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SOME RESULTS ON $(\rho, \mathbf{b}, \mathbf{d})$–VARIATIONAL INEQUALITIES

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Abstract. In this paper, by using a dual gap functional and some working hypotheses, the solution set is investigated for a variational-type inequality governed by $(\rho, \mathbf{b}, \mathbf{d})$-convex path-independent curvilinear integral functional.

1. Introduction

By using gap-type functions, in accordance with Ferris and Mangasarian [5] and following Hiriart-Urruty and Lemaréchal [6], Alshahrani et al. [1] studied the minimum and maximum principle sufficiency properties associated with nonsmooth variational inequalities. Also, based on the works of Burke and Ferris [3], Patriksson [11] and following Marcotte and Zhu [10], the notion of *weak sharp solution* in variational-type inequalities has been strongly studied by many researchers. We make a dishonesty by mentioning only a part: Hu and Song [7], Liu and Wu [9], Zhu [20] and Jayswal and Singh [8].

Optimization problems subject to nonlinear equality and inequality constraints have been formulated and studied by many researchers. But, since so many phenomena are subject to laws involving partial differential equations (PDE)/partial differential inequations (PDI), it generates the need for a consistent analysis of scalar/multiobjective optimization problems with PDE/PDI constraints and multiple/curvilinear integral objective functionals. In the last few years, several multidimensional optimization problems governed by multiple and/or curvilinear integral objective functionals have been investigated, with remarkable results, by Treanţă [16]-[19].

In this paper, motivated and inspired by the aforementioned research works and by taking into account some variational techniques developed in Ansari [2], Clarke [4] and Treanţă [12]-[15], we introduce a new class of variational-type inequalities governed by $(\rho, \mathbf{b}, \mathbf{d})$-*convex path-independent curvilinear integral functionals* (a new concept introduced in this paper). Concretely, under some working hypotheses and using a dual gap functional, we provide some characterizations of their solution sets. The extended concept of *normal cone*, firstly introduced by Marcotte and Zhu [10], represents another novelty of this paper, which plays a crucial role in our investigations. As it is

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well known, the path-independent curvilinear integrals are very important in applications due to their physical meaning (mechanical work). Thus, this paper becomes a relevant research work both from theoretical and practical reasoning. Also, the ideas and techniques developed in this paper may stimulate further research in this dynamic field.

The present paper is structured as follows. Section 2 contains some preliminaries and problem formulation. The main results of this paper are included in section 3. More exactly, the solution set is investigated for an extended variational-type inequality involving $(\rho, \mathbf{b}, \mathbf{d})$-convex path-independent curvilinear integral functional. Finally, section 4 provides the conclusions of this study.

2. Preliminaries and problem description

In this paper, in order to introduce our study, consider the following notations and mathematical objects:

▶ $\Omega \subset \mathbb{R}^m$ is a compact domain and the point $\Omega \ni t = (t^\beta)$, $\beta = \overline{1,m}$, is considered as a multiple parameter of evolution;

▶ consider $\Omega \supset C : t = t(\tau)$, $\tau \in [a,b]$, a piecewise smooth curve joining the different points $t_1 = \left(t_1^1, \dots, t_1^m\right)$, $t_2 = \left(t_2^1, \dots, t_2^m\right)$ in Ω;

▶ let $\overline{\mathscr{X}}$ be the space of piecewise smooth functions $x : \Omega \to \mathbb{R}^n$, endowed with the Euclidean inner product

$$\begin{aligned}\langle x, y\rangle &= \int_C x(t)\cdot y(t)dt^\beta = \int_C \sum_{i=1}^n x^i(t)y^i(t)dt^\beta \\ &= \int_C \sum_{i=1}^n x^i(t)y^i(t)dt^1 + \cdots + \int_C \sum_{i=1}^n x^i(t)y^i(t)dt^m, \quad \forall x, y \in \overline{\mathscr{X}}\end{aligned}$$

and the induced norm;

▶ denote by $\mathscr{X}$ a nonempty, closed and convex subset of $\overline{\mathscr{X}}$, defined as

$$\mathscr{X} = \left\{x \in \overline{\mathscr{X}} : x(t) \in E \subset \mathbb{R}^n,\ x(t_1) = x_1 = \text{given},\ x(t_2) = x_2 = \text{given}\right\};$$

▶ throughout this paper, the summation over the repeated indices is assumed and x, x_α are the simplified notations for $x(t)$, $x_\alpha(t)$ and $x_\alpha(t) = \dfrac{\partial x}{\partial t^\alpha}(t)$;

▶ consider the real-valued continuously differentiable functions (closed Lagrange 1-form densities)

$$f_\beta, g_\beta, h_\beta : J^1(\mathbb{R}^m, \mathbb{R}^n) \to \mathbb{R}, \quad \beta = \overline{1,m},$$

(see $J^1(T,M)$ as the first-order jet bundle associated to T and M) which generate the following path-independent curvilinear integral functionals:

$$F : \overline{\mathscr{X}} \to \mathbb{R}, \quad F(x) = \int_C f_\beta\,(t, x, x_\alpha)\,dt^\beta,$$

$$G : \overline{\mathscr{X}} \to \mathbb{R}, \quad G(x) = \int_C g_\beta\,(t, x, x_\alpha)\,dt^\beta,$$

$$H:\overline{\mathscr{X}}\to\mathbb{R},\quad H(x)=\int_C h_\beta\,(t,x,x_\alpha)\,dt^\beta.$$

Let ρ be a real number, $\mathbf{b}(x,y)$ a symmetric positive real-valued functional on $\overline{\mathscr{X}}\times\overline{\mathscr{X}}$ and $\mathbf{d}(x,y)$ a real-valued functional on $\overline{\mathscr{X}}\times\overline{\mathscr{X}}$.

DEFINITION 2.1.

(i) The scalar functional $F:\overline{\mathscr{X}}\to\mathbb{R}$, $F(x)=\int_C f_\beta\,(t,x,x_\alpha)\,dt^\beta$, is called $(\rho,\mathbf{b},\mathbf{d})$-*convex* on $\mathscr{X}$ if, for any $x,y\in\mathscr{X}$,

$$F(x)-F(y)\geqslant\mathbf{b}(x,y)\int_C\left[\frac{\partial f_\beta}{\partial x}\,(t,y,y_\alpha)\,(x-y)+\frac{\partial f_\beta}{\partial x_\alpha}\,(t,y,y_\alpha)\,D_\alpha(x-y)\right]dt^\beta$$
$$+\rho\mathbf{b}(x,y)\mathbf{d}(x,y),$$

where D_α denotes the total derivative operator.

(ii) The functional F is said to be *strongly* $\mathbf{b}$-*convex,* $\mathbf{b}$-*convex*, or *weakly* $\mathbf{b}$-*convex* on $\mathscr{X}$, according to $\rho\mathbf{d}>0$, $\rho\mathbf{d}=0$, or $\rho\mathbf{d}<0$.

DEFINITION 2.2. The *variational (functional) derivative* $\dfrac{\delta_\beta F}{\delta x}$ of the path-independent curvilinear integral functional $F:\overline{\mathscr{X}}\to\mathbb{R}$, $F(x)=\int_C f_\beta\,(t,x,x_\alpha)\,dt^\beta$, is defined as

$$\frac{\delta_\beta F}{\delta x}=\frac{\partial f_\beta}{\partial x}\,(t,x,x_\alpha)-D_\alpha\frac{\partial f_\beta}{\partial x_\alpha}\,(t,x,x_\alpha)\in\overline{\mathscr{X}}$$

and, for any $\psi\in\overline{\mathscr{X}}$ with $\psi(t_1)=\psi(t_2)=0$, it satisfies the following relation

$$\left\langle\frac{\delta_\beta F}{\delta x},\psi\right\rangle=\int_C\frac{\delta_\beta F}{\delta x}(t)\cdot\psi(t)dt^\beta=\lim_{\varepsilon\to 0}\frac{F(x+\varepsilon\psi)-F(x)}{\varepsilon}.$$

Throughout this paper, it is assumed that the inner product between the variational derivative associated with a path-independent curvilinear integral functional and an element $\psi\in\overline{\mathscr{X}}$ is accompanied by the condition $\psi(t_1)=\psi(t_2)=0$.

By using the previous mathematical tools, we formulate the following *extended variational-type inequality problem*: for some given $\rho,\mathbf{b},\mathbf{d}$ (introduced as above), find $y\in\mathscr{X}$ such that

$$(EVIP)\qquad\mathbf{b}(x,y)\int_C\left[\frac{\partial f_\beta}{\partial x}\,(t,y,y_\alpha)\,(x-y)+\frac{\partial f_\beta}{\partial x_\alpha}\,(t,y,y_\alpha)\,D_\alpha(x-y)\right]dt^\beta$$
$$+\rho\mathbf{b}(x,y)\mathbf{d}(x,y)\geqslant 0,$$

for any $x\in\mathscr{X}$. The *dual extended variational-type inequality problem* associated to $(EVIP)$ is formulated as follows: for some given $\rho,\mathbf{b},\mathbf{d}$ (introduced as above), find $y\in\mathscr{X}$ such that

$$(DEVIP)\qquad\mathbf{b}(x,y)\int_C\left[\frac{\partial f_\beta}{\partial x}\,(t,x,x_\alpha)\,(x-y)+\frac{\partial f_\beta}{\partial x_\alpha}\,(t,x,x_\alpha)\,D_\alpha(x-y)\right]dt^\beta$$

$$+\rho\mathbf{b}(x,y)\mathbf{d}(x,y) \geqslant 0,$$

for any $x \in \mathscr{X}$.

Denote by $\mathscr{X}^*$ and $\mathscr{X}_*$ the solution set associated with $(EVIP)$ and $(DEVIP)$, respectively, and assume they are nonempty.

REMARK 2.1. As can be easily seen, the above extended variational-type inequality problems can be reformulated as follows: *for some given* $\rho, \mathbf{b}, \mathbf{d}$ *(introduced as above), find* $y \in \mathscr{X}$ *such that*

$$(EVIP) \qquad \mathbf{b}(x,y)\left[\left\langle \frac{\delta_\beta F}{\delta y}, x-y \right\rangle + \rho\mathbf{d}(x,y)\right] \geqslant 0, \quad \forall x \in \mathscr{X},$$

respectively: *for some given* $\rho, \mathbf{b}, \mathbf{d}$ *(introduced as above), find* $y \in \mathscr{X}$ *such that*

$$(DEVIP) \qquad \mathbf{b}(x,y)\left[\left\langle \frac{\delta_\beta F}{\delta x}, x-y \right\rangle + \rho\mathbf{d}(x,y)\right] \geqslant 0, \quad \forall x \in \mathscr{X}$$

if and only if

$$dU := D_\alpha\left[\frac{\partial f_\beta}{\partial x_\alpha}(x-y)\right] dt^\beta$$

is an exact total differential and it is satisfied the condition $U(t_1) = U(t_2)$. Throughout this paper, this working hypothesis is assumed.

Further, in order to investigate the solution set $\mathscr{X}^*$, we introduce the following gap functionals.

DEFINITION 2.3. For $x \in \overline{\mathscr{X}}$, the *primal gap functional* associated to $(EVIP)$ is defined as

$$G(x) = \max_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\int_C\left[\frac{\partial f_\beta}{\partial x}(t,x,x_\alpha)(x-y) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x,x_\alpha)D_\alpha(x-y)\right]dt^\beta \right.$$
$$\left. +\rho\mathbf{b}(x,y)\mathbf{d}(x,y)\right\}$$

and, similarly, the *dual gap functional* associated to $(EVIP)$ is defined as

$$H(x) = \max_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\int_C\left[\frac{\partial f_\beta}{\partial x}(t,y,y_\alpha)(x-y) + \frac{\partial f_\beta}{\partial x_\alpha}(t,y,y_\alpha)D_\alpha(x-y)\right]dt^\beta \right.$$
$$\left. +\rho\mathbf{b}(x,y)\mathbf{d}(x,y)\right\}.$$

From now onwards, for $x \in \overline{\mathscr{X}}$, consider the following notations:

$$A(x) = \left\{z \in \mathscr{X} : G(x) = \mathbf{b}(x,z)\int_C\left[\frac{\partial f_\beta}{\partial x}(t,x,x_\alpha)(x-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x,x_\alpha)D_\alpha(x-z)\right]dt^\beta\right.$$

$$+\rho\mathbf{b}(x,z)\mathbf{d}(x,z)\Big\},$$

$$Z(x)=\Big\{z\in\mathscr{X}:H(x)=\mathbf{b}(x,z)\int_C\left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x-z)+\frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha)D_\alpha(x-z)\right]dt^\beta$$
$$+\rho\mathbf{b}(x,z)\mathbf{d}(x,z)\Big\}.$$

REMARK 2.2. By using the previous notations, we can observe the following:

(i)

$$G(x)=\max_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta x},x-y\right\rangle+\rho\mathbf{d}(x,y)\right]\right\},$$
$$H(x)=\max_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta y},x-y\right\rangle+\rho\mathbf{d}(x,y)\right]\right\};$$

(ii) $A(x)=\arg\max_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta x},x-y\right\rangle+\rho\mathbf{d}(x,y)\right]\right\}$, where

$$\arg\max_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta x},x-y\right\rangle+\rho\mathbf{d}(x,y)\right]\right\}$$

denotes the (possibly empty) solution set of $\max_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta x},x-y\right\rangle+\rho\mathbf{d}(x,y)\right]\right\}$;

(iii) $Z(x)=\arg\max_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta y},x-y\right\rangle+\rho\mathbf{d}(x,y)\right]\right\}$;

(iv) if $A(x)=\emptyset$, then $G(x)=\sup_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta x},x-y\right\rangle+\rho\mathbf{d}(x,y)\right]\right\}$; similarly, if $Z(x)=\emptyset$, then $H(x)=\sup_{y\in\mathscr{X}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta y},x-y\right\rangle+\rho\mathbf{d}(x,y)\right]\right\}$.

In order to formulate and prove the main results of this paper, in accordance with Marcotte and Zhu [10], we introduce the following relevant concepts.

DEFINITION 2.4. The *polar set* $\mathscr{X}^\circ$ associated to $\mathscr{X}$ is defined as follows

$$\mathscr{X}^\circ=\left\{y\in\overline{\mathscr{X}}:\langle y,x\rangle\leqslant 0,\ \forall x\in\mathscr{X}\right\}.$$

DEFINITION 2.5. The *projection* of a point $x\in\overline{\mathscr{X}}$ onto the set $\mathscr{X}$ is defined as

$$\mathrm{proj}_{\mathscr{X}}x=\arg\min_{y\in\mathscr{X}}\|x-y\|.$$

DEFINITION 2.6. The *normal cone to* $\mathscr{X}$ *at* $x \in \overline{\mathscr{X}}$, *with respect to* $\rho, \mathbf{b}$ *and* $\mathbf{d}$ (introduced as above), is defined as

$$N_{\mathscr{X}}^{\rho,\mathbf{b},\mathbf{d}}(x) = \left\{ y \in \overline{\mathscr{X}} : \mathbf{b}(z,x)\left[\langle y, z-x\rangle - \rho \mathbf{d}(z,x)\right] \leqslant 0,\ \forall z \in \mathscr{X} \right\}, \quad x \in \mathscr{X},$$
$$N_{\mathscr{X}}^{\rho,\mathbf{b},\mathbf{d}}(x) = \emptyset, \quad x \notin \mathscr{X}$$

and the *tangent cone to* $\mathscr{X}$ *at* $x \in \overline{\mathscr{X}}$, *with respect to* $\rho, \mathbf{b}$ *and* $\mathbf{d}$ (introduced as above), is $T_{\mathscr{X}}^{\rho,\mathbf{b},\mathbf{d}}(x) = \left[N_{\mathscr{X}}^{\rho,\mathbf{b},\mathbf{d}}(x)\right]^{\circ}$.

REMARK 2.3. Taking into account the definition of normal cone at $x \in \overline{\mathscr{X}}$, we notice that: $x^* \in \mathscr{X}^* \Longleftrightarrow -\dfrac{\delta_\beta F}{\delta x^*} \in N_{\mathscr{X}}^{\rho,\mathbf{b},\mathbf{d}}(x^*)$.

3. Main results

In this section, the main results of this paper are formulated and proved. Further, we establish some working assumptions.

Working hypotheses.

(i) The following equalities

$$\mathbf{d}(x^1,x^2) = -\mathbf{d}(x^2,x^1), \quad \forall x^1,x^2 \in \mathscr{X}^*,$$
$$\mathbf{d}(z,x^*) = -\mathbf{d}(x^*,z), \quad \forall z \in \mathscr{X},\ \forall x^* \in \mathscr{X}^*,$$

are fulfilled.

(ii) For any $y \in Z(x)$ and $x,z \in \overline{\mathscr{X}}$, the following relations

$$\mathbf{b}(z,y)(z-y) - \mathbf{b}(x,y)(x-y) = z-x, \quad \mathbf{b}(z,y)\mathbf{d}(z,y) - \mathbf{b}(x,y)\mathbf{d}(x,y) = \mathbf{d}(z,x)$$

are true.

(iii) For any $x,v \in \overline{\mathscr{X}}$ and $\lambda > 0$, there exists

$$\lim_{\lambda \to 0} \frac{\mathbf{d}(x+\lambda v,x)}{\lambda}.$$

(iv) For any $z \in Z(x^*)$, $\bar{x} \in A(x^*)$, $x^* \in \mathscr{X}^*$ and $x \in \mathscr{X}$, the following relations

$$\mathbf{b}(x,z) = \mathbf{b}(x,x^*) = \mathbf{b}(z,x^*) = \mathbf{b}(\bar{x},x^*) \quad [=1],$$
$$\mathbf{d}(x,z) = \mathbf{d}(x,x^*), \quad \mathbf{d}(z,x) = \mathbf{d}(x^*,x) = \mathbf{d}(x^*,\bar{x})$$

are satisfied.

THEOREM 3.1. *Assume the scalar functional* $F(x)=\int_C f_\beta(t,x,x_\alpha)\,dt^\beta$ *is* $(\rho,\mathbf{b},\mathbf{d})$-*convex on* $\mathscr{X}$. *Then:*

1. *for any* $x^1,x^2\in\mathscr{X}^*$, *it follows*

$$\mathbf{b}(x^1,x^2)\int_C\left[\frac{\partial f_\beta}{\partial x}\left(t,x^2,x^2_\alpha\right)(x^1-x^2)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^2,x^2_\alpha\right)D_\alpha(x^1-x^2)\right]dt^\beta$$
$$+\rho\mathbf{b}(x^1,x^2)\mathbf{d}(x^1,x^2)=0;$$

2. *the inclusion* $\mathscr{X}^*\subset\mathscr{X}_*$ *is true.*

Proof. 1. By $x^1\in\mathscr{X}^*$, we get

$$\mathbf{b}(x,x^1)\int_C\left[\frac{\partial f_\beta}{\partial x}\left(t,x^1,x^1_\alpha\right)(x-x^1)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^1,x^1_\alpha\right)D_\alpha(x-x^1)\right]dt^\beta$$
$$+\rho\mathbf{b}(x,x^1)\mathbf{d}(x,x^1)\geqslant 0,\quad\forall x\in\mathscr{X}.$$

Since $x^2\in\mathscr{X}^*\subset\mathscr{X}$, the previous inequality is rewritten as follows

$$\mathbf{b}(x^2,x^1)\int_C\left[\frac{\partial f_\beta}{\partial x}\left(t,x^1,x^1_\alpha\right)(x^2-x^1)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^1,x^1_\alpha\right)D_\alpha(x^2-x^1)\right]dt^\beta$$
$$+\rho\mathbf{b}(x^2,x^1)\mathbf{d}(x^2,x^1)\geqslant 0.\tag{3.1}$$

By hypothesis, the scalar functional $F(x)=\int_C f_\beta(t,x,x_\alpha)\,dt^\beta$ is $(\rho,\mathbf{b},\mathbf{d})$-convex on $\mathscr{X}$. Consequently, it results

$$F(x^1)-F(x^2)$$
$$\geqslant\mathbf{b}(x^1,x^2)\int_C\left[\frac{\partial f_\beta}{\partial x}\left(t,x^2,x^2_\alpha\right)(x^1-x^2)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^2,x^2_\alpha\right)D_\alpha(x^1-x^2)\right]dt^\beta$$
$$+\rho\mathbf{b}(x^1,x^2)\mathbf{d}(x^1,x^2),\tag{3.2}$$

or, equivalently,

$$F(x^2)-F(x^1)$$
$$\geqslant\mathbf{b}(x^2,x^1)\int_C\left[\frac{\partial f_\beta}{\partial x}\left(t,x^1,x^1_\alpha\right)(x^2-x^1)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^1,x^1_\alpha\right)D_\alpha(x^2-x^1)\right]dt^\beta$$
$$+\rho\mathbf{b}(x^2,x^1)\mathbf{d}(x^2,x^1).\tag{3.3}$$

Making the summation $(3.2)+(3.3)$ and using (3.1), we obtain

$$\mathbf{b}(x^1,x^2)\int_C\left[\frac{\partial f_\beta}{\partial x}\left(t,x^2,x^2_\alpha\right)(x^1-x^2)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^2,x^2_\alpha\right)D_\alpha(x^1-x^2)\right]dt^\beta$$

$$+\rho\mathbf{b}(x^1,x^2)\mathbf{d}(x^1,x^2) \leqslant 0. \tag{3.4}$$

Similarly as above, by $x^2 \in \mathscr{X}^*$, we can write

$$\mathbf{b}(x^1,x^2)\int_C \left[\frac{\partial f_\beta}{\partial x}\left(t,x^2,x^2_\alpha\right)\left(x^1-x^2\right)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^2,x^2_\alpha\right)D_\alpha(x^1-x^2)\right]dt^\beta$$
$$+\rho\mathbf{b}(x^1,x^2)\mathbf{d}(x^1,x^2) \geqslant 0. \tag{3.5}$$

Now, taking into account (3.4) and (3.5), the proof is complete.

2. By $x^* \in \mathscr{X}^*$, it results

$$\mathbf{b}(x,x^*)\int_C \left[\frac{\partial f_\beta}{\partial x}\left(t,x^*,x^*_\alpha\right)(x-x^*)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^*,x^*_\alpha\right)D_\alpha(x-x^*)\right]dt^\beta$$
$$+\rho\mathbf{b}(x,x^*)\mathbf{d}(x,x^*) \geqslant 0, \quad \forall x \in \mathscr{X}. \tag{3.6}$$

As well, the $(\rho,\mathbf{b},\mathbf{d})$-convexity property on $\mathscr{X}$ of the scalar functional $F(x)$ (see the summation $(3.2)+(3.3)$ and *Working hypotheses*) implies

$$\mathbf{b}(x^1,x^2)\int_C \left[\frac{\partial f_\beta}{\partial x}\left(t,x^1,x^1_\alpha\right)\left(x^1-x^2\right)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^1,x^1_\alpha\right)D_\alpha(x^1-x^2)\right]dt^\beta$$
$$+\rho\mathbf{b}(x^1,x^2)\mathbf{d}(x^1,x^2)$$
$$\geqslant\mathbf{b}(x^1,x^2)\int_C \left[\frac{\partial f_\beta}{\partial x}\left(t,x^2,x^2_\alpha\right)\left(x^1-x^2\right)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^2,x^2_\alpha\right)D_\alpha(x^1-x^2)\right]dt^\beta$$
$$+\rho\mathbf{b}(x^1,x^2)\mathbf{d}(x^1,x^2), \quad \forall x^1,x^2 \in \mathscr{X}. \tag{3.7}$$

In the following, by using the relations (3.6) and (3.7), we get

$$\mathbf{b}(x,x^*)\int_C \left[\frac{\partial f_\beta}{\partial x}\left(t,x,x_\alpha\right)(x-x^*)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x,x_\alpha\right)D_\alpha(x-x^*)\right]dt^\beta$$
$$+\rho\mathbf{b}(x,x^*)\mathbf{d}(x,x^*) \geqslant 0, \quad \forall x \in \mathscr{X},$$

and the proof is complete. □

REMARK 3.1. The continuity property of the variational derivative $\dfrac{\delta_\beta F}{\delta x}$ implies $\mathscr{X}_* \subset \mathscr{X}^*$. By Theorem 3.1, we conclude $\mathscr{X}^* = \mathscr{X}_*$. As well, the solution set $\mathscr{X}_*$ associated to $(DEVIP)$ is convex and, consequently, the solution set $\mathscr{X}^*$ associated to $(EVIP)$ is a convex set.

THEOREM 3.2. *Assume the scalar functional $H(x)$ is differentiable on $\overline{\mathscr{X}}$ and $F(x)$ is strongly $\mathbf{b}$-convex on $\mathscr{X}$. Then, for any $x,v \in \overline{\mathscr{X}}$, $y \in Z(x)$, the following inequality*

$$\left\langle \frac{\delta_\beta H}{\delta x}, v \right\rangle \geqslant \left\langle \frac{\delta_\beta F}{\delta y}, v \right\rangle$$

is true.

Proof. By Definition 2.3, for $x \in \overline{\mathscr{X}}$, we have

$$H(x) = \max_{y\in\mathscr{X}} \left\{ \mathbf{b}(x,y) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,y,y_\alpha)(x-y) + \frac{\partial f_\beta}{\partial x_\alpha}(t,y,y_\alpha) D_\alpha (x-y) \right] dt^\beta \right.$$
$$\left. + \rho \mathbf{b}(x,y)\mathbf{d}(x,y) \right\}$$

and, in accordance with Remark 2.2, we obtain

$$H(x) = \max_{y\in\mathscr{X}} \left\{ \mathbf{b}(x,y) \left[\left\langle \frac{\delta_\beta F}{\delta y}, x-y \right\rangle + \rho \mathbf{d}(x,y) \right] \right\}, \quad \forall x \in \overline{\mathscr{X}},$$

or, obviously,

$$H(x) = \mathbf{b}(x,y) \left[\left\langle \frac{\delta_\beta F}{\delta y}, x-y \right\rangle + \rho \mathbf{d}(x,y) \right], \quad \forall y \in Z(x). \tag{3.8}$$

Moreover, for any $y \in \mathscr{X}$, $z \in \overline{\mathscr{X}}$, the inequality

$$H(z) \geqslant \mathbf{b}(z,y) \left[\left\langle \frac{\delta_\beta F}{\delta y}, z-y \right\rangle + \rho \mathbf{d}(z,y) \right], \tag{3.9}$$

is true and, using (3.8), (3.9) and *Working hypotheses*, it follows

$$H(z) - H(x) \geqslant \left\langle \frac{\delta_\beta F}{\delta y}, z-x \right\rangle + \rho \mathbf{d}(z,x), \quad \forall y \in Z(x), \quad \forall x,z \in \overline{\mathscr{X}}.$$

For $z = x + \lambda v \in \overline{\mathscr{X}}$, with $\lambda > 0$ and $v(t_1) = v(t_2) = 0$, the aforementioned inequality becomes

$$H(x+\lambda v) - H(x) \geqslant \left\langle \frac{\delta_\beta F}{\delta y}, \lambda v \right\rangle + \rho \mathbf{d}(x+\lambda v, x), \quad \forall y \in Z(x), \quad \forall x,v \in \overline{\mathscr{X}}$$

and, by dividing both sides with $\lambda > 0$, we get

$$\frac{H(x+\lambda v) - H(x)}{\lambda} \geqslant \left\langle \frac{\delta_\beta F}{\delta y}, v \right\rangle + \frac{\rho \mathbf{d}(x+\lambda v, x)}{\lambda}, \quad \forall y \in Z(x), \quad \forall x,v \in \overline{\mathscr{X}}.$$

Further, knowing that F is strongly $\mathbf{b}$-convex on $\mathscr{X}$ and using *Working hypotheses*, by taking the limit for $\lambda \to 0$ of the previous inequality (and using Definition 2.2), the proof is complete. □

THEOREM 3.3. *Assume the scalar functional $H(x)$ is differentiable on $\mathscr{X}^*$ and the scalar functional $F(x)$ is strongly $\mathbf{b}$-convex on $\mathscr{X}$. Also, for any $x^* \in \mathscr{X}^*$, $v \in \overline{\mathscr{X}}$, $z \in Z(x^*)$, the following implication*

$$\left\langle \frac{\delta_\beta H}{\delta x^*}, v \right\rangle \geqslant \left\langle \frac{\delta_\beta F}{\delta z}, v \right\rangle \Longrightarrow \frac{\delta_\beta H}{\delta x^*} = \frac{\delta_\beta F}{\delta z}$$

is fulfilled. Then $Z(x^*) = \mathscr{X}^*$, $\forall x^* \in \mathscr{X}^*$.

Proof. "$\subset$" Consider $z \in Z(x^*)$. In consequence, it follows

$$\begin{aligned} H(x^*) =& \mathbf{b}(x^*,z) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x^*-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x^*-z) \right] dt^\beta \\ & + \rho \mathbf{b}(x^*,z)\mathbf{d}(x^*,z), \quad x^* \in \mathscr{X}^*. \end{aligned} \tag{3.10}$$

By hypothesis, the scalar functional $F(x)$ is strongly $\mathbf{b}$-convex on $\mathscr{X}$ and $x^* \in \mathscr{X}^*$. According to Theorem 3.1 and Remark 3.1, we get $x^* \in \mathscr{X}_*$, that is

$$\begin{aligned} & \mathbf{b}(x,x^*) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,x,x_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x,x_\alpha) D_\alpha(x-x^*) \right] dt^\beta \\ & + \rho \mathbf{b}(x,x^*)\mathbf{d}(x,x^*) \geqslant 0, \end{aligned} \tag{3.11}$$

for any $x \in \mathscr{X}$. By using (3.10), (3.11) and *Working hypotheses*, it results $H(x^*) = 0$, $\forall x^* \in \mathscr{X}^*$, or equivalently,

$$\begin{aligned} & \mathbf{b}(x^*,z) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x^*-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x^*-z) \right] dt^\beta \\ & + \rho \mathbf{b}(x^*,z)\mathbf{d}(x^*,z) = 0, \quad x^* \in \mathscr{X}^*. \end{aligned} \tag{3.12}$$

Taking into account (3.12), for any $x \in \mathscr{X}$ and using *Working hypotheses*, we obtain

$$\begin{aligned} & \mathbf{b}(x,z) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x-z) \right] dt^\beta + \rho \mathbf{b}(x,z)\mathbf{d}(x,z) \\ =& \mathbf{b}(x,x^*) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x-x^*) \right] dt^\beta \\ & + \rho \mathbf{b}(x,x^*)\mathbf{d}(x,x^*). \end{aligned} \tag{3.13}$$

Further, by definition of dual gap functional $H(x)$ associated to $(EVIP)$, for any $\lambda \in [0,1]$ and $x \in \mathscr{X}$, we can write as follows

$$\begin{aligned} & \frac{H(x^* + \lambda(x-x^*)) - H(x^*)}{\lambda} \\ \geqslant & \int_C \left[\frac{\partial f_\beta}{\partial x}(t,x^*,x^*_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^*,x^*_\alpha) D_\alpha(x-x^*) \right] dt^\beta. \end{aligned}$$

By taking the limit for $\lambda \to 0$ of the previous inequality and using Definition 2.2, we get

$$\left\langle \frac{\delta_\beta H}{\delta x^*}, x-x^* \right\rangle \geqslant \int_C \left[\frac{\partial f_\beta}{\partial x}(t,x^*,x^*_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^*,x^*_\alpha) D_\alpha(x-x^*) \right] dt^\beta. \tag{3.14}$$

According to Theorem 3.2 and using the hypothesis, we conclude $\dfrac{\delta_\beta H}{\delta x^*} = \dfrac{\delta_\beta F}{\delta z}$. Therefore, (3.14) becomes

$$\left\langle \frac{\delta_\beta F}{\delta z}, x - x^* \right\rangle \geqslant \int_C \left[\frac{\partial f_\beta}{\partial x}(t,x^*,x^*_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^*,x^*_\alpha) D_\alpha(x-x^*) \right] dt^\beta,$$

or

$$\begin{aligned} &\int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x-x^*) \right] dt^\beta \\ \geqslant &\int_C \left[\frac{\partial f_\beta}{\partial x}(t,x^*,x^*_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^*,x^*_\alpha) D_\alpha(x-x^*) \right] dt^\beta, \end{aligned}$$

which involves

$$\begin{aligned} &\mathbf{b}(x,x^*) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x-x^*) \right] dt^\beta \\ &+ \rho \mathbf{b}(x,x^*)\mathbf{d}(x,x^*) \\ \geqslant &\mathbf{b}(x,x^*) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,x^*,x^*_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^*,x^*_\alpha) D_\alpha(x-x^*) \right] dt^\beta \\ &+ \rho \mathbf{b}(x,x^*)\mathbf{d}(x,x^*). \end{aligned} \tag{3.15}$$

Combining (3.13) and (3.15), it follows

$$\begin{aligned} &\mathbf{b}(x,z) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x-z) \right] dt^\beta + \rho \mathbf{b}(x,z)\mathbf{d}(x,z) \\ \geqslant &\mathbf{b}(x,x^*) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,x^*,x^*_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^*,x^*_\alpha) D_\alpha(x-x^*) \right] dt^\beta + \rho \mathbf{b}(x,x^*)\mathbf{d}(x,x^*). \end{aligned}$$

Since $x^* \in \mathscr{X}^*$, the previous inequality implies

$$\begin{aligned} &\mathbf{b}(x,z) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x-z) \right] dt^\beta \\ &+ \rho \mathbf{b}(x,z)\mathbf{d}(x,z) \geqslant 0, \quad \forall x \in \mathscr{X}, \end{aligned}$$

involving $z \in \mathscr{X}^*$ and, in consequence, $Z(x^*) \subset \mathscr{X}^*$.

"$\supset$" Consider $z, x^* \in \mathscr{X}^*$. By Theorem 3.1, we get

$$\mathbf{b}(x^*,z) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x^*-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x^*-z) \right] dt^\beta + \rho \mathbf{b}(x^*,z)\mathbf{d}(x^*,z) = 0.$$

Since $H(x^*) = 0$, $\forall x^* \in \mathscr{X}^*$, it results

$$H(x^*) = \mathbf{b}(x^*,z) \int_C \left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x^*-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha) D_\alpha(x^*-z) \right] dt^\beta$$

$$+\rho \mathbf{b}(x^*,z)\mathbf{d}(x^*,z),$$

involving $z \in Z(x^*)$. The proof is now complete. □

EXAMPLE 3.1. Let Ω be a square fixed by the diagonally opposite points $0 = (0,0)$ and $2 = (2,2)$ in $\mathbb{R}^2$ and consider $C \subset \Omega$ a piecewise smooth curve connecting the distinct points 0 and 2 in Ω. Also, we introduce

$$\overline{\mathscr{X}} = \{x : \Omega \to [-1,4] : x = \text{piecewise smooth function}\},$$
$$\mathscr{X} = \left\{x \in \overline{\mathscr{X}} : x(t) \in [0,1] \subset [-1,4],\ x(0) = x(0,0) = 0,\ x(2) = x(2,2) = 0\right\}$$

and the vector-valued continuously differentiable function

$$f = (f_1, f_2) : J^1(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^2, \quad f_1(t,x,x_\alpha) = f_2(t,x,x_\alpha) = x^2 + 4x.$$

For $\beta = \overline{1,2}$, we consider the following variational inequality problem: *for* $\rho \in \mathbb{R}$, $\mathbf{b}(x,y) = 1$ *and any real-valued functional* $\mathbf{d}$ *on* $\overline{\mathscr{X}} \times \overline{\mathscr{X}}$ *satisfying*

$$0 < \rho \mathbf{d}(x,y) = const. \leqslant \int_C (x-y)^2 dt^\beta, \quad \forall x,y \in \mathscr{X},$$

find $y \in \mathscr{X}$ *such that*

$$(VIP) \qquad \int_C \left[(f_\beta)_x(t,y,y_\alpha)(x-y) + (f_\beta)_{x_\alpha}(t,y,y_\alpha) D_\alpha(x-y)\right] dt^\beta + \rho \mathbf{d}(x,y) \geqslant 0,$$

for any $x \in \mathscr{X}$.

By direct computation, we obtain the dual gap functional $H : \overline{\mathscr{X}} \to \mathbb{R}$,

$$\begin{aligned} &H(x) \\ &= \int_C h_\beta(t,x,x_\alpha)\, dt^\beta \\ &= \max_{y \in \mathscr{X}} \left\{ \int_C \left[(f_\beta)_x(t,y,y_\alpha)(x-y) + (f_\beta)_{x_\alpha}(t,y,y_\alpha) D_\alpha(x-y)\right] dt^\beta + \rho \mathbf{d}(x,y) \right\} \\ &= \max_{y \in \mathscr{X}} \int_C (2y+4)(x-y) dt^\beta + \rho \mathbf{d}(x,y) = \begin{cases} \displaystyle\int_C 4x dt^\beta + \rho \mathbf{d}(x,y), & -1 \leqslant x < 2 \\ \displaystyle\int_C \frac{(x+2)^2}{2} dt^\beta + \rho \mathbf{d}(x,y), & 2 \leqslant x \leqslant 4. \end{cases} \end{aligned}$$

Also, since

$$F : \overline{\mathscr{X}} \to \mathbb{R}, \quad F(x) = \int_C f_\beta(t,x,x_\alpha)\, dt^\beta,$$

satisfies

$$\begin{aligned} &F(x) - F(y) - \int_C \left[(f_\beta)_x(t,y,y_\alpha)(x-y) + (f_\beta)_{x_\alpha}(t,y,y_\alpha) D_\alpha(x-y)\right] dt^\beta - \rho \mathbf{d}(x,y) \\ &= \int_C (x-y)^2 dt^\beta - \rho \mathbf{d}(x,y) \geqslant 0, \quad \forall x,y \in \mathscr{X}, \end{aligned}$$

it follows that the path-independent curvilinear functional F is strongly $\mathbf{b}$-convex on $\mathscr{X}$, as well.

The closeness conditions imposed for the aforementioned vector-valued functions $f=(f_1,f_2)$ and $h=(h_1,h_2)$ imply $\frac{\partial x}{\partial t^1}=\frac{\partial x}{\partial t^2}$. Consequently, this property must be satisfied by all elements in $\mathscr{X}$. Obviously, the functional $H(x)$ is differentiable on $\mathscr{X}^*$ and, as it can be easily seen, we obtain

$$\mathscr{X}^*=\{y:\Omega\to[0,1]:y(t)=0,\ \forall t\in\Omega\},$$

$$Z(x^*)=\mathscr{X}^*,\quad \forall x^*\in\mathscr{X}^*,\quad \frac{\delta_\beta F}{\delta x}=2x+4.$$

4. Conclusions

In this paper, taking into account the connections between mathematical programming and some variational techniques developed in this paper, a new variational-type inequality has been studied. More precisely, by using a dual gap functional and some working hypotheses, the solution set is investigated for a variational-type inequality governed by $(\rho,\mathbf{b},\mathbf{d})$-convex path-independent curvilinear integral functional.

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