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STRONG CONSISTENCY OF LS ESTIMATOR IN SIMPLE LINEAR EV REGRESSION MODELS

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Abstract. Under some moment conditions on the errors, the sufficient and necessary conditions are given for the rate of the strong consistency of the least squares estimate in a simple linear errors-in-variables regression model.

1. Introduction

Consider the simple linear errors-in-variables (EV) regression model:

$$\eta_k = \theta + \beta x_k + \varepsilon_k, \ \ \xi_k = x_k + \delta_k, \ \ 1 \leqslant k \leqslant n, \tag{1.1}$$

where $\theta, \beta, x_1, \cdots, x_n$ are unknown parameters or constants, $(\varepsilon_k, \delta_k), 1 \leqslant k \leqslant n,$ are random vectors and $\xi_k, \eta_k, 1 \leqslant k \leqslant n,$ are observable. Form (1.1), we have

$$\eta_k = \theta + \beta \xi_k + (\varepsilon_k - \beta \delta_k), \ \ 1 \leqslant k \leqslant n.$$

Then, as a usual regression model of η_k on ξ_k with the errors $\varepsilon_k - \beta\delta_k$, the least squares (LS) estimators of β and θ are given as

$$\hat{\beta}_n = \frac{\sum_{k=1}^n (\xi_k - \bar{\xi}_n)(\eta_k - \bar{\eta}_n)}{\sum_{k=1}^n (\xi_k - \bar{\xi}_n)^2}, \ \ \hat{\theta}_n = \bar{\eta}_n - \hat{\beta}_n \bar{\xi}_n,$$

where $\bar{\xi}_n = n^{-1}\sum_{k=1}^n \xi_k$ and $\bar{\eta}_n = n^{-1}\sum_{k=1}^n \eta_k$, $\bar{\delta}_n$ and $\bar{x}_n$ are defined in the same way.

Based on the above notions, we have

$$\hat{\beta}_n - \beta = \frac{\sum_{k=1}^n (\delta_k - \bar{\delta}_n)\varepsilon_k + \sum_{k=1}^n (x_k - \bar{x}_n)(\varepsilon_k - \beta\delta_k) - \beta\sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2}{\sum_{k=1}^n (\xi_k - \bar{\xi}_n)^2} \tag{1.2}$$

and

$$\hat{\theta}_n - \theta = -\bar{x}_n(\hat{\beta}_n - \beta) - (\hat{\beta}_n - \beta)\bar{\delta}_n + \bar{\varepsilon}_n - \beta\bar{\delta}_n. \tag{1.3}$$

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This model was proposed by Deaton [3] to correct the effects of the sampling errors and was somewhat more practical than the ordinary regression model. Fuller [4] summarized many early works for the EV models. Due to the simple form and wide applicability, the studies for the EV model have attracted much attention for the past two decades. For more details, we refer to Liu and Chen [6], Miao et al. [7], Wang et al. [10], Hu et at. [5], Shen [8], Wang and Hu [9], Wang et al. [11], and Wu et al. [12], and so on. In particular, Liu and Chen [6] obtained the sufficient and necessary conditions of the strong consistency for the unknown parameter β, and Hu et al. [5] for the unknown parameter θ as follows.

THEOREM A. *Under the model (1.1), assume that* $\{(\varepsilon,\delta),(\varepsilon_n,\delta_n),n\geqslant 1\}$ *is a sequence of independent and identically distributed random vectors with* $E\varepsilon=E\delta=0$, *and* $0<E\varepsilon^2,E\delta^2<\infty$. *Then*

$$\hat{\beta}_n\to\beta\ \ a.s.\ if\ and\ only\ if\ \ s_n/n\to\infty,$$

where and in the following, $s_n=\sum_{k=1}^n(x_k-\bar{x}_n)^2,\ n\geqslant 1$. *And further assume that* $\sup_{n\geqslant 1}n\bar{x}_n^2/\max\{n,s_n\}<\infty$, *then*

$$\hat{\theta}_n\to\theta\ \ a.s.\ if\ and\ only\ if\ \ n\bar{x}_n/\max\{n,s_n\}\to 0.$$

Theorem A characterizes the relation between the strong consistency for the unknown parameters and the dispersion of the unknown constants x_n. Theorem A has been generalized and extended by many authors. For example, Miao et al. [7] obtained the rate of strong consistency for the unknown parameter β.

THEOREM B. *Under the model (1.1), assume that* $\{(\varepsilon,\delta),(\varepsilon_n,\delta_n),n\geqslant 1\}$ *is a sequence of independent and identically distributed random vectors with* $E\varepsilon=E\delta=0$, *and* $0<E|\varepsilon|^q,E|\delta|^q<\infty$ *for some* $q\geqslant 2$. *Then* $n^{2-2/q}/s_n\to 0$ *implies that*

$$\frac{\sqrt{s_n}}{n^{1/q}}(\hat{\beta}_n-\beta)\to 0\ \ a.s.$$

When $q\geqslant 2$, $2-2/q\geqslant 2/q$ and hence $\sqrt{s_n}/n^{1/q}\to\infty$, so Theorem B gives the rate of strong consistency for the unknown parameter β. In fact, Miao et al. [7] also gave the rate of strong consistency for the unknown parameter θ, but their result does not include Theorem A.

In the paper, another kind of rate of the strong consistency of LS estimators for the unknown parameters is given. The first one is for the unknown parameter β.

THEOREM 1.1. *Under the model (1.1), assume that* $\{(\varepsilon,\delta),(\varepsilon_n,\delta_n),n\geqslant 1\}$ *is a sequence of independent and identically distributed random vectors with* $E\varepsilon=E\delta=0$, $0<E|\varepsilon|^q,E|\delta|^q<\infty$ *for* $1/p=1/q+1/2$, *where* $1\leqslant p<2$, *and* $E(\delta\varepsilon)\neq\beta E\delta^2$. *Then*

$$n^{1-1/p}(\hat{\beta}_n-\beta)\to 0\ \ a.s.\ if\ and\ only\ if\ \ n^{2-1/p}/s_n\to 0.$$

For the unknown parameter θ, we have

THEOREM 1.2. *Under the assumptions of Theorem 1.1, if* $\sup_{n\geqslant 1}\min\{n,s_n\}\bar{x}_n^2/s_n^* < \infty$*, then*

$$n^{1-1/p}(\hat{\theta}_n-\theta)\to 0 \text{ a.s. if and only if } n^{2-1/p}\bar{x}_n/s_n^*\to 0, \tag{1.4}$$

where $s_n^*=\max\{n,s_n\}$.

REMARK 1.1. When $1<p<2$, $n^{1-1/p}\to\infty$ and hence Theorem 1.1 and Theorem 1.2 also give the rates of strong consistency for the unknown parameters β and θ, respectively.

REMARK 1.2. We compare Theorem 1.1 to Theorem B under the same moment conditions. In this case, $1/p=1/q+1/2$ and hence $n^{2-2/q}/s_n\to 0$ implies $n^{2-1/p}/s_n\to 0$ and $n^{1-1/p}/(\sqrt{s_n}/n^{1/q})\to 0$. So the rate of strong consistency in Theorem 1.1 is weaker than that in Theorem B. However, the restrictive condition on s_n in Theorem 1.1 is also weaker than that in Theorem B when $1<p<2$ (or $q>2$ equivalently). Therefore, the two theorems are not included in each other when $1<p<2$ (or $q>2$ equivalently). Of course, Theorem B includes Theorem 1.1 when $p=1$ (or $q=2$ equivalently).

REMARK 1.3. Theorem 1.1 obtains the sufficient and necessary condition for the rate of the strong consistency for the unknown parameter β, but Theorem B does not. We guess that the necessary condition is also $n^{2-2/q}/s_n\to 0$ in Theorem B. In fact, Chen et al. [2] got the results recently.

REMARK 1.4. Under correlated and heterogeneous errors, Wang and Hu [9] and Wang et al. [11] obtained the different convergence rate of the strong consistency, but did not discuss the sufficient and necessary condition. To our knowledge, no literature obtains the sufficient and necessary condition for the rate of the strong consistency for the unknown parameter. So the main results in the paper are new.

REMARK 1.5. The condition $\sup_{n\geqslant 1}\min\{n,s_n\}\bar{x}_n^2/s_n^*<\infty$ in Theorem 1.2 is slightly weaker than $\sup_{n\geqslant 1}n\bar{x}_n^2/s_n^*<\infty$ in Theorem A.

2. Lemmas and proofs of main results

To prove the main results, the strong law of large numbers for weighted sums of random variables is needed. One can refer to Bai and Cheng [1].

LEMMA 2.1. *Let* $1\leqslant p<2$, $1<q,r<\infty$ *with* $1/p=1/q+1/r$. *Let* $\{X,X_n,n\geqslant 1\}$ *be a sequence of independent and identically distributed random variables,* $\{a_{nk},n\geqslant 1,1\leqslant k\leqslant n\}$ *be an array of constants with*

$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{k=1}^{n}|a_{nk}|^r<\infty.$$

Assume that $EX=0$ *and* $E|X|^q<\infty$*, then*

$$n^{-1/p}\sum_{k=1}^{n}a_{nk}X_k\to 0 \ \ a.s.$$

Proof of Theorem 1.1. By Theorem A, we only prove the case $1 < p < 2$.

Sufficiency. Assume that $n^{2-1/p}/s_n \to 0$. From (1.2), to prove $n^{1-1/p}(\hat{\beta}_n - \beta) \to 0$ a.s., it suffices to prove that

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k \to 0 \text{ a.s.} \tag{2.1}$$

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (x_k - \bar{x}_n)(\varepsilon_k - \beta\delta_k) \to 0 \text{ a.s.} \tag{2.2}$$

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 \to 0 \text{ a.s.} \tag{2.3}$$

$$s_n^{-1} \sum_{k=1}^{n} (\xi_k - \bar{\xi}_n)^2 \to 1 \text{ a.s.} \tag{2.4}$$

By the Kolmogorov strong law of large numbers,

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k = \frac{n^{2-1/p}}{s_n} \cdot \left(\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k\delta_k - \bar{\varepsilon}_n\bar{\delta}_n \right) \to 0 \times [E(\varepsilon\delta) - 0] = 0 \text{ a.s.}$$

and

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 = \frac{n^{2-1/p}}{s_n} \cdot \left(\frac{1}{n} \sum_{k=1}^{n} \delta_k^2 - \bar{\delta}_n^2 \right) \to 0 \times (E\delta^2 - 0) = 0 \text{ a.s.}$$

Hence, (2.1) and (2.3) hold. Set $a_{nk} = n(x_k - \bar{x}_n)/s_n$ for $n \geqslant 1$ and $1 \leqslant k \leqslant n$. Then

$$\sup_{n \geqslant 1} n^{-1} \sum_{k=1}^{n} |a_{nk}|^2 = \sup_{n \geqslant 1} \frac{n}{s_n} = \sup_{n \geqslant 1} \left(\frac{n^{2-1/p}}{s_n} \cdot n^{-1+1/p} \right) < \infty.$$

Therefore by Lemma 2.1 with $r = 2$,

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (x_k - \bar{x}_n)\varepsilon_k = n^{-1/p} \sum_{k=1}^{n} a_{nk}\varepsilon_k \to 0 \text{ a.s.} \tag{2.5}$$

and

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (x_k - \bar{x}_n)\delta_k = n^{-1/p} \sum_{k=1}^{n} a_{nk}\delta_k \to 0 \text{ a.s.} \tag{2.6}$$

Then (2.2) holds from (2.5) and (2.6). Note that

$$s_n^{-1} \sum_{k=1}^{n} (\xi_k - \bar{\xi}_n)^2 = 1 + 2s_n^{-1} \sum_{k=1}^{n} (x_k - \bar{x}_n)\delta_k + s_n^{-1} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2.$$

Then by (2.6) and (2.3), (2.4) holds.

Necessity. Suppose that $n^{2-1/p}/s_n \to 0$ does not hold. Taking subsequence when necessary, we may assume that

$$n^{2-1/p}/s_n \to c \in (0, \infty] \text{ as } n \to \infty. \tag{2.7}$$

By (2.7) and the Kolmogorov strong law of large numbers,

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k = \frac{n^{2-1/p}}{s_n} \cdot \frac{1}{n} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k \to cE(\varepsilon\delta) \text{ a.s.} \tag{2.8}$$

and

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 = \frac{n^{2-1/p}}{s_n} \cdot \frac{1}{n} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 \to cE\delta^2 \text{ a.s.} \tag{2.9}$$

Set $a_{nk} = n(x_k - \bar{x}_n)/s_n$ for $n \geqslant 1$ and $1 \leqslant k \leqslant n$. Note that $n^{1-1/p}(\hat{\beta}_n - \beta) \to 0$ a.s. implies $\hat{\beta}_n \to \beta$ a.s. since $1 < p < 2$, then $n/s_n \to 0$ from Theorem A. Hence,

$$\sup_{n\geqslant 1} n^{-1} \sum_{k=1}^{n} a_{nk}^2 = \sup_{n\geqslant 1} \frac{n}{s_n} < \infty.$$

Therefore by Lemma 2.1 with $r = 2$,

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (x_k - \bar{x}_n)\varepsilon_k = n^{-1/p} \sum_{k=1}^{n} a_{nk}\varepsilon_k \to 0 \text{ a.s.}$$

and

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (x_k - \bar{x}_n)\delta_k = n^{-1/p} \sum_{k=1}^{n} a_{nk}\delta_k \to 0 \text{ a.s.,} \tag{2.10}$$

which follow that

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (x_k - \bar{x}_n)(\varepsilon_k - \beta\delta_k) \to 0 \text{ a.s.} \tag{2.11}$$

By (2.10), $n/s_n \to 0$, and the Kolomogorov strong law of large numbers,

$$\begin{aligned} & s_n^{-1} \sum_{k=1}^{n} (\xi_k - \bar{\xi}_n)^2 \\ =& 1 + \frac{2}{s_n} \sum_{k=1}^{n} (x_k - \bar{x}_n)\delta_k + \frac{1}{s_n} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 \\ =& 1 + 2n^{-1+1/p} \cdot n^{-1/p} \sum_{k=1}^{n} \frac{n(x_k - \bar{x}_n)}{s_n}\delta_k + \frac{n}{s_n} \cdot \frac{1}{n} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 \\ & \to 1 \text{ a.s.} \end{aligned} \tag{2.12}$$

Thus by (1.2), (2.9), (2.11), and (2.12),

$$n^{1-1/p}(\hat{\beta}_n - \beta) \to c[E(\varepsilon\delta) - \beta E\delta^2] \text{ a.s.,}$$

which leads to a contradiction with $n^{1-1/p}(\hat{\beta}_n - \beta) \to 0$ a.s., so we have $n^{2-1/p}/s_n \to 0$. The proof is completed. □

To prove Theorem 1.2, the follow lemma is also needed, and (2.14) is interesting itself.

LEMMA 2.2. *Under the model (1.1), assume that* $\{(\varepsilon,\delta),(\varepsilon_n,\delta_n),n\geqslant 1\}$ *is a sequence of independent and identically distributed random vectors with* $E\varepsilon = E\delta = 0$, $0 < E\varepsilon^2, E\delta^2 < \infty$. *Then*

$$\min\{1,E\delta^2\} \leqslant \liminf_{n\to\infty}\frac{1}{s_n^*}\sum_{k=1}^{n}(\xi_k-\bar{\xi}_n)^2 \leqslant \limsup_{n\to\infty}\frac{1}{s_n^*}\sum_{k=1}^{n}(\xi_k-\bar{\xi}_n)^2 \leqslant 1+E\delta^2 \ \ a.s., \tag{2.13}$$

where $s_n^* = \max\{n,s_n\}$ *as Theorem 1.2, and*

$$\limsup_{n\to\infty}|\hat{\beta}_n-\beta| \leqslant \frac{|E(\varepsilon\delta)|+|\beta|E\delta^2}{\min\{1,E\delta^2\}} \ \ a.s. \tag{2.14}$$

Proof. It is clear that

$$\frac{1}{s_n^*}\sum_{k=1}^{n}(\xi_k-\bar{\xi}_n)^2 = \frac{s_n}{s_n^*}+\frac{2}{s_n^*}\sum_{k=1}^{n}(x_k-\bar{x}_n)\delta_k+\frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2. \tag{2.15}$$

Set $a_{nk} = n(x_k-\bar{x}_n)/s_n^*$, then

$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{k=1}^{n}a_{nk}^2 = \sup_{n\geqslant 1}\frac{ns_n}{(s_n^*)^2} \leqslant 1 < \infty.$$

Therefore by Lemma 2.1 with $p=1$, and $q=r=2$,

$$\frac{1}{s_n^*}\sum_{k=1}^{n}(x_k-\bar{x}_n)\delta_k = \frac{1}{n}\sum_{k=1}^{n}a_{nk}\delta_k \to 0 \ \ \text{a.s.} \tag{2.16}$$

By the Kolmogorov strong law of large numbers,

$$\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2 = \frac{1}{n}\sum_{k=1}^{n}\delta_k^2-\bar{\delta}_n^2 \to E\delta^2 \ \ \text{a.s.}, \tag{2.17}$$

which and the definition of s_n^* follow that

$$\begin{aligned}\min\{1,E\delta^2\} &\leqslant \liminf_{n\to\infty}\left(\frac{s_n}{s_n^*}+\frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2\right)\\ &\leqslant \limsup_{n\to\infty}\left(\frac{s_n}{s_n^*}+\frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2\right) \leqslant 1+E\delta^2 \ \ \text{a.s.}\end{aligned} \tag{2.18}$$

Then by (2.15), (2.16), and (2.18), (2.13) holds.

By the Kolmogorov strong law of large numbers,

$$\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k = \frac{1}{n}\sum_{k=1}^{n}\delta_k\varepsilon_k-\bar{\delta}_n\bar{\varepsilon}_n \to E(\varepsilon\delta) \ \ \text{a.s.},$$

and hence

$$\limsup_{n\to\infty}\left|\frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k\right| = \limsup_{n\to\infty}\left|\frac{n}{s_n^*}\cdot\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k\right| \leqslant |E(\varepsilon\delta)| \ \text{ a.s.} \tag{2.19}$$

By the same argument as (2.16),

$$\frac{1}{s_n^*}\sum_{k=1}^{n}(x_k-\bar{x}_n)(\varepsilon_k-\beta\delta_k)\to 0 \ \text{ a.s.} \tag{2.20}$$

Therefore, (2.14) follows from (1.2), (2.13), (2.17), (2.19), and (2.20). □

Proof of Theorem 1.2. By the Marcinkiewicz-Zygmund strong law of large numbers,

$$n^{1-1/p}\bar{\varepsilon}_n\to 0 \ \text{ a.s. and } \ n^{1-1/p}\bar{\delta}_n\to 0 \ \text{ a.s.},$$

and hence by Lemma 2.2,

$$n^{1-1/p}(\hat{\beta}_n-\beta)\bar{\delta}_n = n^{1-1/p}\bar{\delta}_n\cdot(\hat{\beta}_n-\beta)\to 0 \ \text{ a.s.}$$

Then by (1.3), to prove (1.4), it is equivalent to prove that

$$n^{1-1/p}\cdot\bar{x}_n(\hat{\beta}_n-\beta)\to 0 \text{ a.s. if and only if } n^{2-1/p}\bar{x}_n/s_n^*\to 0. \tag{2.21}$$

Sufficiency. Assume that $n^{2-1/p}\bar{x}_n/s_n^*\to 0$. By the Kolmogorov strong law of large numbers,

$$n^{1-1/p}\cdot\frac{\bar{x}_n}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k = \frac{n^{2-1/p}\bar{x}_n}{s_n^*}\cdot\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k\to 0\times E(\varepsilon\delta)=0 \ \text{ a.s.} \tag{2.22}$$

and

$$n^{1-1/p}\cdot\frac{\bar{x}_n}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2 = \frac{n^{2-1/p}\bar{x}_n}{s_n^*}\cdot\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2\to 0 \ \text{ a.s.} \tag{2.23}$$

Set $a_{nk}=n\bar{x}_n(x_k-\bar{x}_n)/s_n^*$, note that

$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{k=1}^{n}a_{nk}^2 = \sup_{n\geqslant 1}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{n\bar{x}_n(x_k-\bar{x}_n)}{s_n^*}\right|^2 = \sup_{n\geqslant 1}\frac{n\bar{x}_n^2 s_n}{(s_n^*)^2} = \sup_{n\geqslant 1}\frac{\min\{n,s_n\}\bar{x}_n^2}{s_n^*}<\infty.$$

Hence by Lemma 1.2 with $r=2$,

$$n^{1-1/p}\cdot\frac{\bar{x}_n}{s_n^*}\sum_{k=1}^{n}(x_k-\bar{x}_n)(\varepsilon_k-\beta\delta_k) = \frac{1}{n^{1/p}}\sum_{k=1}^{n}\frac{n\bar{x}_n(x_k-\bar{x}_n)}{s_n^*}(\varepsilon_k-\beta\delta_k)\to 0 \ \text{ a.s.} \tag{2.24}$$

By (1.2), (2.22), (2.23), (2.24), and Lemma 2.2,

$$n^{1-1/p}\cdot\bar{x}_n(\hat{\beta}_n-\beta)\to 0 \ \text{ a.s.}$$

Necessity. Suppose that $n^{2-1/p}\bar{x}_n/s_n^* \to 0$ does not hold. Taking subsequence when necessary, we may assume that

$$n^{2-1/p}\bar{x}_n/s_n^* \to c \neq 0 \text{ as } n \to \infty. \tag{2.25}$$

By the Kolmogorov strong law of large numbers,

$$n^{1-1/p} \cdot \frac{\bar{x}_n}{s_n^*} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k = \frac{n^{2-1/p}\bar{x}_n}{s_n^*} \cdot \frac{1}{n} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k \to cE(\varepsilon\delta) \text{ a.s.} \tag{2.26}$$

and

$$n^{1-1/p} \cdot \frac{\bar{x}_n}{s_n^*} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 = \frac{n^{2-1/p}\bar{x}_n}{s_n^*} \cdot \frac{1}{n} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 \to cE\delta^2 \text{ a.s.} \tag{2.27}$$

Hence, by (1.2), (2.24), (2.26), (2.27), and Lemma 2.2,

$$\liminf_{n\to\infty} |\bar{x}_n(\hat{\beta}_n - \beta)| \geqslant \frac{|c[E(\varepsilon\delta) - \beta E\delta^2]|}{1 + E\delta^2} \text{ a.s.,}$$

which leads to a contradiction with $n^{1-1/p} \cdot \bar{x}_n(\hat{\beta}_n - \beta) \to 0$ a.s., so we have $n^{2-1/p}\bar{x}_n/s_n^* \to 0$. We complete the proof. □

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