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MORREY TYPE TEICHMÜLLER SPACE AND HIGHER BERS MAPS

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Abstract. In this paper, we focus on the set of univalent analytic functions f with $\log f' \in H_K^2$. Motivated by the study of BMO-Teichmüller spaces and Morrey type spaces, we establish serval equivalent characterizations of Morrey type domains. Furthermore, we show that the higher Bers maps, induced by the higher Schwarzian differential operators, are holomorphic in Morrey type Teichmüller spaces. Finally, one of connected components in the small pre-logarithmic derivative model of the Morrey type Teichmüller space is also obtained.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disc in the extended complex plane $\widehat{\mathbb{C}}$. Denote by $\mathbb{D}^*$ the exterior of $\overline{\mathbb{D}}$ and $S^1 = \partial\mathbb{D}$ the boundary of $\mathbb{D}$. Here use $\mathscr{A}(\mathbb{D})$ to denote the set of all analytic functions defined in $\mathbb{D}$. Throughout this paper, the notation $a \lesssim b$ stands for the fact that there is a constant $C > 0$ such that $a < Cb$ and the notation $a \approx b$ indicates that $a \lesssim b \lesssim a$. Let $M(\mathbb{D}^*)$ be the open unit ball of the Banach space $L^\infty(\mathbb{D}^*)$ of all Beltrami differentials $\mu(z)$ on $\mathbb{D}^*$, which have finite L_∞-norms. For $\mu(z) \in M(\mathbb{D}^*)$, there exists a unique quasiconformal mapping $f^\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ whose complex dilatation is μ in $\mathbb{D}^*$ and is zero in $\mathbb{D}$, normalized by

$$f^\mu(0) = (f^\mu)'(0) - 1 = (f^\mu)''(0) = 0.$$

We say that two Beltrami coefficients μ_1 and μ_2 in $M(\mathbb{D}^*)$ are equivalent and denoted by $\mu_1 \sim \mu_2$, if $f^{\mu_1}|_{\mathbb{D}} = f^{\mu_2}|_{\mathbb{D}}$. Then the universal Teichmüller space T is the space of equivalent classes and can be represented as

$$T = M(\mathbb{D}^*)/\sim = \{[\mu] : \mu \in M(\mathbb{D}^*)\},$$

where $[\mu]$ is the equivalent class among $\mu \in M(\mathbb{D}^*)$.

A Beltrami differential $\mu(z) \in M(\mathbb{D}^*)$ is vanishing at the boundary of $\mathbb{D}^*$, if for any $\varepsilon > 0$, there exists $r > 1$ such that $\|\mu|_{|z|<r}\|_\infty < \varepsilon$. Moreover, the small Teichmüller space is defined by

$$T^0 = \{[\mu] : \mu \in M^0(\mathbb{D}^*)\}.$$

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It is a subspace of the universal Teichmüller space T, where $M^0(\mathbb{D}^*)$ consists of all vanishing Beltrami differentials.

Denote by S_Q the class of all univalent analytic functions f in $\mathbb{D}$ normalized by $f(0)=f'(0)-1=0$, which can be quasiconformal extended to $\widehat{\mathbb{C}}$. It is known that

$$T(1)=\{\log f' : f \text{ belongs to } S_Q\}$$

is an alternative model called the pre-logarithmic derivative model of universal Teichmüller space. In [29], $T(1)$ is a disconnected subset of Bloch space $\mathfrak{B}^1$. The connected components of $T(1)$ include $T_b=\{\log f'\in T(1): f(\mathbb{D}) \text{ is bounded}\}$ and $T_\theta=\{\log f'\in T(1): f(e^{i\theta})=\infty\}$, $\theta\in[0,2\pi)$. Analogously, the small pre-logarithmic derivative model of universal Teichmüller space is defined by

$$T^0(1)=\{\log f'\in T(1): \log f'\in \mathfrak{B}_0^1\}.$$

It is well known that $\log f'$ is in the small Bloch space $\mathfrak{B}_0^1$ if and only if its complex dilatation $\mu_f(z)$ belongs to $M^0(\mathbb{D}^*)$.

Recently, some other subspaces of Teichüller spaces, combined with BMO spaces, VMO spaces, Q_K spaces, $F(p,q,s)$ spaces and Dirichlet Morrey spaces, have been widely studied (see [1], [4], [5], [8], [9], [19], [12], [25], [20] and [21] for more details).

In this paper, motivated by the study of BMO Teichmüller spaces and Morrey type spaces, we introduce Morrey type Teichmüller spaces as follows.

Let $K:[0,\infty)\to[0,\infty)$ be a right-continuous and nondecreasing function satisfying $K(t)=K(1)$ for $t\geqslant 1$, $K(2t)\approx K(t)$ and the following conditions:

$$\int_0^{\frac{1}{e}} K\left(\log\frac{1}{r}\right)dr<\infty; \tag{1.1}$$

$$\int_0^1 \varphi_K(s)\frac{ds}{s}<\infty; \tag{1.2}$$

and

$$\int_1^\infty \varphi_K(s)\frac{ds}{s^{1+p}}<\infty, \quad 0<p<2; \tag{1.3}$$

where

$$\varphi_K(s)=\sup_{0<t\leqslant 1} K(st)/K(t), \quad 0<s<\infty. \tag{1.4}$$

It is not difficult to verify that $K(t)=t^q, 0\leqslant q\leqslant 1$, satisfies $(1.1)-(1.3)$.

For $1\leqslant n<\infty$, the Hardy space H^n consists of all $f\in\mathscr{A}(\mathbb{D})$ with

$$\|f\|_{H^n}^n=\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^n d\theta<\infty.$$

The Morrey type space H_K^2 consists of all functions $f\in H^2$ with

$$\|f\|_{H_K^2}=\sup_{I\subset\partial\mathbb{D}}\left(\frac{1}{K(|I|)}\int_I|f(\zeta)-f_I|^2\frac{|d\zeta|}{2\pi}\right)^{\frac{1}{2}}<\infty,$$

where

$$f_I = \frac{1}{|I|}\int_I f(\zeta)\frac{|d\zeta|}{2\pi}$$

is the average of f over I and

$$|I| = \frac{1}{2\pi}\int_I |d\zeta|$$

is the length of subarc $I \subset S^1$.

The small Morrey type space $H^2_{K,0}$ consists of all functions $f \in H^2_K$ satisfying

$$\lim_{|I|\to 0}\frac{1}{K(|I|)}\int_I |f(\zeta)-f_I|^2\frac{|d\zeta|}{2\pi} = 0.$$

The analytic Morrey (small Morrey) space is the Morrey (small Morrey) type space when $K(t)=t^{\lambda}$ $(0<\lambda\leqslant 1)$. Specially, $H^2_K = \mathrm{BMOA}$ and $H^2_{K,0} = \mathrm{VMOA}$ when $K(t)=t$. The properties of analytic Morrey spaces can be found in [14, 15, 24].

Next, we introduce (vanishing) K-Carleson measures and α-Bloch (small α-Bloch) spaces.

Let

$$S_{\mathbb{D}}(I) = \{r\zeta \in \mathbb{D} : 1-|I| \leqslant r < 1, \zeta \in I\}$$

and

$$S_{\mathbb{D}^*}(I) = \{r\zeta \in \mathbb{D}^* : 1 \leqslant r < 1+|I|, \zeta \in I\}$$

be Carleson squares in $\mathbb{D}$ and $\mathbb{D}^*$, respectively.

A non-negative measure μ on $\mathbb{D}$ is called K-Carleson measure if

$$\|\mu\|_{\mathbb{D},K} = \sup_{I\subset\partial\mathbb{D}}\left(\frac{\mu(S_{\mathbb{D}}(I))}{K(|I|)}\right)^{\frac{1}{2}} < \infty.$$

Moreover, if in addition

$$\lim_{|I|\to 0}\left(\frac{\mu(S_{\mathbb{D}}(I))}{K(|I|)}\right) = 0,$$

μ is called vanishing K-Carleson measure on $\mathbb{D}$. When $K(t)=t^{\lambda}$ $(0<\lambda\leqslant 1)$, the (vanishing) K-Carleson measure is the (vanishing) λ-Carleson measure. In particular, it is the classical Carleson measure when $K(t)=t$. Similarly, we can define the (vanishing) K-Carleson measure on $\mathbb{D}^*$. Let $CM_K(\mathbb{D})$ $(CM_{K,0}(\mathbb{D}))$ and $CM_K(\mathbb{D}^*)$ $(CM_{K,0}(\mathbb{D}^*))$ be the set of all (vanishing) K-Carleson measures on $\mathbb{D}$ and $\mathbb{D}^*$, respectively.

For $\alpha \in (0,\infty)$, the α-Bloch space $\mathfrak{B}^{\alpha}$ [28] is defined as

$$\mathfrak{B}^{\alpha} := \left\{h \in \mathscr{A}(\mathbb{D}) : \|h\|_{\mathfrak{B}^{\alpha}} := \sup_{z\in\mathbb{D}}|h'(z)|(1-|z|^2)^{\alpha} < \infty\right\}.$$

The small α-Bloch space $\mathfrak{B}^{\alpha}_0$ [28] is the subspace of $\mathfrak{B}^{\alpha}$ consisting of functions h satisfying

$$\lim_{|z|\to 1^-}|h'(z)|(1-|z|^2)^{\alpha} = 0.$$

Clearly, $\mathfrak{B}^\alpha$ ($\mathfrak{B}_0^\alpha$) is the classical Bloch (small Bloch) space when $\alpha = 1$.

In [23], a characterization of pre-logarithmic derivatives $\log f'$ in the Morrey space is described by Schwarzian derivatives S_f for univalent analytic functions f in $\mathbb{D}$, which are defined as

$$S_f(z) = N_f'(z) - \frac{1}{2}N_f^2(z),$$

where

$$N_f(z) = (\log f')'(z) = \frac{f''(z)}{f'(z)}$$

are pre-Schwarz derivatives of f.

THEOREM A. *([23]) Let $K(t) = t^\lambda$ $(0 < \lambda \leqslant 1)$ and f be a univalent analytic function in $\mathbb{D}$. Then the following statements hold.*

(1) If $\log f' \in H_K^2(H_{K,0}^2)$, then $\log f' \in \mathfrak{B}^1$ and

$$|S_f(z)|^2(1-|z|^2)^3 dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}));$$

(2) If $|S_f(z)|^2(1-|z|^2)^3 dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$ and $\log f' \in \mathfrak{B}_0^1$, then $\log f' \in H_K^2(H_{K,0}^2)$.

In this paper, we generalize Theorem A to

THEOREM 1.1. *Let $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function satisfying $K(t) = K(1)$ for $t \geqslant 1$, $K(2t) \approx K(t)$ and $(1.1)-(1.3)$. Suppose that f is a univalent analytic function in $\mathbb{D}$. Then the following statements hold.*

(1) If $\log f' \in H_K^2(H_{K,0}^2)$, then $\log f' \in \mathfrak{B}^1$ and

$$|S_f(z)|^2(1-|z|^2)^3 dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}));$$

(2) If $|S_f(z)|^2(1-|z|^2)^3 dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$ and $\log f' \in \mathfrak{B}_0^1$, then $\log f' \in H_K^2(H_{K,0}^2)$.

Let f be a univalent analytic function in $\mathbb{D}$, normalized by $f(0) = f'(0) - 1 = 0$, which can be extended to a quasiconformal mapping in the extended complex plane $\widehat{\mathbb{C}}$. There is a quasisymmetric homeomorphism $h = f^{-1} \circ g : S^1 \to S^1$, called the conformal welding corresponding to f [13], if $g : \mathbb{D}^* \to \widehat{\mathbb{C}} - \overline{f(\mathbb{D})}$ is a conformal mapping with $g(\infty) = \infty$.

For a quasisymmetric homeomorphism $h : S^1 \to S^1$, there are two important kernel functions [11]:

$$\phi_h(\zeta, z) = \frac{1}{2\pi i}\int_{S^1} \frac{h(w)}{(1-\zeta w)^2(1-zh(w))} dw, \quad (\zeta, z) \in \mathbb{D}\times\mathbb{D}$$

and

$$U(f,\zeta,z) = \frac{f'(\zeta)f'(z)}{[f(\zeta)-f(z)]^2} - \frac{1}{(\zeta-z)^2}, \quad (\zeta,z) \in \mathbb{D}\times\mathbb{D},$$

where the kernel $U(f,\zeta,z)$ is called Grunsky kernel function. Let

$$\phi_h(z)=\left(\frac{1}{\pi}\int_{\mathbb{D}}|\phi_h(\zeta,z)|^2 d\xi d\eta\right)^{\frac{1}{2}},\quad z\in\mathbb{D}$$

and

$$U(f,z)=\left(\frac{1}{\pi}\int_{\mathbb{D}}|U(f,\zeta,z)|^2 d\xi d\eta\right)^{\frac{1}{2}},\quad z\in\mathbb{D}.$$

One of main results is the following

THEOREM 1.2. *Let K be the same as that in Theorem 1.1. Suppose that f is a univalent analytic function in $\mathbb{D}$, normalized by $f(0)=f'(0)-1=0$, which can be extended to a quasiconformal mapping in the extended complex plane $\widehat{\mathbb{C}}$ and $\log f'\in\mathfrak{B}_0^1$. Then the following statements are equivalent:*

(1) $\log f'\in H_K^2(H_{K,0}^2)$;

(2) $|S_f(z)|^2(1-|z|^2)^3 dxdy\in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$;

(3) f can be extended to a quasiconformal mapping to the whole plane such that its complex dilatation μ satisfies $\frac{|\mu(z)|^2}{(|z|^2-1)}dxdy\in CM_K(\mathbb{D}^*)(CM_{K,0}(\mathbb{D}^*))$;

(4) $U^2(f,z)(1-|z|^2)dxdy\in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$;

(5) $\phi_h^2(\bar{z})(1-|z|^2)dxdy\in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$, *where h is the conformal welding corresponding to f.*

According to Theorem 1.2, we introduce the Morrey type Teichmüller space as follows. Let $\mathscr{L}(\mathbb{D}^*)$ be the Banach space of all essentially bounded measurable functions μ on $\mathbb{D}^*$ with $\lambda_\mu=\frac{|\mu(z)|^2}{|z|^2-1}dxdy$ being K-Carleson measures. The norm of $\mu\in\mathscr{L}(\mathbb{D}^*)$ is defined by the form as

$$\|\mu\|_{\mathscr{L}}=\|\mu\|_\infty+\|\lambda_\mu\|_{\mathbb{D}^*,K}<\infty.$$

We set

$$\mathfrak{M}(\mathbb{D}^*)=M(\mathbb{D}^*)\cap\mathscr{L}(\mathbb{D}^*)\ \text{ and }\ \mathfrak{M}^0(\mathbb{D}^*)=M^0(\mathbb{D}^*)\cap\mathscr{L}(\mathbb{D}^*).$$

Then, μ belongs to $\mathfrak{M}(\mathbb{D}^*)$ if and only if

$$\|\mu\|_{\mathfrak{M}}=\ln\left(\frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}\right)+\|\lambda_\mu\|_{\mathbb{D}^*,K}<\infty.$$

Here,

$$T_{MT}=\{[\mu]\in T:\mu\in\mathfrak{M}(\mathbb{D}^*)\}\ \text{ and }\ T_{MT}^0=\{[\mu]\in T^0:\mu\in\mathfrak{M}^0(\mathbb{D}^*)\}$$

are called the Morrey type Teichmüller space and the small Morrey type Teichmüller space, respectively.

The higher Bers maps are studied in [3], which are defined by higher Schwarzian derivatives introduced in [18]. The higher Schwarzian derivatives $\sigma_n(f)$ $(n \geqslant 3)$ of a univalent function f are generalizations of the classical Schwarzian derivative S_f with $\sigma_3(f) = S_f$ and

$$\sigma_{n+1}(f)(z) = \sigma_n'(f)(z) - (n-1)N_f(z)\sigma_n(f)(z), \quad n \geqslant 3.$$

Let $\mathscr{N}_{K,n}$ $(n \geqslant 3)$ be the space of all $f \in \mathscr{A}(\mathbb{D})$ satisfying

$$\|f\|^2_{\mathscr{N}_{K,n}} = \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^{2n-3} \frac{(1-|a|^2)}{|1-\bar{a}z|^2} dxdy < \infty.$$

The holomorphy of higher Bers map in the BMO Teichmüller space is obtained in [22] as follows.

THEOREM B. *([22]) Let $K(t) = t$ and $n \geqslant 3$. Then the higher Bers map $\beta_n : \mathfrak{M}(\mathbb{D}^*) \to \mathscr{N}_{K,n}$ is holomorphic. Moreover, the differential $D_0\beta_n$ at the origin is given by the following correspondence*

$$\mu \mapsto \frac{(-1)^n n!}{\pi} \int_{\mathbb{D}^*} \frac{\mu(w)}{(z-w)^{n+1}} dudv.$$

Analogously, the higher Bers maps are well-defined in the Morrey type Teichmüller space and we obtain their holomorphy as the following theorem, which generalizes Theorem B.

THEOREM 1.3. *Let K be the same as that in Theorem 1.1 and $n \geqslant 3$. Then the higher Bers map $\beta_n : \mathfrak{M}(\mathbb{D}^*) \to \mathscr{N}_{K,n}$ is holomorphic. Moreover, the differential $D_0\beta_n$ at the origin is given by the following correspondence*

$$\mu \mapsto \frac{(-1)^n n!}{\pi} \int_{\mathbb{D}^*} \frac{\mu(w)}{(z-w)^{n+1}} dudv.$$

Here we call the space

$$T^0_{MT}(1) := \{\log f' \in T^0(1) : \log f' \in H^2_K\},$$

the small pre-logarithmic derivative model of the Morrey type Teichmüller space. Then we draw the following conclusion.

THEOREM 1.4. *The small pre-logarithmic derivative model $T^0_{MT}(1)$ has a connected component*

$$T^0_{MT,b}(1) = \{\log f' \in T^0_{MT}(1) : f(\mathbb{D}) \text{ is bounded}\}.$$

The structure of this paper is arranged as follows. Firstly, serval equivalent characterizations of Morrey type domains are obtained in Theorem 1.1 and Theorem 1.2 and their proofs are given in section 2. In section 3, the well-defined of higher Bers maps in the Morrey type Teichüller space are discussed (refer to Theorem 1.3), which generalizes Theorem B. Next, we draw one connected component of $T^0_{MT}(1)$ which is the small pre-logarithmic derivative model of the Morrey type Teichmüller space, and see Theorem 1.4 for details. Finally, some remarks are presented in Section 5.

2. Morrey type Teichmüller space

In this section, we shall prove Theorem 1.1 and 1.2. Some lemmas are needed. The following results, due to [26], give some characterizations of the space H_K^2.

LEMMA 2.1. [26] *Let K be the same as that in Theorem 1.1. Suppose $f \in H^2$. Then $f \in H_K^2$ $(H_{K,0}^2)$ if and only if $d\mu(z) = |f'(z)|^2(1-|z|^2)dxdy$ is a K-Carleson measure (vanishing K-Carleson measure).*

LEMMA 2.2. [26] *Let K be the same as that in Theorem 1.1. Then $d\mu(z)$ is a K-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2} d\mu(z) < \infty;$$

$d\mu(z)$ is a vanishing K-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2} d\mu(z) < \infty$$

and

$$\lim_{|a|\to 1^-} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2} d\mu(z) = 0.$$

LEMMA 2.3. [26] *Let K be the same as that in Theorem 1.1 and let $f \in H^2$. Then $d\mu(z) = |f(z)|^2(1-|z|^2)dxdy$ is a (vanishing) K-Carleson measure if and only if $d\nu(z) = |f'(z)|^2(1-|z|^2)^3 dxdy$ is a (vanishing) K-Carleson measure. Moreover,*

$$\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2} |f(z)|^2(1-|z|^2)dxdy \approx$$

$$\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2} |f'(z)|^2(1-|z|^2)^3 dxdy.$$

We will need the following lemma in our proof of Theorem 1.2.

LEMMA 2.4. [2] *Let f be a univalent analytic function in $\mathbb{D}$ with $\partial f(\mathbb{D})$ being a Jordan curve in $\widehat{\mathbb{C}}$ and $\log f' \in \mathfrak{B}_0^1$. Then f can be extended to a quasiconformal mapping $\widetilde{f}$ to the extended plane $\widehat{\mathbb{C}}$ whose complex dilatation satisfies*

$$|\mu(z)| = \frac{1}{2}\left|S_f\left(\frac{1}{\bar{z}}\right)\right|\left(1-\frac{1}{|z|^2}\right)^2, \quad 1 < |z| < R < 2.$$

In [11], it is also introduced the following kernel:

$$\varphi_h(\zeta,z) = \frac{1}{2\pi i}\int_{S^1} \frac{h(w)}{(\zeta-w)^2(1-zh(w))} dw, \quad (\zeta,z) \in \mathbb{D}\times\mathbb{D}.$$

Then two operators are defined as follows:

$$T_h^-(\eta(\zeta)) = \frac{1}{\pi}\int_{\mathbb{D}} \phi_h(\zeta,\bar{z})\eta(z)dxdy, \quad \eta \in \mathscr{H}^2, \zeta \in \mathbb{D},$$

$$T_h^+(\eta(\zeta)) = \frac{1}{\pi}\int_{\mathbb{D}} \varphi_h(\zeta,\bar{z})\eta(z)dxdy, \quad \eta \in \mathscr{H}^2, \zeta \in \mathbb{D},$$

where $\mathscr{H}^2$ is analytic Hilbert space with respect to the inner product:

$$\langle \varphi, \phi \rangle = \frac{1}{\pi}\int_{\mathbb{D}} \varphi(w)\overline{\phi(w)}dudv.$$

It is proved in [11] that T_h^- and T_h^+ are bounded operators.

The following results, which establish the relationship between Schwarzaian derivatives, kernel functions and complex dilations, are important in our proof of Theorem 1.2.

LEMMA 2.5. [19] *Let h be the conformal welding corresponding to f. Then*

$$U(f,z) \leqslant \phi_h(\bar{z}) \leqslant \|T_h^+\| U(f,z), \quad z \in \mathbb{D}.$$

LEMMA 2.6. [19] *Let h be the conformal welding corresponding to f. Denote by ν the complex dilatation of a quasiconformal extension of h^{-1} to $\mathbb{D}$. Then*

$$\frac{(1-|z^2|)^2}{36}|S_f(z)|^2 \leqslant U^2(f,z) \leqslant \phi_h^2(\bar{z}) \leqslant \frac{1}{\pi}\int_{\mathbb{D}} \frac{|\nu(w)|^2}{1-|\nu(w)|^2}\frac{1}{|1-\bar{z}w|^4}dudv.$$

The following result [27] will be very useful in our proof of Theorem 1.2.

LEMMA 2.7. [27] *Suppose that $k > -1$, $r,t > 0$, and $r+t-k > 2$. If $t < k+2 < r$, then there exists a universal constant $C > 0$ such that for all $z, \zeta \in \mathbb{D}$,*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^k}{|1-\overline{w}z|^r |1-\overline{w}\zeta|^t}dudv \leqslant C\frac{(1-|z|^2)^{2+k-r}}{|1-\overline{\zeta}z|^t},$$

where $w = u+iv$.

We now start our proof of Theorem 1.1 and 1.2.

Proof of Theorem 1.1. Since f is a univalent analytic function in $\mathbb{D}$, it always has the famous distortion theorem $|N_f(z)|(1-|z|^2) \leqslant 6$ [17]. Using $S_f(z) = N_f'(z) - \frac{1}{2}N_f^2(z)$, we get $|S_f(z)|^2 \lesssim |N_f'(z)|^2 + |N_f(z)|^4$ by Cauchy-Schwarzian inequality. This implies that

$$\begin{aligned}|S_f(z)|^2(1-|z|^2)^3 &\lesssim |N_f'(z)|^2(1-|z|^2)^3 + |N_f(z)|^4(1-|z|^2)^3\\ &\lesssim |N_f'(z)|^2(1-|z|^2)^3 + |N_f(z)|^2(1-|z|^2).\end{aligned}$$

Note that $\log f' \in H_K^2(H_{K,0}^2)$, then $|N_f(z)|^2(1-|z|^2)dxdy$ is a K-Carleson measure (vanishing K-Carleson measure) by Lemma 2.1. Using Lemma 2.3, $|N_f'(z)|^2(1-|z|^2)^3dxdy$ is a K-Carleson measure (vanishing K-Carleson measure). Therefore, $|S_f(z)|^2(1-|z|^2)^3dxdy$ is K-Carleson measure (vanishing K-Carleson measure). That is $|S_f(z)|^2(1-|z|^2)^3dm(z) \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$.

Conversely, set

$$J = \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} |N_f'(z)|^2(1-|z|^2)^3dxdy.$$

Note that

$$S_f(z) = N_f'(z) - \frac{1}{2}N_f^2(z),$$

so

$$J \lesssim \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} |S_f(z)|^2(1-|z|^2)^3dxdy + \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} |N_f(z)|^4(1-|z|^2)^3dxdy.$$

Since $\log f' \in \mathfrak{B}_0^1$, for any $\varepsilon > 0$, there exists $0 < r_\varepsilon < 1$ such that $|N_f(z)|(1-|z|^2) < \varepsilon$ as $|z| > r_\varepsilon$. By Lemma 2.3, it has

$$\begin{aligned}\frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)\cap\{z:|z|>r_\varepsilon\}} |N_f(z)|^4(1-|z|^2)^3dxdy &\leqslant \varepsilon^2 \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} |N_f(z)|^2(1-|z|^2)dxdy \\ &\lesssim \varepsilon^2 \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} |N_f'(z)|^2(1-|z|^2)^3dxdy.\end{aligned}$$

By $|N_f(z)|(1-|z|^2) \leqslant 6$, we have

$$\begin{aligned}\frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)\cap\{z:|z|\leqslant r_\varepsilon\}} |N_f(z)|^4(1-|z|^2)^3dxdy &\leqslant 6^4 \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)\cap\{z:|z|\leqslant r_\varepsilon\}} (1-|z|^2)^{-1}dxdy \\ &\lesssim \frac{\int_{S_{\mathbb{D}}(I)} dxdy}{K(|I|)} \frac{1}{1-r_\varepsilon^2} \\ &\lesssim \frac{1}{1-r_\varepsilon^2}.\end{aligned}$$

Here we choose sufficiently small $\varepsilon > 0$ such that

$$(1-C\varepsilon^2)J \lesssim \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} |S_f(z)|^2(1-|z|^2)^3dxdy + \frac{1}{1-r_\varepsilon^2},$$

where C is a positive constant depending on the above inequalities. Since $|S_f(z)|^2(1-|z|^2)^3dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$, then $|N_f'(z)|^2(1-|z|^2)^3dxdy$ is K-Carleson measure (vanishing K-Carleson measure). By Lemma 2.1, 2.2 and 2.3, it is obtained $\log f' \in H_K^2(H_{K,0}^2)$.

Proof of Theorem 1.2. Note that (1) $\Leftrightarrow$ (2) by Theorem 1.1. And it follows from Lemma 2.5 that (4) $\Leftrightarrow$ (5). Lemma 2.6 gives (5) $\Rightarrow$ (2). Thus it remains to show that (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (5).

Here $(2) \Rightarrow (3)$ is proved. By $\log f' \in \mathfrak{B}_0^1$ and Lemma 2.4, f can be extended to a quasiconformal mapping $\widetilde{f}$ to the extended plane $\widehat{\mathbb{C}}$ and its complex dilatation satisfies

$$|\mu(z)| = \frac{1}{2}\left|S_f\left(\frac{1}{\bar{z}}\right)\right|\left(1-\frac{1}{|z|^2}\right)^2, \quad 1<|z|<R<2.$$

Then

$$\frac{|\mu(z)|^2}{|z|^2-1} = \frac{\left|S_f\left(\frac{1}{\bar{z}}\right)\right|^2(|z|^2-1)^3}{4|z|^8} \leqslant \left|S_f\left(\frac{1}{\bar{z}}\right)\right|^2(|z|^2-1)^3, \quad 1<|z|<R<2.$$

From $|S_f(z)|^2(1-|z|^2)^3dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$, we have $|S_f(\frac{1}{\bar{z}})|^2(|z|^2-1)^3dxdy \in CM_K(\mathbb{D}^*)(CM_{K,0}(\mathbb{D}^*))$. Therefore, we obtain

$$\frac{|\mu(z)|^2}{|z|^2-1}dxdy \in CM_K(\mathbb{D}^*)(CM_{K,0}(\mathbb{D}^*)).$$

Then it shows that $(3) \Rightarrow (5)$. Since f is a univalent function in $\mathbb{D}$ and can be extended to a quasiconformal mapping to the extended plane $\widehat{\mathbb{C}}$ such that its complex dilatation μ satisfies $\frac{|\mu(z)|^2}{(|z|^2-1)}dxdy \in CM_K(\mathbb{D}^*)(CM_{K,0}(\mathbb{D}^*))$. Next $\frac{|\mu(\frac{1}{\bar{z}})|^2}{(1-|z|^2)}dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$ needs to be proved. For $I \subset \partial\mathbb{D}$ and $|I| > \frac{1}{2}$, it has

$$\begin{aligned}
&\frac{1}{K(|I|)}\int_{S_{\mathbb{D}}(I)}\frac{|\mu(\frac{1}{\bar{z}})|^2}{1-|z|^2}dxdy \\
&\leqslant \frac{1}{K(\frac{1}{2})}\int_{\{z\in S_{\mathbb{D}}(I):|z|<\frac{1}{2}\}}\frac{|\mu(\frac{1}{\bar{z}})|^2}{1-|z|^2}dxdy + \frac{1}{K(\frac{1}{2})}\int_{\{z\in S_{\mathbb{D}}(I):|z|\geqslant\frac{1}{2}\}}\frac{|\mu(\frac{1}{\bar{z}})|^2}{1-|z|^2}dxdy \\
&\leqslant C + \frac{1}{K(\frac{1}{2})}\int_{S_{\mathbb{D}^*}(S^1)}\frac{|\mu(z)|^2}{|z|^2-1}\frac{1}{|z|^2}dxdy \leqslant C + \frac{1}{K(\frac{1}{2})}\int_{S_{\mathbb{D}^*}(S^1)}\frac{|\mu(z)|^2}{|z|^2-1}dxdy.
\end{aligned}$$

For $|I| \leqslant \frac{1}{2}$, a new subarc $J \subset \partial\mathbb{D}$ has the same midpoint with I and satisfies $|J| = 2|I|$. Then, if $z \in S_{\mathbb{D}}(I)$, it has $\frac{1}{\bar{z}} \in S_{\mathbb{D}^*}(I)$. According to $K(2t) \approx K(t)$, then we have

$$\begin{aligned}
\frac{1}{K(|I|)}\int_{S(I)}\frac{|\mu(\frac{1}{\bar{z}})|^2}{1-|z|^2}dxdy &\lesssim \frac{1}{K(|J|)}\int_{S_{\mathbb{D}^*}(J)}\frac{|\mu(z)|^2}{|z|^2-1}\frac{1}{|z|^2}dxdy \\
&\lesssim \frac{1}{K(|J|)}\int_{S_{\mathbb{D}^*}(J)}\frac{|\mu(z)|^2}{|z|^2-1}dxdy.
\end{aligned}$$

In summary, it has

$$\frac{\mid\mu(\frac{1}{\bar{z}})\mid^2}{(1-|z|^2)}dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D})).$$

Denote by $\widetilde{H} = g^{-1}\circ f$ a quasiconformal extension of h^{-1} to $\mathbb{D}^*$. Notice that $\widetilde{H}$ has the same complex dilation μ as $\widetilde{f}$, where $\widetilde{f} = g^{-1}\circ f|_{\mathbb{D}^*}$ is a quasiconformal mapping of $\mathbb{D}^*$ onto itself. Note that $\widehat{H} = j\circ\widetilde{H}\circ j$, where $j(z) = \frac{1}{\bar{z}}$. By a simple computation, then $\widehat{H}$ is a quasiconformal extension of h^{-1} to $\mathbb{D}$ with its complex dilatation $\nu(z)$ satisfying $|\nu(z)| = |\mu(\frac{1}{\bar{z}})|$. Therefore, $\frac{|\nu(z)|^2}{1-|z|^2}dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$. Then $\phi_h^2(\bar{z})(1-|z|^2)dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$ needs to be proved. Here set $\gamma^a = \frac{a-z}{1-\bar{a}z}$, where $a\in\mathbb{D}$. By Lemma 2.1 and a simple computation, it is only proved that

$$\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}} \phi_h^2(\bar{w})(1-|\gamma^a(w)|^2)dudv < \infty$$

when $\frac{|\nu(z)|^2}{1-|z|^2}dxdy \in CM_K(\mathbb{D})$ and in addition

$$\lim_{|a|\to 1^-} \frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}} \phi_h^2(\bar{w})(1-|\gamma^a(w)|^2)dudv = 0,$$

when $\frac{|\nu(z)|^2}{1-|z|^2}dxdy \in CM_{K,0}(\mathbb{D})$. By Lemma 2.6, we obtain

$$\begin{aligned}
&\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}} \phi_h^2(\bar{w})(1-|\gamma^a(w)|^2)dudv\\
&\leqslant \frac{1}{\pi}\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}(1-|\gamma^a(w)|^2)dudv\int_{\mathbb{D}}\frac{|\nu(z)|^2}{1-|\nu(z)|^2}\frac{1}{|1-z\bar{w}|^4}dxdy\\
&\lesssim \frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}\frac{|\nu(z)|^2}{1-|z|^2}\frac{1-|a|^2}{|1-\bar{a}z|^2}dxdy\\
&\quad\times\int_{\mathbb{D}}\frac{(1-|\gamma^a(w)|^2)|1-\bar{a}z|^2(1-|z|^2)}{(1-|a|^2)|1-z\bar{w}|^4}dudv.
\end{aligned}$$

From Lemma 2.7, it has

$$\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}} \phi_h^2(\bar{w})(1-|\gamma^a(w)|^2)dudv \lesssim \frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}\frac{|\nu(z)|^2}{1-|z|^2}\frac{1-|a|^2}{|1-\bar{a}z|^2}dxdy.$$

By Lemma 2.1, it is obtained that $\phi_h^2(\bar{z})(1-|z|^2)dxdy \in CM_K(\mathbb{D})(CM_{K,0}(\mathbb{D}))$. Therefore, this theorem is proved completely.

Next, we recall that the Douady-Earle extension $w = E(h)(z)$ of the quasisymmetric homeomorphism h is defined as the equation, for $z,w\in\mathbb{D}$,

$$F(z,w) = \frac{1}{2\pi}\int_{S^1}\frac{(h(t)-w)(1-|w|^2)}{(1-\bar{w}h(t))\,|z-t|^2}\,|dt| = 0,$$

(see [6]).

In [10], they obtained the upper bound for the maximal dilatation of $E(h)$ in terms of the cross-ratio distortion of h. Given a quadruple $Q = \{a,b,c,d\}$ consisting of four points a,b,c and d on the unit circle S^1 arranged in counterclockwise order, one cross-ratio of Q is defined by

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

Given a quasisymmetric homeomorphism h, the cross-ratio distortion norm of h is defined as

$$\|h\|_{cr} = \sup_{cr(Q)=1} |\ln cr(h(Q))|,$$

where

$$cr(h(Q)) = \frac{(h(b)-h(a))(h(d)-h(c))}{(h(c)-h(b))(h(d)-h(a))}.$$

LEMMA 2.8. *For any quasisymmetric homeomorphism h of S^1, there exists a universal constant $C > 0$ such that the maximal dilatation*

$$\ln\left(\frac{1+\|\mu_{E(h)}\|_\infty}{1-\|\mu_{E(h)}\|_\infty}\right) \leqslant C\|h\|_{cr},$$

where $E(h)(z)$ is the Douady-Earle extension of h.

Let $\nu(w)$ be the complex dilatation of the inverse mapping $E(h)^{-1}$ of the Douay-Earle extension $E(h)$ of quasisymmetric homeomorphism h.

PROPOSITION 1. *Let K be the same as that in Theorem 1.1. Let h be a quasisymmetric homeomorphism h of S^1 and ν be the complex dilatation of the inverse mapping $E(h)^{-1}$ of the Douay-Earle extension $E(h)$ of h. Then there exists a universal constant $B > 0$ such that*

$$\|\nu\|_{\mathfrak{M}} \leqslant B(\|h\|_{cr} + \|(1-|w|^2)\phi_h^2(\bar{w})\|_{\mathbb{D},K})$$

Proof. It follows from [4] and [11] that there is a constant $A > 0$ which depends only on the quasisymmetric constant of h such that

$$\frac{|\nu(w)|^2}{1-|\nu(w)|^2} \leqslant A(1-|w|^2)^2\phi_h^2(\bar{w}).$$

Then we have $\|\lambda_\mu\|_{\mathbb{D}^*,K} \leqslant A\|(1-|w|^2)\phi_h^2(\bar{w})\|_{\mathbb{D},K}$. Since the maximal dilatation of the inverse mapping $E(h)^{-1}$ is equal to the maximal dilatation of $E(h)$, using Lemma 2.8, there exists a universal constant $C > 0$ such that

$$\ln\left(\frac{1+\|\nu\|_\infty}{1-\|\nu\|_\infty}\right) = \ln\left(\frac{1+\|\mu_{E(h)}\|_\infty}{1-\|\mu_{E(h)}\|_\infty}\right) \leqslant C\|h\|_{cr}$$

Therefore, there is a universal constant $B > 0$ such that

$$\|\nu\|_{\mathfrak{M}} \leqslant B(\|h\|_{cr} + \|(1-|w|^2)\phi_h^2(\bar{w})\|_{\mathbb{D},K}),$$

where $B = \max\{A, C\}$.

3. The holomorphy of higher Bers maps

In order to prove Theorem 1.3, the following result is needed.

LEMMA 3.1. *Let K be the same as that in Theorem 1.1. If $\mu \in \mathfrak{M}(\mathbb{D}^*)$, then $\sigma_n(f^\mu) \in \mathscr{N}_{K,n}$ for $n \geqslant 3$.*

Proof. Owing to $\mu \in \mathfrak{M}(\mathbb{D}^*)$, there exists a quasiconformal mapping f satisfied with Condition (3) of Theorem 1.2. According to the proof of Theorem 1.2 and Lemma 2.1, it is obtained that $\sigma_3(z) \in \mathscr{N}_{K,3}$. Suppose that $\sigma_n(f^\mu) \in \mathscr{N}_{K,n}, n \geqslant 3$. By Lemma 2.2 and 2.3, we have

$$\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} |\sigma_n'(z)|^2 (1-|z|^2)^{2n-1} dxdy < \infty.$$

Since $\sigma_{n+1}(f)(z) = \sigma_n'(f)(z) - (n-1)N_f(z)\sigma_n(f)(z), n \geqslant 3$, it has

$$|\sigma_{n+1}(z)| \leqslant |\sigma_n'(f)(z)| + |(n-1)N_f(z)\sigma_n(f)(z)|.$$

Therefore

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} |\sigma_{n+1}(z)|^2 (1-|z|^2)^{2n-1} dxdy \\ \lesssim &\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} |\sigma_n'(z)|^2 (1-|z|^2)^{2n-1} dxdy \\ &+ \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} |N_f(z)\sigma_n(f)(z)|^2 (1-|z|^2)^{2n-1} dxdy \end{aligned}$$

Since f is a univalent analytic function in $\mathbb{D}$, we have $|N_f(z)|(1-|z|^2) \leqslant 6$. Therefore, it has $\sigma_{n+1}(f^\mu) \in \mathscr{N}_{K,n+1}$. Using the mathematical induction, this lemma is proved completely.

Next, we will generalize the following lemma:

LEMMA C. ([19]) *Let $K(t) = t$. Then the Bers map $\beta_3 : \mathfrak{M}(\mathbb{D}^*) \to \mathscr{N}_{K,3}$ is continuous and for any $\mu, \nu \in \mathfrak{M}(\mathbb{D}^*)$, it has*

$$\|\beta_3(\mu) - \beta_3(\nu)\|_{\mathscr{N}_{K,3}} \lesssim \|\mu - \nu\|_{\mathscr{L}}.$$

The following general result shows that the Bers map $\beta_3 : \mathfrak{M}(\mathbb{D}^*) \to \mathscr{N}_{K,3}(\mathbb{D})$ is Lipschitz continuous.

LEMMA 3.2. *Let K be the same as that in Theorem 1.1. Then the Bers map $\beta_3 : \mathfrak{M}(\mathbb{D}^*) \to \mathscr{N}_{K,3}$ is continuous and for any $\mu, \nu \in \mathfrak{M}(\mathbb{D}^*)$, it has*

$$\|\beta_3(\mu) - \beta_3(\nu)\|_{\mathscr{N}_{K,3}} \lesssim \|\mu - \nu\|_{\mathscr{L}}.$$

Proof. In [1], it is proved that for any two elements $\mu, \nu \in M(\mathbb{D}^*)$, it has

$$|\beta_3(\mu)-\beta_3(\nu)|^2(1-|z|^2)^2 \lesssim \int_{\mathbb{D}^*} \frac{|\mu(\zeta)-\nu(\zeta)|^2+\|\mu-\nu\|_\infty^2|\mu(\zeta)|^2}{|\zeta-z|^4}d\xi d\eta.$$

Therefore, we have

$$\begin{aligned}
&\|\beta_3(\mu)-\beta_3(\nu)\|^2_{\mathcal{N}_{K,3}}\\
=&\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}|\beta_3(\mu)-\beta_3(\nu)|^2(1-|z|^2)^3\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy\\
\lesssim&\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}\left(\int_{\mathbb{D}^*}\frac{|\mu(\zeta)-\nu(\zeta)|^2}{|\zeta-z|^4}d\xi d\eta\right)(1-|z|^2)\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy\\
&+\|\mu-\nu\|_\infty^2\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}\left(\int_{\mathbb{D}^*}\frac{|\mu(\zeta)|^2}{|\zeta-z|^4}d\xi d\eta\right)(1-|z|^2)\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy.
\end{aligned}$$

Let $\zeta=\frac{1}{\overline{\tau}}$. Then it has

$$\begin{aligned}
&\|\beta_3(\mu)-\beta_3(\nu)\|^2_{\mathcal{N}_{K,3}}\\
\lesssim&\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}\frac{|\mu(\frac{1}{\overline{\tau}})-\nu(\frac{1}{\overline{\tau}})|^2}{1-|\tau|^2}\frac{1-|a|^2}{|1-\overline{a}\tau|^2}dudv\\
&\times\int_{\mathbb{D}}\frac{(1-|z|^2)(1-|\tau|^2)|1-\overline{a}\tau|^2}{|1-\overline{\tau}z|^4|1-\overline{a}z|^2}dxdy\\
&+\|\mu-\nu\|_\infty^2\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}\frac{|\mu(\frac{1}{\overline{\tau}})|^2}{1-|\tau|^2}\frac{1-|a|^2}{|1-\overline{a}\tau|^2}dudv\\
&\times\int_{\mathbb{D}}\frac{(1-|z|^2)(1-|\tau|^2)|1-\overline{a}\tau|^2}{|1-\overline{\tau}z|^4|1-\overline{a}z|^2}dxdy.
\end{aligned}$$

By Lemma 2.7 and the proof of $(3)\Rightarrow(5)$ in Theorem 1.2, we obtain

$$\|\beta_3(\mu)-\beta_3(\nu)\|_{\mathcal{N}_{K,3}}\lesssim\|\mu-\nu\|_{\mathscr{L}}.$$

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. The proof of Theorem 1.3 is organized two parts. The continuality of the higher Bers map is first proved. Then it shows that the higher Bers map is holomorphic.

For any $\mu,\nu\in\mathfrak{M}(\mathbb{D}^*)$, let f denote the quasiconformal mapping whose complex dilatation is equal to μ in $\mathbb{D}^*$ and is zero in $\mathbb{D}$, and let g denote the quasiconformal mapping whose complex dilatation is equal to ν in $\mathbb{D}^*$ and is zero in $\mathbb{D}$, both normalized

$$f(0)=f'(0)-1=f''(0)=0 \quad \text{and} \quad g(0)=g'(0)-1=g''(0)=0.$$

By the definition of the higher Schwarzian derivative, we have

$$\|\sigma_{n+1}(f)-\sigma_{n+1}(g)\|_{\mathcal{N}_{K,n+1}}\leqslant\|\sigma_n'(f)-\sigma_n'(g)\|_{\mathcal{N}_{K,n+1}}+(n-1)\|N_f\sigma_n(f)-N_g\sigma_n(g)\|_{\mathcal{N}_{K,n+1}}.$$

By Lemma 2.2 and 2.3, it has

$$\|\sigma_n'(f)-\sigma_n'(g)\|_{\mathcal{N}_{K,n+1}} \approx \|\sigma_n(f)-\sigma_n(g)\|_{\mathcal{N}_{K,n}}.$$

Since f is a univalent analytic function in $\mathbb{D}$, we obtain

$$\|N_f\|_{\mathfrak{B}^1} = \sup_{z\in\mathbb{D}}(1-|z|^2)|N_f(z)| \leqslant 6.$$

Note that

$$|N_f\sigma_n(f)-N_g\sigma_n(g)| \leqslant |N_f||\sigma_n(f)-\sigma_n(g)|+|\sigma_n(g)||N_f-N_g|.$$

Consequently, it has

$$\|N_f\sigma_n(f)-N_g\sigma_n(g)\|_{\mathcal{N}_{K,n+1}} \leqslant \|N_f\|_{\mathfrak{B}^1}\|\sigma_n(f)-\sigma_n(g)\|_{\mathcal{N}_{K,n}}+\|\sigma_n(g)\|_{\mathcal{N}_{K,n}}\|N_f-N_g\|_{\mathfrak{B}^1}.$$

By Theorem 3.1 in Chapter II in [13], there is a constant $C>0$ such that

$$\|N_f-N_g\|_{\mathfrak{B}^1} \leqslant C\|\mu-\nu\|_\infty.$$

By Lemma 3.1, we have

$$\|\sigma_{n+1}(f)-\sigma_{n+1}(g)\|_{\mathcal{N}_{K,n+1}} \lesssim \|\sigma_n(f)-\sigma_n(g)\|_{\mathcal{N}_{K,n}}+\|\mu-\nu\|_\infty.$$

Repeating this process $n-3$ times, it has

$$\|\sigma_{n+1}(f)-\sigma_{n+1}(g)\|_{\mathcal{N}_{K,n+1}} \lesssim \|\sigma_3(f)-\sigma_3(g)\|_{\mathcal{N}_{K,3}}+\|\mu-\nu\|_\infty.$$

By Lemma 3.2, we obtain

$$\|\sigma_{n+1}(f)-\sigma_{n+1}(g)\|_{\mathcal{N}_{K,n+1}} \lesssim \|\mu-\nu\|_{\mathcal{L}}.$$

Therefore, the higher Bers map is continuous.

Then, it needs to show that the higher Bers map $\beta_n:\mathfrak{M}(\mathbb{D}^*)\to\mathcal{N}_{K,n}(\mathbb{D})$ is holomorphic. That is sufficient to show that for any $\mu\in\mathfrak{M}(\mathbb{D}^*)$ and $\nu\in\mathcal{L}(\mathbb{D}^*)$, $\beta_n(\mu+t\nu)$ is holomorphic in a small neighborhood of $t=0$ in the complex plane. By $\mu\in\mathfrak{M}(\mathbb{D}^*)$, there exists a positive constant ε such that for any t with $|t|<2\varepsilon$,

$$\|\mu+t\nu\|_\infty<1 \quad \text{and} \quad \|\mu+t\nu\|_{\mathcal{L}}<\infty.$$

Here abbreviate the function $\beta_n(\mu+t\nu)$ by $\psi(t)$. For fixed $z\in\mathbb{D}$, the function $\psi(t)$ is holomorphic in $|t|<2\varepsilon$ [3]. For $|t|<\varepsilon$, $|t_0|<\varepsilon$, it follows from Cauchy's formula that

$$\left|\frac{\psi(t)(z)-\psi(t_0)(z)}{t-t_0}-\frac{d}{dt}\Big|_{t=t_0}\psi(t)(z)\right| = \frac{|t-t_0|}{2\pi}\left|\int_{|s|=2\varepsilon}\frac{\psi(s)(z)}{(s-t)(s-t_0)^2}ds\right|$$

$$\leqslant \frac{|t-t_0|}{2\pi\varepsilon^3}\int_{|s|=2\varepsilon}|\psi(s)(z)||ds|.$$

Consequently, by Fubini's theorem, it has

$$\begin{aligned}
&\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}\left|\frac{\psi(t)(z)-\psi(t_0)(z)}{t-t_0}-\frac{d}{dt}|_{t=t_0}\psi(t)(z)\right|^2(1-|z|^2)^{2n-3}\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy\\
\leqslant&\frac{(1-|a|^2)}{K(1-|a|)}\frac{|t-t_0|^2}{4\pi^2\varepsilon^6}\int_{\mathbb{D}}\left(\int_{|s|=2\varepsilon}|\psi(s)(z)||ds|\right)^2(1-|z|^2)^{2n-3}\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy\\
\lesssim&|t-t_0|^2\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}\int_{|s|=2\varepsilon}|\psi(s)(z)|^2|ds|(1-|z|^2)^{2n-3}\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy\\
=&|t-t_0|^2\int_{|s|=2\varepsilon}\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}|\psi(s)(z)|^2(1-|z|^2)^{2n-3}\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy|ds|\\
\lesssim&|t-t_0|^2.
\end{aligned}$$

Thus, it deduces that the limit

$$\lim_{t\to t_0}\frac{\psi(t)-\psi(t_0)}{t-t_0}=\frac{d}{dt}|_{t=t_0}\psi(t)$$

exists in $\mathscr{N}_{K,n}$. This implies that $\beta_n:\mathfrak{M}(\mathbb{D}^*)\to\mathscr{N}_{K,n}$ is holomorphic.

It was shown by Buss in Theorem 3.4 in [3] that

$$\frac{d}{dt}|_{t=0}\psi(t)(z)=\frac{(-1)^n n!}{\pi}\int_{\mathbb{D}^*}\frac{\mu(w)}{(z-w)^{n+1}}dudv$$

This theorem is proved completely.

4. The connected component $T^0_{MT,b}(1)$ of $T^0_{MT}(1)$

In this section, we will prove Theorem 1.4. Since the small pre-logarithmic derivative model $T^0_{MT}(1)$ is a subset of $T^0(1)$, each $f\in T^0_{MT}(1)$ is a univalent analytic function in $\mathbb{D}$, normalized by $f(0)=f'(0)-1=0$, which can be extended to a quasiconformal mapping in $\widehat{\mathbb{C}}$. Here assume $f(z_0)=\infty$ for $z_0\in\overline{\mathbb{D}^*}$. Consider the pre-Bers projection mapping L_{z_0} on $\mathfrak{M}^0(\mathbb{D}^*)$ by setting $L_{z_0}(\mu)=\log(f^\mu)'$. Then the following lemma is natural.

LEMMA 4.1. $\cup_{z_0\in\overline{\mathbb{D}^*}}L_{z_0}(\mathfrak{M}^0(\mathbb{D}^*))=T^0_{MT}(1)$.

Proof. For any $\mu\in\mathfrak{M}^0(\mathbb{D}^*)\subset M^0(\mathbb{D}^*)$, there exists a unique quasiconformal mapping $f^\mu:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ whose complex dilatation is equal to μ in $\mathbb{D}^*$ and is zero in $\mathbb{D}$, normalized by $f^\mu(0)=(f^\mu)'(0)-1=0$ and $f(z_0)=\infty$ for $z_0\in\overline{\mathbb{D}^*}$. It is known that $\log(f^\mu)'\in\mathfrak{B}^1_0$. It follows from Theorem 1.2 that $\log(f^\mu)'\in H^2_K$. Therefore, it has $\log(f^\mu)'\in T^0_{MT}(1)$.

Conversely, for $\log f' \in T^0_{MT}(1) \subset T^0(1)$, we show that its complex dilatation $\mu_f(z)$ satisfies $\mu_f(z) \in M^0(\mathbb{D}^*)$ in $\mathbb{D}^*$, normalized by $f^\mu(0) = (f^\mu)'(0) - 1 = 0$. By Theorem 1.2 and $\log(f^\mu)' \in H^2_K$, we can know that its complex dilatation $\mu_f(z)$ satisfies $\frac{|\mu_f(z)|^2}{(|z|^2-1)}dxdy \in CM_K(\mathbb{D}^*)$. Since $\log f' \in T^0(1) \subset \mathfrak{B}^1_0$, it has $\mu_f(z) \in M^0(\mathbb{D}^*)$. Further, we have $\mu_f(z) \in \mathfrak{M}^0(\mathbb{D}^*)$. Here we assume $f(z_0) = \infty$ for $z_0 \in \overline{\mathbb{D}^*}$. It has $\log f' \in L_{z_0}(\mathfrak{M}^0(\mathbb{D}^*))$. This lemma is proved completely.

To prove Theorem 1.4, we need the following result which has its own interest.

THEOREM 4.1. *Let K satisfy (1.1)–(1.4). For $z_0 \in \overline{\mathbb{D}^*}$, the pre-Bers projection mapping $L_{z_0} : \mathfrak{M}^0(\mathbb{D}^*) \to H^2_K$ is holomorphic.*

Proof. It follows from Lemma 4.1 that the mapping $L_{z_0} : \mathfrak{M}^0(\mathbb{D}^*) \to H^2_K$ is well defined. First it shows that $L_{z_0} : \mathfrak{M}^0(\mathbb{D}^*) \to H^2_K$ is continuous. For $\mu, \nu \in \mathfrak{M}^0(\mathbb{D}^*)$, it follows from Theorem 3.1 in Chapter II in [13] that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{(f^\mu)''}{(f^\mu)'} - \frac{(f^\nu)''}{(f^\nu)'}\right| \lesssim \|\mu-\nu\|_\infty.$$

By Lemma 3.2, it has

$$\|\beta_3(\mu)-\beta_3(\nu)\|_{\mathcal{N}_{K,3}} \lesssim \|\mu-\nu\|_{\mathscr{L}}.$$

It follows from Lemmas 2.1, 2.2 and 2.3 that

$$\begin{aligned}
&\|\log(f^\mu)' - \log(f^\nu)'\|^2_{H^2_K}\\
&\approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} \left|\frac{(f^\mu)''}{(f^\mu)'} - \frac{(f^\nu)''}{(f^\nu)'}\right|^2 (1-|z|^2)dxdy\\
&\approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} \left|\left(\frac{(f^\mu)''}{(f^\mu)'}\right)' - \left(\frac{(f^\nu)''}{(f^\nu)'}\right)'\right|^2 (1-|z|^2)^3dxdy\\
&\lesssim \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} \left|S_{f^\mu} - S_{f^\nu}\right|^2 (1-|z|^2)^3dxdy\\
&\quad + \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} \left|\left(\frac{(f^\mu)''}{(f^\mu)'}\right)^2 - \left(\frac{(f^\nu)''}{(f^\nu)'}\right)^2\right|^2 (1-|z|^2)^3dxdy\\
&\lesssim \|\beta_3(\mu)-\beta_3(\nu)\|^2_{\mathcal{N}_{K,3}} + \sup_{z\in\mathbb{D}}\left\{(1-|z|^2)^2\left|\frac{(f^\mu)''}{(f^\mu)'} - \frac{(f^\nu)''}{(f^\nu)'}\right|^2\right\}\\
&\quad \times \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} \left|\frac{(f^\mu)''}{(f^\mu)'} + \frac{(f^\nu)''}{(f^\nu)'}\right|^2 (1-|z|^2)dxdy\\
&\lesssim \|\mu-\nu\|^2_{\mathscr{L}} + \|\mu-\nu\|^2_\infty(\|\log(f^\mu)'\|^2_{H^2_K(\mathbb{D})} + \|\log(f^\nu)'\|^2_{H^2_K(\mathbb{D})})\\
&\lesssim \|\mu-\nu\|^2_{\mathscr{L}}.
\end{aligned}$$

Therefore, $L_{z_0}:\mathfrak{M}^0(\mathbb{D}^*)\to H_K^2$ is continuous.

Then, it needs to show that the pre-Bers projection mapping $L_{z_0}:\mathfrak{M}^0(\mathbb{D}^*)\to H_K^2$ is holomorphic. That is sufficient to show that for any $\mu\in\mathfrak{M}^0(\mathbb{D}^*)$ and $\nu\in\mathscr{L}(\mathbb{D}^*)$, $\beta_n(\mu+t\nu)$ is holomorphic in a small neighborhood of $t=0$ in the complex plane. By $\mu\in\mathfrak{M}^0(\mathbb{D}^*)$, there exists a positive constant ε such that for any t with $|t|<2\varepsilon$,

$$\|\mu+t\nu\|_\infty<1 \quad \text{and} \quad \|\mu+t\nu\|_{\mathscr{L}}<\infty.$$

Here abbreviate the function $L_{z_0}(\mu+t\nu)$ by $\phi(t)$. For fixed $z\in\mathbb{D}$, the function $\phi(t)$ is holomorphic in $|t|<2\varepsilon$ [16]. For $|t|<\varepsilon$, $|t_0|<\varepsilon$, it follows from Cauchy's formula that

$$\begin{aligned}\left|\frac{\phi(t)(z)-\phi(t_0)(z)}{t-t_0}-\frac{d}{dt}|_{t=t_0}\phi(t)(z)\right| &= \frac{|t-t_0|}{2\pi}\left|\int_{|s|=2\varepsilon}\frac{\phi(s)(z)}{(s-t)(s-t_0)^2}ds\right|\\ &\leqslant \frac{|t-t_0|}{2\pi\varepsilon^3}\int_{|s|=2\varepsilon}|\phi(s)(z)||ds|.\end{aligned}$$

Consequently, by Lemma 2.1 and Fubini's theorem, it has

$$\begin{aligned}&\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}\left|\frac{\phi(t)(z)-\phi(t_0)(z)}{t-t_0}-\frac{d}{dt}|_{t=t_0}\phi(t)(z)\right|^2(1-|z|^2)\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy\\ \leqslant&\frac{(1-|a|^2)}{K(1-|a|)}\frac{|t-t_0|^2}{4\pi^2\varepsilon^6}\int_{\mathbb{D}}\left(\int_{|s|=2\varepsilon}|\phi(s)(z)||ds|\right)^2(1-|z|^2)\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy\\ \lesssim&\frac{(1-|a|^2)}{K(1-|a|)}|t-t_0|^2\int_{\mathbb{D}}\int_{|s|=2\varepsilon}|\phi(s)(z)|^2|ds|(1-|z|^2)\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy\\ =&|t-t_0|^2\int_{|s|=2\varepsilon}\frac{(1-|a|^2)}{K(1-|a|)}\int_{\mathbb{D}}|\phi(s)(z)|^2(1-|z|^2)\frac{(1-|a|^2)}{|1-\overline{a}z|^2}dxdy|ds|\\ \lesssim&|t-t_0|^2.\end{aligned}$$

Thus, it deduces that the limit

$$\lim_{t\to t_0}\frac{\phi(t)-\phi(t_0)}{t-t_0}=\frac{d}{dt}|_{t=t_0}\phi(t)$$

exists in H_K^2. This implies that $L_{z_0}:\mathfrak{M}^0(\mathbb{D}^*)\to H_K^2$ is holomorphic.
We now start our proof of Theorem 1.4.

Proof of Theorem 1.4. For $\log f'\in T^0_{MT}(1)$, by Theorem 1.2, f can be extended to a quasiconformal mapping to the whole plane such that its complex dilatation μ satisfies $\frac{|\mu(z)|^2}{(|z|^2-1)}dxdy\in CM_K(\mathbb{D}^*)$. Let f^t be the quasiconformal mapping in $\widehat{\mathbb{C}}$ satisfied with $f^{-1}(\infty)=(f^t)^{-1}(\infty)$ and its complex dilatation $\mu_{f^t}=t\mu_f$. Consider the mapping $t\longmapsto\log(f^t)', 0\leqslant t\leqslant 1$. Since $\|\log(f^\mu)'-\log(f^\nu)'\|_{H_K^2}\lesssim\|\mu-\nu\|_{\mathscr{L}}$, we obtain $\|\log(f^t)'-\log f'\|_{H_K^2}\lesssim|1-t|\cdot\|\mu\|_{\mathscr{L}}$. Due to $\log f'\in H_K^2$, it has $\log(f^t)'\in H_K^2$. For f^{t_1}, f^{t_2}, we conclude from Theorem 4.2 that

$$\|\log(f^{t_1})'-\log(f^{t_2})'\|_{H_K^2}\lesssim|t_1-t_2|\cdot\|\mu\|_{\mathscr{L}}.$$

On the other hand, by Theorem 3.1 in Chapter II in [13], we get

$$\|\log(f^{t_1})' - \log(f^{t_2})'\|_{\mathfrak{B}} \lesssim |t_1 - t_2| \cdot \|\mu\|_\infty.$$

Thus, we deduce that

$$\|\log(f^{t_1})' - \log(f^{t_2})'\|_{\mathfrak{B},H^2_K} \lesssim |t_1 - t_2| \cdot \|\mu\|_{\mathscr{L}}.$$

This means that the path $t \longmapsto \log(f^t)', 0 \leqslant t \leqslant 1$ is continuous in $\mathfrak{B}_0 \cap H^2_K(\mathbb{D})$. Therefore, the mapping $t \longmapsto \log(f^t)', 0 \leqslant t \leqslant 1$, is continuous in $T^0_{MT}(1)$. Consequently, each $\log f' \in T^0_{MT}(1)$ can be connected by a continuous path to a Möbius transformation γ satisfied with $\log \gamma' \in T^0_{MT}(1)$. Since $\gamma(\mathbb{D})$ is bounded, we have $\log \gamma' \in T^0_{MT}(1)$. Moreover, it has the path $\rho \longmapsto \log \gamma'_\rho$ connecting the point $\log \gamma'$ to the point 0 in $T^0_{MT}(1)$ by [1], where $\gamma_\rho = \gamma(\rho z)$. Therefore, $T^0_{MT,b}(1) = \{\log f' \in T^0_{MT}(1) : f(\mathbb{D}) \text{ is bounded}\}$ is a connected component of $T^0_{MT}(1)$.

5. Some remarks

In this section, we consider the higher Bers map in the Q_K Teichmüller space and study the mutual relations between the Q_K Teichmüller space and the Morrey type Teichmüller space.

Let K be the same as that in Theorem 1.1. The Q_K space consists of all $f \in \mathscr{A}(\mathbb{D})$ for which

$$\|f\|_{Q_K} = \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 K\big(1 - |\gamma^a(z)|^2\big)\, dxdy \right)^{\frac{1}{2}} < \infty.$$

Then for any positive integer $n \in \mathbb{N}$, an analytic function f belongs to Q_K if and only if

$$\sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f^{(n)}(z)|^2 (1-|z|^2)^{2n-2} K\big(1 - |\gamma^a(z)|^2\big)\, dxdy \right)^{\frac{1}{2}} < \infty.$$

Here, the $\mathscr{Q}_{K,n}$ space consists of all $f \in \mathscr{A}(\mathbb{D})$ with

$$\|f\|_{\mathscr{Q}_{K,n}} = \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{2n-2} K\big(1 - |\gamma^a(z)|^2\big)\, dxdy \right)^{\frac{1}{2}} < \infty.$$

A non-negative measure μ on $\mathbb{D}$ is called the K-type Carleson measure if

$$\|\mu\|_{\mathbb{D},K\text{-type}} = \sup_{I\subset\partial\mathbb{D}} \left(\int_{S_{\mathbb{D}}(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu \right)^{\frac{1}{2}} < \infty.$$

Similarly, the K-type Carleson measure on $\mathbb{D}^*$ can be defined as above.

It is known that a positive measure μ on $\mathbb{D}$ is a K-type Carleson measure if and only if [26]

$$\sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} K\big(1 - |\gamma^a(z)|^2\big)\, d\mu \right)^{\frac{1}{2}} < \infty.$$

In addition, an analytic function f belongs to Q_K if and only if $|f^{(n)}(z)|^2(1-|z|^2)^{2n-2}dxdy$ is a K-type Carleson measure.

Let $\mathscr{L}'(\mathbb{D}^*)$ the Banach space of all essentially bounded measurable functions μ on $\mathbb{D}^*$ with $\lambda_\mu = \frac{|\mu(z)|^2}{|z|^2-1}dxdy$ being K-type Carleson measures. The norm of $\mu \in \mathscr{L}'(\mathbb{D}^*)$ is defined by

$$\|\mu\|_{\mathscr{L}'} = \|\mu\|_\infty + \|\lambda_\mu\|_{\mathbb{D}^*,K\text{-type}} < \infty.$$

The Q_K-Teichmüller space T_{Q_k} is defined by

$$T_{Q_k} = \{[\mu] \in T : \mu \in \mathfrak{M}'(\mathbb{D}^*)\},$$

where

$$\mathfrak{M}'(\mathbb{D}^*) = M(\mathbb{D}^*) \cap \mathscr{L}'(\mathbb{D}^*).$$

LEMMA 5.1. *Let K be the same as that in Theorem 1.1. If $\mu \in \mathfrak{M}'(\mathbb{D}^*)$, then $\sigma_n(f^\mu) \in \mathscr{Q}_{K,n-1}$ for $n \geqslant 3$.*

Proof. Using the proof of [25, Theorems 1] and similar to the proof of Lemma 3.1, Lemma 5.1 is proved easily. Here we omit its proof.

LEMMA 5.2. *Let K be the same as that in Theorem 1.1. Then the Bers map $\beta_3 : \mathfrak{M}'(\mathbb{D}^*) \to \mathscr{Q}_{K,2}$ is continuous and for any $\mu, \nu \in \mathfrak{M}'(\mathbb{D}^*)$, it has*

$$\|\beta_3(\mu) - \beta_3(\nu)\|_{\mathscr{Q}_{K,2}} \lesssim \|\mu - \nu\|_{\mathscr{L}'}.$$

Proof. Using the proof of [12, Theorems 2.2] and similar to the proof of Lemma 3.2, Lemma 5.2 is proved easily. Here we also omit its proof.

Owing to Lemmas 5.1 and 5.2, we have the following Theorem which is similar to Theorem 1.3 and also omit the proof.

THEOREM 5.1. *Let K be the same as that in Theorem 1.1 and $n \geqslant 3$. Then the higher Bers map $\beta_n : \mathfrak{M}'(\mathbb{D}^*) \to \mathscr{Q}_{K,n-1}$ is holomorphic. Moreover, the differential $D_0\beta_n$ at the origin is given by the following correspondence*

$$\mu \mapsto \frac{(-1)^n n!}{\pi} \int_{\mathbb{D}^*} \frac{\mu(w)}{(z-w)^{n+1}} dudv.$$

Next, the mutual relations of higher Schwarzian derivatives between the Q_K Teichmüller space and the Morrey type Teichmüller space are discussed. We need to introduce the fractional order derivative of f. For $b > 1$, the α-order derivative is defined as follows:

$$f^{(\alpha)}(z) = \frac{\Gamma(b+\alpha)}{\Gamma(b)} \int_{\mathbb{D}} \frac{(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+\alpha}} \overline{w}^{[\alpha-1]} f'(w)dudv, \quad b+\alpha > 0,$$

where Γ denotes the Gamma function and let $[\alpha]$ be the smallest integer which is larger than or equal to α. It is known that $f^{(\alpha)}$ is just the derivative of order n of f for $\alpha = n \in \mathbb{N}$. The following lemmas are needed.

LEMMA 5.3. [7] *Let* K *be the same as that in Theorem 1.1. Then there exists a weight* $K_1 \approx K$ *such that, for a small enough* q_1, $0 < q_1 < p$, $\frac{K_1(t)}{t^{q_1}}$ *is non-decreasing.*

LEMMA 5.4. [26] *Let* K *be the same as that in Theorem 1.1. Then there exists a* q_2, $0 < q_2 < p$, *such that* $\frac{K(t)}{t^{q_2}}$ *is non-increasing.*

LEMMA 5.5. [26] *Suppose that* K *is the same as that in Theorem 1.1,* $b+\alpha \geqslant 1+p$, $b \geqslant \max\{p, \frac{(1+p)}{2}\}$ *and* $\alpha > \frac{1}{2}$. *Let* ψ *be measurable on* $\mathbb{D}$ *and an operator on* $L^2(\mathbb{D})$ *is defined as:*

$$\Psi(\psi)(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+\alpha}} |\psi(w)| dudv.$$

If $d\mu(z) = |\psi(z)|^2 dxdy$ *is a* K*-type Carleson measure, then*

$$|\Psi(\psi)(z)|^2 (1-|z|^2)^{2(\alpha-1)} dxdy$$

is a K*-type Carleson measure.*

THEOREM 5.2. *Let* K *be the same as that in Theorem 1.1.*

(1) If $\mu \in \mathfrak{M}(\mathbb{D}^*)$, *then the fractional order derivative* $\sigma_n^{(\frac{q_1+1}{2})}(f^\mu) \in \mathcal{Q}_{K,n}$ *for* $n \geqslant 3$.

(2) If $\mu \in \mathfrak{M}'(\mathbb{D}^*)$, *then the fractional order derivative* $\sigma_n^{(\frac{3-q_2}{2})}(f^\mu) \in \mathcal{N}_{K,n+1}$ *for* $n \geqslant 3$.

Proof. First, we show (1). If $\mu \in \mathfrak{M}(\mathbb{D}^*)$, then $\sigma_n(f^\mu) \in \mathcal{N}_{K,n}$ for $n \geqslant 3$ and

$$\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)}{K(1-|a|)} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-3} dxdy < \infty.$$

Using Lemma 2.2, we have

$$\sup_{I\in\partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-3} dxdy < \infty.$$

For any $I \in \partial\mathbb{D}$, using Lemma 5.3,

$$\begin{aligned}
&\int_{S_{\mathbb{D}}(I)} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-3-q_1} K\left(\frac{1-|z|}{|I|}\right) dxdy \\
\lesssim &\int_{S_{\mathbb{D}}(I)} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-3-q_1} K_1\left(\frac{1-|z|}{|I|}\right) dxdy \\
\lesssim &\frac{1}{|I|^{q_1}} \int_{S_{\mathbb{D}}(I)} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-3} dxdy
\end{aligned}$$

$$\lesssim \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-3} dxdy$$
$$\lesssim \sup_{I \in \partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-3} dxdy < \infty.$$

Then $|\sigma_n(f^\mu)|^2(1-|z|^2)^{2n-3-q_1}dxdy$ is a K-type Carleson measure. Similar to [26, Theorems 3.2], using Lemma 5.5, $\left|\sigma_n^{\left(\frac{q_1+1}{2}\right)}(f^\mu)\right|^2 (1-|z|^2)^{2n-2}dxdy$ is a K-type Carleson measure. Therefore, we have

$$\sigma_n^{\left(\frac{q_1+1}{2}\right)}(f^\mu) \in \mathscr{Q}_{K,n}.$$

Next, (2) is needed to show. If $\mu \in \mathfrak{M}'(\mathbb{D}^*)$, then $\sigma_n(f^\mu) \in \mathscr{Q}_{K,n-1}$ for $n \geqslant 3$ and

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-4} K\big(1-|\gamma^a(z)|^2\big) dxdy < \infty.$$

Using the above result, we have

$$\sup_{I \in \partial\mathbb{D}} \int_{S_{\mathbb{D}}(I)} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-4} K\left(\frac{1-|z|}{|I|}\right) dxdy < \infty.$$

For any $I \in \partial\mathbb{D}$, using Lemmas 5.4 and 5.5,

$$\frac{1}{K(|I|)} \int_{S_{\mathbb{D}}(I)} \left|\sigma_n^{\left(\frac{3-q_2}{2}\right)}(f^\mu)\right|^2 (1-|z|^2)^{2n-1} dxdy$$
$$\lesssim \frac{|I|^{q_2}}{K(|I|)} \int_{S_{\mathbb{D}}(I)} \left|\sigma_n^{\left(\frac{3-q_2}{2}\right)}(f^\mu)\right|^2 (1-|z|^2)^{2n-1-q_2} K\left(\frac{1-|z|}{|I|}\right) dxdy$$
$$\lesssim \int_{S_{\mathbb{D}}(I)} \left|\sigma_n^{\left(\frac{3-q_2}{2}\right)}(f^\mu)\right|^2 (1-|z|^2)^{2n-1-q_2} K\left(\frac{1-|z|}{|I|}\right) dxdy$$
$$\lesssim \int_{S_{\mathbb{D}}(I)} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-4} K\left(\frac{1-|z|}{|I|}\right) dxdy$$
$$\lesssim \sup_{I \in \partial\mathbb{D}} \int_{S_{\mathbb{D}}(I)} |\sigma_n(f^\mu)|^2 (1-|z|^2)^{2n-4} K\left(\frac{1-|z|}{|I|}\right) dxdy < \infty.$$

Therefore, we have

$$\sigma_n^{\left(\frac{3-q_2}{2}\right)}(f^\mu) \in \mathscr{N}_{K,n+1}.$$

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