} \frac{f_n(s)}{p_1^n} = \sum_{k=1}^{\infty} q_k s^k =: Q(s) < \infty, \ \forall 0 \leqslant s < 1.$$

Furthermore, $Q(s)$ *is the unique solution of the functional equation*

$$Q(f(s)) = p_1 Q(s), \ \ Q(0) = 0, \ \ Q(s) > 0$$

for all $s \in (0,1)$.

The harmonic moments were given in [13]. We use the following special case.

LEMMA 2. *Assume that* $1 > p_1 > 0$, $\mathbb{E}(Z_1^2) < \infty$. *Then,*

$$\lim_{n\to\infty} \Gamma(r)A_n(r)\mathbb{E}(Z_n^{-r}) = \begin{cases} \int_0^\infty Q(\exp(-v))v^{r-1}dv, & \alpha < r; \\ \int_1^m Q(\phi(v))v^{r-1}dv, & \alpha = r; \\ \int_0^\infty \phi(v)v^{r-1}dv, & \alpha > r, \end{cases}$$

where α is the Schröder index, ϕ is the Laplace transformation of $V = \lim_{n\to\infty} Z_n/m^n$, Γ is the $\Gamma-$ function defined as $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx$ and

$$A_n(r) = \begin{cases} p_1^{-n}, & \alpha < r; \\ \frac{1}{n}p_1^{-n}, & \alpha = r; \\ m^{rn}, & \alpha > r. \end{cases}$$

The proof of Theorem 2.

Conditioning on N_t, using the independence between $\{N_t\}$ and $\{Z_n\}$, one has

$$\mathbb{E}(\mathbb{R}_t) = \sum_{n=0}^{\infty} \mathbb{E}(R_n)\mathbb{P}(N_t = n) = m,$$

and

$$\mathbb{V}\text{ar}(\mathbb{R}_t) = \sum_{n=0}^{\infty} \mathbb{V}\text{ar}(R_n)\mathbb{P}(N_t = n) = \sum_{n=0}^{\infty} \mathbb{E}(R_n - m)^2\mathbb{P}(N_t = n),$$

where $R_n = Z_{n+1}/Z_n$.

Note that $Z_{n+1} = X_1 + \cdots + X_{Z_n}$, where $\{X_j\}$ are independent and have the same distribution with Z_1. In addition, $\{X_j\}$ are independent of Z_n. Conditioning on Z_n, we obtain

$$\begin{aligned} \mathbb{V}\text{ar}(\mathbb{R}_t) &= \sum_n \sum_k \mathbb{E}\left(k^{-1}\sum_{j=1}^{k} X_j - m\right)^2 \mathbb{P}(Z_n = k)\mathbb{P}(N_t = n) \\ &= \sum_n \sum_k k^{-1}\sigma^2\mathbb{P}(Z_n = k)\mathbb{P}(N_t = n) = \sigma^2 \sum_n \mathbb{E}(Z_n^{-1})\mathbb{P}(N_t = n). \end{aligned}$$

Letting $r = 1$ in Lemma 2, one has

$$\lim_{n\to\infty}(A(n,\alpha))^{-1}\mathbb{E}(Z_n^{-1}) = \left\{\begin{array}{ll} \int_0^\infty Q(\exp(-v))dv, & \alpha < 1; \\ \int_1^m Q(\phi(v))dv, & \alpha = 1; \\ \int_0^\infty \phi(v)dv, & \alpha > 1 \end{array}\right\} =: C(\alpha),$$

where

$$A(n,\alpha) = \begin{cases} p_1^n, & \alpha < 1; \\ np_1^n, & \alpha = 1; \\ m^{-n}, & \alpha > 1. \end{cases}$$

For any $\varepsilon > 0$, choose N large enough such that for all $n \geqslant N$, we have

$$\mathbb{E}(Z_n^{-1}) \in ((C(\alpha) - \varepsilon)A(n,\alpha), (C(\alpha) + \varepsilon)A(n,\alpha)). \tag{7}$$

We can divide $\mathbb{V}\mathrm{ar}(\mathbb{R}_t)$ into the following three parts.

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\mathbb{R}_t) &= \sigma^2\sum_{n=0}^{N}(\mathbb{E}(Z_n^{-1})-C(\alpha)A(n,\alpha))\mathbb{P}(N_t=n)\\ &\quad+\sigma^2\sum_{n\geqslant N+1}(\mathbb{E}(Z_n^{-1})-C(\alpha)A(n,\alpha))\mathbb{P}(N_t=n)\\ &\quad+\sigma^2\sum_{n=0}^{\infty}C(\alpha)A(n,\alpha)\mathbb{P}(N_t=n)\\ &=: I_1(\alpha,t)+I_2(\alpha,t)+I_3(\alpha,t). \end{aligned}\tag{8}$$

According to (7),

$$|I_2(\alpha,t)|\leqslant\sigma^2\sum_{n\geqslant N+1}|\mathbb{E}(Z_n^{-1})-C(\alpha)A(n,\alpha)|\mathbb{P}(N_t=n)\leqslant\varepsilon I_3(\alpha,t).\tag{9}$$

For t large enough such that $\lambda t>N$, one has

$$|I_1(\alpha,t)|\leqslant C\mathbb{P}(N_t\leqslant N)\leqslant C(N+1)\frac{(\lambda t)^N}{N!}e^{-\lambda t}\to 0,\tag{10}$$

as $t\to\infty$.

Now we deal with $I_3(\alpha,t)$. If $\alpha<1$, $A(n,\alpha)=p_1^n$, then

$$\begin{aligned}I_3(\alpha,t) &= C(\alpha)\sigma^2\sum_{n=0}^{\infty}p_1^n\mathbb{P}(N_t=n)=C(\alpha)\sigma^2\sum_{n=0}^{\infty}\frac{(\lambda t p_1)^n}{n!}e^{-\lambda t}\\ &= C(\alpha)\sigma^2\exp(\lambda t(p_1-1)).\end{aligned}\tag{11}$$

If $\alpha=1$, $A(n,\alpha)=np_1^n$, then

$$\begin{aligned}I_3(\alpha,t) &= C(\alpha)\sigma^2\sum_{n=0}^{\infty}np_1^n\mathbb{P}(N_t=n)=C(\alpha)\sigma^2\sum_{n=0}^{\infty}n\frac{(\lambda t p_1)^n}{n!}e^{-\lambda t}\\ &= C(\alpha)\sigma^2\exp(\lambda t(p_1-1))\sum_{n=0}^{\infty}n\frac{(\lambda t p_1)^n}{n!}e^{-\lambda t p_1}\\ &= C(\alpha)\sigma^2\lambda t p_1\exp(\lambda t(p_1-1)).\end{aligned}\tag{12}$$

If $\alpha>1$, $A(n,\alpha)=m^{-n}$, then

$$\begin{aligned}I_3(\alpha,t) &= C(\alpha)\sigma^2\sum_{n=0}^{\infty}m^{-n}\mathbb{P}(N_t=n)=C(\alpha)\sigma^2\sum_{n=0}^{\infty}\frac{(\lambda t m^{-1})^n}{n!}e^{-\lambda t}\\ &= C(\alpha)\sigma^2\exp(\lambda t(m^{-1}-1)).\end{aligned}\tag{13}$$

We complete the proof of Theorem 2 by (8)–(13).

We give an example to illustrate Theorem 2. We choose corresponding branching law to satisfy $\alpha<1,\alpha=1$ and $\alpha>1$ respectively.

EXAMPLE 1. Choose three generating functions,

$$f(s)=\frac{s}{(4-3s^2)^{1/2}},\quad g(s)=\frac{s}{2-s},\quad h(s)=\frac{s}{(\sqrt{2}-(\sqrt{2}-1)s^{1/2})^2},$$

then corresponding Schröder index $\alpha_1=0.5<1,\alpha_2=1,\alpha_3=2>1$ and

$$\begin{aligned}C(\alpha_1,t)&=12\pi\exp(-0.5\lambda t),\\ C(\alpha_2,t)&=(\ln 2)\lambda t\exp(-0.5\lambda t),\\ C(\alpha_3,t)&=(2-\sqrt{2})\exp\left(\left(\frac{1}{\sqrt{2}}-1\right)\lambda t\right).\end{aligned}$$

Proof. For generating function $f(s)$, we know

$$f'(s)=\frac{4}{(4-3s^2)^{3/2}},\quad f''(s)=\frac{36s}{(4-3s^2)^{5/2}}.$$

Then

$$p_1=f'(0)=\frac{4}{8}=0.5,\;\; m_1=f'(1)=4,\;\; \sigma_1^2=f''(1)+m_1-m_1^2=24.$$

Thus $\alpha_1=-\log_4 0.5=0.5<1$. By iteration,

$$f_n(s)=\frac{s}{(4^n-(4^n-1)s^2)^{1/2}}.$$

So we have

$$Q_1(s)=\lim_{n\to\infty}\frac{2^n s}{(4^n-(4^n-1)s^2)^{1/2}}=\frac{s}{\sqrt{1-s^2}}.$$

Consequently,

$$\int_0^\infty Q_1(e^{-v})dv=\int_0^\infty\frac{e^{-v}}{\sqrt{1-e^{-2v}}}dv=\frac{\pi}{2}.$$

For generating function $g(s)$, one has

$$g'(s)=\frac{2}{(2-s)^2},\quad g''(s)=\frac{4}{(2-s)^3}.$$

Then

$$p_1=g'(0)=\frac{2}{4}=0.5,\;\; m_2=g'(1)=2,\;\; \sigma_2^2=g''(1)+m_2-m_2^2=2.$$

Thus $\alpha_2=-\log_2 0.5=1$. By iteration,

$$g_n(s)=\frac{s}{2^n-(2^n-1)s}.$$

So we have

$$Q_2(s) = \lim_{n\to\infty} \frac{2^n s}{2^n - (2^n - 1)s} = \frac{s}{1-s}.$$

According to Theorem 2, we need to calculate the Laplace transformation of W which is determined by

$$\phi_2(v) = \lim_{n\to\infty} g_n(\exp(-v/2^n)) = \lim_{n\to\infty} \frac{\exp(-v/2^n)}{2^n - (2^n - 1)\exp(-v/2^n)} = \frac{1}{1+v}.$$

Consequently,

$$\int_1^2 Q_2(\phi_2(v))dv = \int_1^2 \frac{\frac{1}{1+v}}{1 - \frac{1}{1+v}} dv = \ln(2).$$

Finally, for generating function $h(s)$, one has

$$h'(s) = \frac{\sqrt{2}}{(\sqrt{2} - (\sqrt{2}-1)s^{1/2})^3}, \quad h''(s) = \frac{3(2-\sqrt{2})s^{-1/2}}{2(\sqrt{2} - (\sqrt{2}-1)s^{1/2})^4}.$$

Then

$$p_1 = h'(0) = \frac{\sqrt{2}}{2^{3/2}} = 0.5, \;\; m_3 = h'(1) = \sqrt{2}, \;\; \sigma_3^2 = h''(1) + m_3 - m_3^2 = 1 - \frac{\sqrt{2}}{2}.$$

Thus $\alpha_3 = -\log_{\sqrt{2}} 0.5 = 2 > 1$. By iteration,

$$h_n(s) = \frac{s}{((\sqrt{2})^n - ((\sqrt{2})^n - 1)s^{1/2})^2}.$$

So we have

$$\begin{aligned}\phi_3(v) &= \lim_{n\to\infty} h_n\left(\exp\left(-v/2^{n/2}\right)\right) = \lim_{n\to\infty} \frac{\exp\left(-v/2^{n/2}\right)}{\left(2^{n/2} - (2^{n/2}-1)\exp\left(-v/2^{n/2+1}\right)\right)^2} \\ &= \frac{4}{(2+v)^2}.\end{aligned}$$

Consequently,

$$\int_0^\infty \phi_3(v)dv = 2.$$

We complete the proof of Example 1.

4. Asymptotic normality of $\sqrt{Y_t}(\mathbb{R}_t - m)$

In this section, we deal with the asymptotic normality of $\sqrt{Y_t}(\mathbb{R}_t - m)$.

The proof of Theorem 3.
Conditioning on Y_t,

$$G_t(x) - \Phi(x) = \sum_{k=1}^{\infty} \mathbb{P}\left(\frac{\sqrt{k}(S_k - m)}{k\sigma} \leqslant x\right)\mathbb{P}(Y_t = k) - \Phi(x)$$
$$= \sum_{k=1}^{\infty} (L_k(x) - \Phi(x))\mathbb{P}(Y_t = k), \tag{14}$$

where $L_k(x)$ is defined at the beginning of Section 2. According to (1), for any $\varepsilon \in (0,1)$, there exist $N = N(\varepsilon) > 0$ such that for any $k \geqslant N$, we have

$$L_k(x) \in (\Phi(x) - \varepsilon, \Phi(x) + \varepsilon). \tag{15}$$

We can divide (14) into the following two parts.

$$J_1(\varepsilon,t) = \sum_{k<N} (L_k(x) - \Phi(x))\mathbb{P}(Y_t = k),$$
$$J_2(\varepsilon,t) = \sum_{k\geqslant N} (L_k(x) - \Phi(x))\mathbb{P}(Y_t = k).$$

The rest of the proof is straightforward via (15).

The proof of Theorem 4.
Conditioning on Y_t, according to the Berry–Esseen bound for i.i.d. random variables, we obtain

$$\sup_{x\in\mathbb{R}} |G_t(x) - \Phi(x)| \leqslant \sum_{k=1}^{\infty} |L_k(x) - \Phi(x)|\mathbb{P}(Y_t = k) \leqslant C\sum_{k=1}^{\infty} k^{-1/2}\mathbb{P}(Y_t = k) = C\mathbb{E}(Y_t^{-1/2}).$$

The rest of the proof is similar to that of Theorem 2.

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