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COORDINATE STRONGLY s–CONVEX FUNCTIONS AND RELATED RESULTS

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Abstract. In this article, we give non-trivial examples of coordinate s-convex functions which are not s-convex functions. Also, we present a new class of coordinate strongly s-convex functions. We prove that every strongly s-convex function is coordinate strongly s-convex function but the converse is not generally true. Furthermore, we establish Jensen type inequality for strongly s-convex functions. We present Jensen and Hermite-Hadamard type inequalities for coordinate strongly s-convex functions.

1. Introduction

Convexity plays one of the important role in optimization theory and has applications in pure and applied sciences. Convex function is defined as [25]:

DEFINITION 1.1. A function $\Psi : [a,b] \to \mathbb{R}$ is called convex if

$$\Psi(\tau u + (1-\tau)v) \leqslant \tau\Psi(u) + (1-\tau)\Psi(v) \tag{1}$$

holds for all $u, v \in [a,b]$ and $\tau \in [0,1]$.

In the last few decade, several generalizations have been made for convexity. For examples quasi-convex [10], φ-convex [12], λ-convex [13], approximately convex [16], midconvex [17], pseudo-convex [19], strongly convex [24], h-convex [31], delta-convex [26], Schur convexity [28] and others [21, 8, 9, 1, 2, 18, 30, 25].

Dragomir [11] generalized convex functions to the concept of coordinate convex functions.

DEFINITION 1.2. Let us consider the bidimensional interval $[a,b] \times [c,d]$ in $\mathbb{R}^2$. A mapping $\Psi : [a,b] \times [c,d] \to \mathbb{R}$ is said to be coordinate convex if the partial mappings $\Psi_y : [a,b] \to \mathbb{R}$ defined as $\Psi_y(u) = \Psi(u,y)$ and $\Psi_x : [c,d] \to \mathbb{R}$ defined as $\Psi_x(v) = \Psi(x,v)$ are convex for all $x \in [a,b]$ and $y \in [c,d]$.

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REMARK 1.3. Dragomir proved that every convex function is coordinate convex, but the converse is not generally true [11].

The definition of s-convex function is given in [23] (see also [14, 7]).

DEFINITION 1.4. Let Ψ be a function which is real valued and defined on some interval $[a,b]$ and $0 < s < \infty$ be a fixed positive number, then the function Ψ is s-convex if

$$\Psi(\tau u + (1-\tau)v) \leqslant \tau^s \Psi(u) + (1-\tau)^s \Psi(v) \tag{2}$$

holds for all $u, v \in [a,b]$, $\tau \in [0,1]$.

The following lemma gives a close relation between convex and s-convex functions [7].

LEMMA 1.5. *Suppose* $\Psi : [a,b] \to \mathbb{R}$ *is convex function.*

(a): *If* Ψ *is non-negative, then* Ψ *is s-convex for* $s \in (0,1]$.

(b): *If* Ψ *is non-positive, then* Ψ *is s-convex for* $s \in [1,\infty)$.

The following definitions are related to the concept of coordinate s-convex functions [5].

DEFINITION 1.6. Let us consider the bidimensional interval $[a,b] \times [c,d]$ in $[0,\infty) \times [0,\infty)$. A function $\Psi : [a,b] \times [c,d] \to \mathbb{R}$ is said to be coordinate s-convex if the partial mappings $\Psi_y : [a,b] \to \mathbb{R}$ defined as $\Psi_y(u) = \Psi(u,y)$ and $\Psi_x : [c,d] \to \mathbb{R}$ defined as $\Psi_x(v) = \Psi(x,v)$ are s-convex for all $x \in [a,b]$ and $y \in [c,d]$ with some fixed $s \in (0,1]$.

The following remark is related to the concept of coordinate s-convex functions.

REMARK 1.7. A function $\Psi : [a,b] \times [c,d] \to \mathbb{R}$, is coordinate s-convex with some fixed $0 < s \leqslant 1$ if

$$\begin{aligned} &\Psi(\tau x_1 + (1-\tau)y_1, \xi x_2 + (1-\xi)y_2) \\ &\leqslant \tau^s \xi^s \Psi(x_1,x_2) + \tau^s (1-\xi)^s \Psi(x_1,y_2) + (1-\tau)^s \xi^s \Psi(y_1,x_2) \\ &\quad + (1-\tau)^s (1-\xi)^s \Psi(y_1,y_2) \end{aligned} \tag{3}$$

holds for all $x_1, y_1 \in [a,b]$, $x_2, y_2 \in [c,d]$ and $\tau, \xi \in [0,1]$.

REMARK 1.8. Darus and Alomari proved interesting result that every s-convex function is coordinate s-convex but the converse is not generally true [5].

In this article we present some non-trivial examples for coordinate s-convex functions.

The main focus of this article is to introduce general class of strong convexity and present related inequalities. The definition of strong convexity is given in [24].

DEFINITION 1.9. Let c be a positive real number. A function $\Psi : [a,b] \to \mathbb{R}$ is said to be strongly convex with respect to c if

$$\Psi(\tau u + (1-\tau)v) \leqslant \tau\Psi(u) + (1-\tau)\Psi(v) - c\tau(1-\tau)(u-v)^2 \tag{4}$$

holds for all $u,v \in [a,b]$ and $\tau \in [0,1]$.

REMARK 1.10. In [27, p.268] it is given that for strongly convex function Ψ we have

$$\Psi(u) - \Psi(v) \geqslant \Psi'(v)(u-v) + c(u-v)^2$$

if Ψ is differentiable function and

$$\Psi'' \geqslant 2c \tag{5}$$

if Ψ is twice differentiable function.

The following definitions of coordinate strongly convex functions are given in [4].

DEFINITION 1.11. A function $\Psi : [a,b] \times [c,d] \to \mathbb{R}$ is coordinate strongly convex if $\Psi_y : [a,b] \to \mathbb{R}$ defined as $\Psi_y(u) = \Psi(u,y)$ and $\Psi_x : [c,d] \to \mathbb{R}$ defined as $\Psi_x(v) = \Psi(x,v)$ are strongly convex for all $x \in [a,b]$ and $y \in [c,d]$.

The concept of strongly convex function on the coordinate can be defined as:

DEFINITION 1.12. ([4]) Let c_1 and c_2 be positive real numbers. A function $\Psi : [a,b] \times [c,d] \to \mathbb{R}$ is said to be coordinate strongly convex with respect to (c_1,c_2) if

$$\begin{aligned} &\Psi(\tau x_1 + (1-\tau)y_1, \xi x_2 + (1-\xi)y_2) \\ \leqslant &\tau\xi\Psi(x_1,x_2) + \tau(1-\xi)\Psi(x_1,y_2) + (1-\tau)\xi\Psi(y_1,x_2) + (1-\tau)(1-\xi)\Psi(y_1,y_2) \\ &- c_1\tau(1-\tau)(x_1-y_1)^2 - 2c_2\xi(1-\xi)(x_2-y_2)^2 \end{aligned} \tag{6}$$

holds for all $x_1,y_1 \in [a,b]$ and $x_2,y_2 \in [c,d]$ and $\tau,\xi \in [0,1]$.

REMARK 1.13. Adil Khan et al. [4] proved that every strongly convex function is coordinate strongly convex, but the converse is not generally true. For the converse the authors presented some examples.

To close this section, we provide the definition of strong s-convexity which has been given in [15].

DEFINITION 1.14. A function $\Psi : [a,b] \to [0,\infty)$ is said to be strongly s-convex with respect to $c > 0$ and $s \in (0,1]$ if

$$\Psi(\tau u + (1-\tau)v) \leqslant \tau^s\Psi(u) + (1-\tau)^s\Psi(v) - c\tau(1-\tau)(u-v)^2 \tag{7}$$

holds for all $u,v \in [a,b]$ and $\tau \in [0,1]$.

For more details related to convex, strongly convex, s-convex, strongly s-convex and coordinate s-convex functions and related inequalities we recommend [4, 3, 7, 15, 22, 5, 20, 29, 32, 33, 34].

First we present some non-trivial examples of coordinate s-convex but which are not s-convex functions. We introduce a new class of coordinate strongly s-convex functions. We prove that every strongly s-convex function is coordinate strongly s-convex, but the converse is not generally true. This paper is also devoted to establish Jensen type inequalities for strongly s-convex, coordinate s-convex and coordinate strongly s-convex functions. At the end of paper Hermite-Hadamard type inequality has been presented for coordinate strong s-convexity.

2. Main results

In the following examples we present coordinate s-convex functions which are not s-convex.

EXAMPLE 2.1. Let $\Psi : [0,5]\times[0,5] \to \mathbb{R}$ be a function defined by $\Psi(x,y) = \sqrt{xy}$. Obviously this function is not coordinate convex and therefore Ψ is not convex. Consider the partial mappings $\Psi_y(x) : [0,5] \to \mathbb{R}$ and $\Psi_x(y) : [0,5] \to \mathbb{R}$. We show that these mappings are s-convex for $s \in \left(0, \frac{1}{2}\right)$.

Now consider the partial mapping $\Psi_y(x)$, we prove that

$$\sqrt{((1-\tau)x_2+\tau x_1)y} \leqslant \tau^s\sqrt{x_1y}+(1-\tau)^s\sqrt{x_2y} \tag{8}$$

for all $\tau \in [0,1]$, $x_1, x_2, y \in [0,5]$ and $s \in (0, \frac{1}{2})$.

For $\tau = 0$ or $\tau = 1$ clearly (8) holds. Therefore we prove that (8) holds for $\tau \in (0,1)$. Squaring (8) on both sides, we have

$$\begin{gathered}((1-\tau)x_2+\tau x_1)y \leqslant \tau^{2s}x_1y+(1-\tau)^{2s}x_2y+2\tau^s(1-\tau)^s\sqrt{x_1x_2}y \\ i.e.\ (\tau^{2s}-\tau)x_1y+((1-\tau)^{2s}-(1-\tau))x_2y+2\tau^s(1-\tau)^s\sqrt{x_1x_2}y \geqslant 0.\end{gathered}$$

Let $\tau = \frac{1}{p}$, $p > 1$ then $\tau^{2s-1} = p^{1-2s} > 1$ for $s \in (0, \frac{1}{2})$. Thus, $\tau^{2s-1} > 1$

$$\tau < \tau^{2s}, \quad \text{for } s \in (0, 1/2). \tag{9}$$

Since $1-\tau \in (0,1)$, therefore by similar procedure, we have

$$1-\tau < (1-\tau)^{2s}, \quad \text{for } s \in (0, 1/2). \tag{10}$$

We conclude that (8) holds for $0 < s < \frac{1}{2}$, $0 \leqslant x_1, x_2, y \leqslant 5$ and $0 < \tau < 1$. Hence $\Psi_y(x)$ is s-convex function for $s \in (0, \frac{1}{2})$.

Similarly, $\Psi_x(y)$ is s-convex function for $s \in (0, \frac{1}{2})$, so $\Psi(x,y)$ is coordinate s-convex function for $s \in (0, \frac{1}{2})$.

Now we show that Ψ is not s-convex function.

Let $x=(x_1,x_2)$ and $y=(y_1,y_2)$, consider

$$\begin{aligned} \Psi((1-\tau)y+\tau x) &\leqslant \tau^s\Psi(x)+(1-\tau)^s\Psi(y) \\ i.e.\ \sqrt{(1-\tau)y_1+\tau x_1}\sqrt{(1-\tau)y_2+\tau x_2} &\leqslant \tau^s\sqrt{x_1x_2}+(1-\tau)^s\sqrt{y_1y_2}. \end{aligned} \tag{11}$$

Substituting $x_1=y_2=1$ and $x_2=y_1=0$ in (11), we have $\sqrt{\tau(1-\tau)}\leqslant 0$, which is not true for all $\tau\in(0,1)$. Hence Ψ is not s-convex function.

EXAMPLE 2.2. Consider $\Psi:[0,10]\times[0,10]\to\mathbb{R}$ defined by $\Psi(x,y)=(x-1)^2(y-1)^2$, then Ψ is coordinate s-convex but not s-convex function.

Clearly $\frac{\partial^2\Psi(x,y)}{\partial x^2}=2(y-1)^2\geqslant 0$ and $\frac{\partial^2\Psi(x,y)}{\partial y^2}=2(x-1)^2\geqslant 0$, for all $x,y\in[0,10]$. Therefore, by Lemma 1.5 we can say that the partial mappings $\Psi_x(y)=(x-1)^2(y-1)^2$ and $\Psi_y(x)=(x-1)^2(y-1)^2$ are s-convex for $0<s\leqslant 1$. Hence, $\Psi(x,y)$ is coordinates s-convex function.

Now, to show that the function $\Psi(x,y)=(x-1)^2(y-1)^2$ is not s-convex function. On contrary suppose that Ψ is s-convex function, then

$$\begin{aligned} &((1-\tau)y_1+\tau x_1-1)^2((1-\tau)y_2+\tau x_2-1)^2 \\ \leqslant&\tau^s(x_1-1)^2(x_2-1)^2+(1-\tau)^s(y_1-1)^2(y_2-1)^2. \end{aligned} \tag{12}$$

Substituting $x_1=1$, $x_2=0$, $y_1=3$, $y_2=1$ and $\tau=\frac{1}{2}$, we obtain $\frac{1}{4}\leqslant 0$ for $s\in(0,\infty)$, which is contradiction. Thus, the function Ψ is not s-convex.

Now, we give the definition of coordinate strongly s-convex function.

DEFINITION 2.3. A function $\Psi:[a,b]\times[c,d]\to\mathbb{R}$ is coordinate strongly s-convex if $\Psi_y:[a,b]\to\mathbb{R}$ defined as $\Psi_y(u)=\Psi(u,y)$ and $\Psi_x:[c,d]\to\mathbb{R}$ defined as $\Psi_x(v)=\Psi(x,v)$ are strongly s-convex for all $x\in[a,b]$ and $y\in[c,d]$.

The following remark is related to the concept of coordinate strong s-convexity.

REMARK 2.4. Let c_1 and c_2 be positive real numbers and s be fixed positive real number. If $\Psi:[a,b]\times[c,d]\to\mathbb{R}$ is coordinate strongly s-convex with respect to (c_1,c_2) that is Ψ_x is strongly s-convex with respect to c_2 and Ψ_y is strongly s-convex with respect to c_1, then the inequality

$$\begin{aligned} &\Psi(\tau x_1+(1-\tau)y_1,\xi x_2+(1-\xi)y_2) \\ \leqslant&\tau^s\xi^s\Psi(x_1,x_2)+\tau^s(1-\xi)^s\Psi(x_1,y_2)+(1-\tau)^s\xi^s\Psi(y_1,x_2) \\ &+(1-\tau)^s(1-\xi)^s\Psi(y_1,y_2)-c_1\tau(1-\tau)(x_1-y_1)^2-2c_2\xi(1-\xi)(x_2-y_2)^2 \end{aligned} \tag{13}$$

holds for all $x_1,y_1\in[a,b]$, $x_2,y_2\in[c,d]$ and $\tau,\xi\in[0,1]$.

The following lemma will play a key role to prove strong s-convexity of a function.

LEMMA 2.5. *Suppose $\Psi:[a,b]\to\mathbb{R}$ is strongly convex with respect to c.*

(a): *If Ψ is non-negative, then Ψ is strongly s-convex for $s \in (0,1]$.*

(b): *If Ψ is non-positive, then Ψ is strongly s-convex for $s \in [1,\infty)$.*

Proof. **(a):** If Ψ is non-negative strongly convex with respect to c, then

$$\Psi((1-\tau)y+\tau x)+c\tau(1-\tau)(x-y)^2 \leqslant (1-\tau)\Psi(y)+\tau\Psi(x) \tag{14}$$

for $x,y \in [a,b]$ and $0<\tau<1$.

Since $\tau \in [0,1]$, so $\tau^s \geqslant \tau$ and $(1-\tau)^s \geqslant 1-\tau$ for $s \in (0,1]$.

$$\tau\Psi(x) \leqslant \tau^s\Psi(x) \quad \text{and} \quad 1-\tau\Psi(x) \leqslant (1-\tau)^s\Psi(x).$$

Therefore (14), can be written as

$$\Psi(\tau x+(1-\tau)y) \leqslant \tau^s\Psi(x)+(1-\tau)^s\Psi(y)-c\tau(1-\tau)(x-y)^2.$$

This shows that Ψ is strongly s-convex for $s \in (0,1]$.

Similarly, we can prove part **(b)**. □

LEMMA 2.6. *Every strongly s-convex function $\Psi : [a,b]\times[c,d] \to \mathbb{R}$ is coordinate strongly s-convex, but the converse is not generally true.*

Proof. Suppose that $\Psi : [a,b]\times[c,d] \to \mathbb{R}$ is strongly s-convex with respect to c. Consider $\Psi_x : [c,d] \to \mathbb{R}$ defined by $\Psi_x(v) = \Psi(x,v)$, so for all $v,w \in [c,d]$ and $0 \leqslant \tau \leqslant 1$, we have

$$\begin{aligned}\Psi_x(\tau v+(1-\tau)w) &= \Psi(x,\tau v+(1-\tau)w),\\ &\leqslant \tau^s\Psi(x,v)+(1-\tau)^s\Psi(x,w)-c\tau(1-\tau)(v-w)^2,\\ &= \tau^s\Psi_x(v)+(1-\tau)^s\Psi_x(w)-c\tau(1-\tau)(v-w)^2.\end{aligned}$$

Hence Ψ_x is strongly s-convex function on $[c,d]$.

Similarly consider $\Psi_y : [a,b] \to \mathbb{R}$ defined by $\Psi_y(u) = \Psi(u,y)$, so for all $u,w \in [a,b]$ and $\tau \in [0,1]$, we have

$$\begin{aligned}\Psi_y(\tau u+(1-\tau)w) &= \Psi(\tau u+(1-\tau)w,y),\\ &\leqslant \tau^s\Psi(u,y)+(1-\tau)^s\Psi(w,y)-c\tau(1-\tau)(u-w)^2,\\ &= \tau^s\Psi_y(u)+(1-\tau)^s\Psi_y(w)-c\tau(1-\tau)(u-w)^2.\end{aligned}$$

This shows that strongly s-convex function is coordinate strong s-convex. □

Now we give examples of coordinate strongly s-convex functions which are not strongly s-convex functions.

EXAMPLE 2.7. Consider $\Psi : [-15,10]\times[-15,10] \to \mathbb{R}$ is defined by $\Psi(x,y) = x^2y^2+x^2+y^2$, we show that Ψ is coordinate strongly s-convex function but not strongly s-convex.

Obviously $\frac{\partial^2\Psi(x,y)}{\partial x^2} = 2y^2+2 \geqslant 2$ and $\frac{\partial^2\Psi(x,y)}{\partial y^2} = 2x^2+2 \geqslant 2$, for all $x,y \in [-15,10]$. Therefore, by Lemma 2.5 and (5) we can say that the partial mappings $\Psi_x(y) = x^2y^2 + x^2+y^2$ and $\Psi_y(x) = x^2y^2+x^2+y^2$ are strongly s-convex for $0<s\leqslant 1$. Hence, $\Psi(x,y)$ is coordinates strongly s-convex for $0<s\leqslant 1$.

Now we show that Ψ is not strongly s-convex function. On contrary suppose that Ψ is strongly s-convex, by using the definition of strong s-convexity, we have

$$\begin{aligned}&\{\tau x_1+y_1(1-\tau)\}^2\{\tau x_2+y_2(1-\tau)\}^2+\{\tau x_1+y_1(1-\tau)\}^2+\{\tau x_2+y_2(1-\tau)\}^2\\ \leqslant&\tau^s\left(x_1^2+x_2^2+x_1^2x_2^2\right)+\left(y_1^2+y_2^2+y_1^2y_2^2\right)(1-\tau)^s-c\tau(1-\tau)\{(x_1-y_1)^2+(x_2-y_2)^2\}\end{aligned}$$

for all $x_1,y_1,x_2,y_2 \in [-15,10]$ and for some $0<s\leqslant 1$.

Setting $x_1=-1$, $x_2=1$, $y_1=-2$, $y_2=2$ and $\tau=\frac{1}{2}$, we have $\frac{99}{4} \leqslant \frac{27}{2^s}-\frac{c}{2}$ for $s\in(0,1]$ and $c>0$. This shows that Ψ is not strongly s-convex function.

EXAMPLE 2.8. Consider $\Psi : [3,20]\times[3,20] \to \mathbb{R}$ is defined by $\Psi(x,y) = (x-2)^2(y-2)^2$, we show that Ψ is coordinate strongly s-convex function but not strongly s-convex.

Clearly $\frac{\partial^2\Psi(x,y)}{\partial x^2} = 2(y-2)^2 \geqslant 2$ and $\frac{\partial^2\Psi(x,y)}{\partial y^2} = 2(x-2)^2 \geqslant 2$, for all $x,y\in[3,20]$. Therefore, by Lemma 2.5 and (5) we can say that the partial mappings $\Psi_x(y) = (x-2)^2(y-2)^2$ and $\Psi_y(x) = (x-2)^2(y-2)^2$ are strongly s-convex functions for $s\in(0,1]$. Particularly, for $s=\frac{9}{10}$, Ψ is coordinates strongly $\frac{9}{10}$-convex function.

Now we show that Ψ is not strongly s-convex function for $s=\frac{9}{10}$. On contrary suppose that Ψ is strongly $\frac{9}{10}$-convex function, so by using the definition of strong $\frac{9}{10}$-convexity, we have

$$\begin{aligned}&((1-\tau)y_1+\tau x_1-2)^2((1-\tau)y_2+\tau x_2-2)^2\\ \leqslant&\tau^{\frac{9}{10}}(x_1-2)^2(x_2-2)^2+(1-\tau)^{\frac{9}{10}}(y_1-2)^2(y_2-2)^2\\ &-c\tau(1-\tau)\{(x_2-y_2)^2+(x_1-y_1)^2\}\end{aligned} \tag{15}$$

for all $x_1,y_1,x_2,y_2\in[3,20]$.

Substituting $x_1=3$, $x_2=4$, $y_1=6$, $y_2=3$ and $\tau=\frac{1}{2}$, we have $14.062 \leqslant 10.717-\frac{5c}{2}$ for $c>0$, which is contradiction. Thus, the function $\Psi(x,y)$ is not strongly $\frac{9}{10}$-convex function.

Jensen type inequality for strong s-convexity is presented in the following theorem.

THEOREM 2.9. *Suppose $\Psi : [a,b] \to \mathbb{R}$ is strongly s-convex function with respect to c for $s\in(0,1]$. Also, let $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in [a,b]^n$ and $\mathbf{q} = (q_1,q_2,\ldots,q_n)$ be positive n-tuple such that $\sum_{i=1}^{n} q_i = 1$ and $\overline{x} = \sum_{i=1}^{n} q_ix_i$. Then the following inequality holds*

$$\Psi\left(\sum_{i=1}^{n} q_ix_i\right) \leqslant \sum_{i=1}^{n} q_i^s\Psi(x_i) - c\sum_{i=1}^{n} q_i(x_i-\overline{x})^2. \tag{16}$$

Proof. By using induction method we will prove that (16) holds for $n \geqslant 2$. For $n = 2$, (16) can be written as

$$\Psi(q_1x_1 + q_2x_2) \leqslant q_1^s\Psi(x_1) + q_2^s\Psi(x_2) - c\left[q_1(x_1 - \overline{x})^2 + q_2(x_2 - \overline{x})^2\right], \tag{17}$$

where $\overline{x} = q_1x_1 + q_2x_2$. To prove that the inequality (17) is true for $n = 2$ it is enough to show that

$$q_1(1 - q_1)(x_1 - x_2)^2 = q_1(x_1 - \overline{x})^2 + (1 - q_1)(x_2 - \overline{x})^2. \tag{18}$$

Let $x_1, x_2 \in [\lambda, \xi]$ and $q_1, q_2 \in [0, 1]$ such that $q_1 + q_2 = 1$, then by putting the value of $\overline{x} = q_1x_1 + (1 - q_1)x_2$ in the right side of (18), we have

$$\begin{aligned}
&q_1(x_1 - \overline{x})^2 + (1 - q_1)(x_2 - \overline{x})^2\\
=&q_1(x_1 - q_1x_1 - x_2 + q_1x_2)^2 + (1 - q_1)(x_2 - q_1x_1 - x_2 + q_1x_2)^2\\
=&q_1\{(1 - q_1)x_1 - x_2(1 - q_1)\}^2 + q_1^2(1 - q_1)(x_1 - x_2)^2\\
=&q_1(1 - q_1)^2(x_1 - x_2)^2 + q_1^2 - q_1^3(x_1 - x_2)^2\\
=&(q_1 + q_1^3 - 2q_1^2 + q_1^2 - q_1^3)(x_1 - x_2)^2\\
=&q_1(1 - q_1)(x_1 - x_2)^2.
\end{aligned}$$

Suppose that for $n - 1$ the inequality (16) holds.

Consider $x_1, x_2, ..., x_n \in [\lambda, \xi]$ and $q_1, q_2, ..., q_n \geqslant 0$ with $\sum\limits_{i=1}^{n} q_i = 1$. If there is some $q_i = 0$, then by hypothesis inequality (16) holds. Now suppose $q_i \neq 0$ for all $i = 1, 2, ..., n$. Suppose that $\alpha_1 = \sum\limits_{i=1}^{n-1} q_i$, $\alpha_2 = q_n$, $z_1 = \frac{1}{\alpha_1}\sum\limits_{i=1}^{n-1} q_ix_i$, $z_2 = x_n$ and $\overline{x} = \sum\limits_{i=1}^{n} q_ix_i$. Since $\sum\limits_{i=1}^{n-1} \frac{q_i}{\alpha_1} = 1$ and using the hypothesis, we have

$$\Psi(z_1) = \Psi\left(\sum_{i=1}^{n-1} \frac{q_i}{\alpha_1}x_i\right) \leqslant \sum_{i=1}^{n-1}\left(\frac{q_i}{\alpha_1}\right)^s\Psi(x_i) - c\sum_{i=1}^{n-1}\left(\frac{q_i}{\alpha_1}\right)(x_i - z_1)^2. \tag{19}$$

Then, we can write as:

$$\begin{aligned}
&\Psi\left(\sum_{i=1}^{n} q_ix_i\right)\\
=&\Psi(\alpha_1z_1 + \alpha_2z_2)\\
\leqslant&\alpha_1^s\Psi(z_1) + \alpha_2^s\Psi(z_2) - c\left\{\alpha_1(z_1 - (\alpha_1z_1 + \alpha_2z_2))^2 + \alpha_2(z_2 - (\alpha_1z_1 + \alpha_2z_2))^2\right\}\\
\leqslant&\sum_{i=1}^{n-1} q_i^s\Psi(x_i) + q_n^s\Psi(x_n) - c\alpha_1^s\sum_{i=1}^{n-1}\left(\frac{q_i}{\alpha_1}\right)(x_i - z_1)^2 - c\left\{\alpha_1(z_1 - \overline{x})^2 + q_n(x_n - \overline{x})^2\right\}\\
=&\sum_{i=1}^{n} q_i^s\Psi(x_i) - c\left\{\frac{\alpha_1^s}{\alpha_1}\sum_{i=1}^{n-1} q_i(x_i^2 + z_1^2 - 2x_iz_1) + \alpha_1(z_1^2 + \overline{x}^2 - 2z_1\overline{x}) + q_n(x_n - \overline{x})^2\right\}
\end{aligned}$$

$$\leqslant \sum_{i=1}^{n} q_i^s \Psi(x_i) - c\left\{\sum_{i=1}^{n-1} q_i(x_i^2 + z_1^2 - 2x_i z_1) + \alpha_1(z_1^2 + \overline{x}^2 - 2z_1\overline{x}) + q_n(x_n - \overline{x})^2\right\},$$

(as $\alpha_1^s/\alpha_1 \geqslant 1$ for all $s \in (0,1)$)

$$= \sum_{i=1}^{n} q_i^s \Psi(x_i) - c\left\{\left(\sum_{i=1}^{n-1} q_i x_i^2 + \alpha_1 z_1^2 - 2\sum_{i=1}^{n-1} q_i x_i z_1 + \alpha_1 z_1^2 + \alpha_1 \overline{x}^2 - 2\alpha_1 z_1 \overline{x}\right) + q_n(x_n - \overline{x})^2\right\}$$

$$= \sum_{i=1}^{n} q_i^s \Psi(x_i) - c\left\{\left(\sum_{i=1}^{n-1} q_i x_i^2 + \sum_{i=1}^{n-1} q_i \overline{x}^2 - 2\sum_{i=1}^{n-1} q_i x_i \overline{x}\right) + q_n(x_n - \overline{x})^2\right\}$$

$$= \sum_{i=1}^{n} q_i^s \Psi(x_i) - c\left\{\sum_{i=1}^{n-1} q_i(x_i^2 + \overline{x}^2 - 2x_i\overline{x}) + q_n(x_n - \overline{x})^2\right\}$$

$$= \sum_{i=1}^{n} q_i^s \Psi(x_i) - c\left\{\sum_{i=1}^{n-1} q_i(x_i - \overline{x})^2 + q_n(x_n - \overline{x})^2\right\}$$

$$= \sum_{i=1}^{n} q_i^s \Psi(x_i) - c\sum_{i=1}^{n} q_i(x_i - \overline{x})^2. \quad \square$$

The following results are different formulations of the above Jensen type inequality.

COROLLARY 2.10. *Suppose* $\Psi : [a,b] \to \mathbb{R}$ *is strongly* s*-convex function with respect to* c*. Also, let* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [a,b]^n$ *and* $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ *be a positive* $n-$*tuple such that* $Q_n = \sum_{i=1}^{n} q_i$ *and* $\overline{x} = \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i$*, then*

$$\Psi\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right) \leqslant \frac{1}{Q_n^s}\sum_{i=1}^{n} q_i^s \Psi(x_i) - \frac{c}{Q_n}\sum_{i=1}^{n} q_i(x_i - \overline{x})^2.$$

COROLLARY 2.11. *Suppose* $\Psi : [a,b] \to \mathbb{R}$ *is strongly* s*-convex function with respect to* c*. Also, let* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [a,b]^n$ *and* $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ *be a positive* $n-$*tuple such that* $Q_n = \sum_{i=1}^{n} q_i^{\frac{1}{s}}$ *and* $\overline{x} = \frac{1}{Q_n}\sum_{i=1}^{n} q_i^{\frac{1}{s}} x_i$*, then*

$$\Psi\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i^{\frac{1}{s}} x_i\right) \leqslant \frac{1}{Q_n^{\frac{1}{s}}}\sum_{i=1}^{n} q_i \Psi(x_i) - \frac{c}{Q_n}\sum_{i=1}^{n} q_i(x_i - \overline{x})^2.$$

Jensen type inequality for coordinate strong s-convexity is presented in the following theorem.

THEOREM 2.12. *Let* $\Psi : [a,b] \times [c,d] \to \mathbb{R}$ *be coordinate strongly* s*-convex function with respect to* c_1 *and* c_2*. Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [a,b]^n$*,* $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in$

$[c,d]^m$, $\mathbf{q}=(q_1,q_2,\ldots,q_n)$ be a positive $n-$tuple and $\mathbf{w}=(w_1,w_2,\ldots,w_m)$ be a positive $m-$tuple such that $Q_n=\sum_{i=1}^{n}q_i=1$ and $W_m=\sum_{j=1}^{m}w_j=1$. Also let $\overline{x}=\sum_{i=1}^{n}q_ix_i$ and $\overline{y}=\sum_{j=1}^{m}w_iy_j$, then we have

$$\begin{aligned}
&\Psi(\overline{x},\overline{y})+c_1\sum_{i=1}^{n}q_i(x_i-\overline{x})^2+c_2\sum_{j=1}^{m}w_j(y_i-\overline{y})^2\\
\leqslant&\frac{1}{2}\left\{\sum_{i=1}^{n}q_i^s\Psi(x_i,\overline{y})+\sum_{j=1}^{m}w_j^s\Psi(\overline{x},y_j)+c_1\sum_{i=1}^{n}q_i(x_i-\overline{x})^2+c_2\sum_{j=1}^{m}w_j(y_i-\overline{y})^2\right\}\\
\leqslant&\sum_{i=1}^{n}\sum_{j=1}^{m}q_i^sw_j^s\Psi(x_i,y_j).
\end{aligned}\tag{20}$$

Proof. By using Jensen type inequality for strong s-convexity, we have

$$\Psi(x_i,\overline{y})+c_2\sum_{j=1}^{m}w_j(y_j-\overline{y})^2\leqslant\sum_{j=1}^{m}w_j^s\Psi(x_i,y_j),\tag{21}$$

and

$$\Psi(\overline{x},y_j)+c_1\sum_{i=1}^{n}q_i(x_i-\overline{x})^2\leqslant\sum_{i=1}^{n}q_i^s\Psi(x_i,y_j).\tag{22}$$

Multiplying (21) and (22) by $q_i^s\geqslant 0$ and $w_j^s\geqslant 0$ respectively and summing from $i=1,2,\ldots,n,$ and $j=1,2,\ldots,m$ respectively, we obtain

$$\sum_{i=1}^{n}q_i^s\Psi(x_i,\overline{y})+c_2\sum_{j=1}^{m}w_j(y_j-\overline{y})^2\leqslant\sum_{i=1}^{n}\sum_{j=1}^{m}q_i^sw_j^s\Psi(x_i,y_j),\tag{23}$$

and

$$\sum_{j=1}^{m}w_j^s\Psi(\overline{x},y_j)+c_1\sum_{i=1}^{n}q_i(x_i-\overline{x})^2\leqslant\sum_{i=1}^{n}\sum_{j=1}^{m}q_i^sw_j^s\Psi(x_i,y_j).\tag{24}$$

Adding (23) and (24), we have

$$\begin{aligned}
&\frac{1}{2}\left\{\sum_{i=1}^{n}q_i^s\Psi(x_i,\overline{y})+\sum_{j=1}^{m}w_j^s\Psi(\overline{x},y_j)+c_1\sum_{i=1}^{n}q_i(x_i-\overline{x})^2+c_2\sum_{j=1}^{m}w_j(y_i-\overline{y})^2\right\}\\
\leqslant&\sum_{i=1}^{n}\sum_{j=1}^{m}q_i^sw_j^s\Psi(x_i,y_j),
\end{aligned}\tag{25}$$

hence the right side of (20). Similarly, for the left side of (20), again using Jensen type inequality we have

$$\Psi(\overline{x},\overline{y})+c_1\sum_{i=1}^{n}q_i(x_i-\overline{x})^2\leqslant\sum_{i=1}^{n}q_i^s\Psi(x_i,\overline{y}),\tag{26}$$

and

$$\Psi(\bar{x},\bar{y})+c_2\sum_{j=1}^{m} w_j(y_j-\bar{y})^2 \leqslant \sum_{j=1}^{m} w_j^s \Psi(\bar{x},y_j). \tag{27}$$

Adding $c_1\sum_{i=1}^{n} q_i(x_i-\bar{x})$ and $c_2\sum_{j=1}^{m} w_j(y_j-\bar{y})$ to both sides of (27) and (26) respectively, adding both inequalities we obtain

$$\begin{aligned}&\Psi(\bar{x},\bar{y})+c_1\sum_{i=1}^{n} q_i(x_i-\bar{x})^2+c_2\sum_{j=1}^{m} w_j(y_i-\bar{y})^2\\ &\leqslant\frac{1}{2}\left\{\sum_{i=1}^{n} q_i^s\Psi(x_i,\bar{y})+\sum_{j=1}^{m} w_j^s\Psi(\bar{x},y_j)+c_1\sum_{i=1}^{n} q_i(x_i-\bar{x})^2+c_2\sum_{j=1}^{m} w_j(y_i-\bar{y})^2\right\}.\end{aligned} \tag{28}$$

Combining (25) and (28), we deduced (20). □

Now we give the following corollary related to the Jensen type inequality for coordinate s-convex function. This result is improved version for convex function on the coordinate given in [6].

COROLLARY 2.13. *Under the assumptions of Theorem 2.12, the following inequalities hold*

$$\Psi(\bar{x},\bar{y})\leqslant\frac{1}{2}\left\{\sum_{i=1}^{n} q_i^s\Psi(x_i,\bar{y})+\sum_{j=1}^{m} w_j^s\Psi(\bar{x},y_j)\right\}\leqslant\sum_{i=1}^{n}\sum_{j=1}^{m} q_i^s w_j^s\Psi(x_i,y_j). \tag{29}$$

We end this paper with the Hermite-Hadamard type inequality for strong s-convexity on the coordinate.

THEOREM 2.14. *Let $\Psi:[a,b]\times[c,d]\to\mathbb{R}$ be coordinate strongly s-convex function with respect to (c_1,c_2), then*

$$\begin{aligned}&4^{s-1}\left[\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{c_1}{12}(b-a)^2+\frac{c_2}{12}(d-c)^2\right]\\ &\leqslant\frac{2^{s-1}}{2}\left\{\frac{1}{b-a}\int_a^b\Psi\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d\Psi\left(\frac{a+b}{2},y\right)dy\right.\\ &\quad\left.+\frac{c_1}{12}(b-a)^2+\frac{c_2}{12}(d-c)^2\right\}\\ &\leqslant\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\Psi(x,y)\,dxdy\\ &\leqslant\frac{1}{2(s+1)}\left[\frac{1}{b-a}\int_a^b\{\Psi(x,c)+\Psi(x,d)\}dx\right.\end{aligned} \tag{30}$$

$$+\frac{1}{d-c}\int_c^d \{\Psi(a,y)+\Psi(b,y)\}dy\Bigg] - \frac{c_1}{12}(b-a)^2 - \frac{c_2}{12}(d-c)^2$$

$$\leqslant \frac{\Psi(a,c)+\Psi(b,c)+\Psi(a,d)+\Psi(b,d)}{(s+1)^2} - \frac{1}{4(s+1)}\left\{c_1(b-a)^2 - c_2(d-c)^2\right\}.$$

Proof. By Hermite-Hadamard type inequality for strong s-convexity, we have [5]:

$$2^{s-1}\left[\Psi_x\left(\frac{c+d}{2}\right) + \frac{c_2}{12}(d-c)^2\right] \leqslant \frac{1}{d-c}\int_c^d \Psi_x(y)dy$$

$$\leqslant \frac{\Psi_x(c)+\Psi_x(d)}{s+1} - \frac{c_2}{6}(d-c)^2.$$

Then

$$2^{s-1}\left[\Psi\left(x,\frac{c+d}{2}\right) + \frac{c_2}{12}(d-c)^2\right] \leqslant \frac{1}{d-c}\int_c^d \Psi(x,y)dy$$

$$\leqslant \frac{\Psi(x,c)+\Psi(x,d)}{s+1} - \frac{c_2}{6}(d-c)^2.$$

By integrating with respect to x, we obtain

$$2^{s-1}\left[\frac{1}{b-a}\int_a^b \Psi\left(x,\frac{c+d}{2}\right)dx + \frac{c_2}{12}(d-c)^2\right]$$

$$\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \Psi(x,y)\,dxdy \tag{31}$$

$$\leqslant \frac{1}{s+1}\left[\frac{1}{b-a}\int_a^b \{\Psi(x,c)+\Psi(x,d)\}dx\right] - \frac{c_2}{6}(d-c)^2.$$

By similar arguments, we also have

$$2^{s-1}\left[\frac{1}{d-c}\int_c^d \Psi\left(\frac{a+b}{2},y\right)dy + \frac{c_1}{12}(b-a)^2\right]$$

$$\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \Psi(x,y)\,dxdy \tag{32}$$

$$\leqslant \frac{1}{s+1}\left[\frac{1}{d-c}\int_c^d \{\Psi(a,y)+\Psi(b,y)\}dy\right] - \frac{c_1}{6}(b-a)^2.$$

Adding (31) and (32), we obtain the third and second inequalities of (30).

Also, using Hermite-Hadamard type inequality for strong s-convexity, we deduce

$$2^{s-1}\left[\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{c_1}{12}(b-a)^2\right]\leqslant\frac{1}{b-a}\int_a^b\Psi\left(x,\frac{c+d}{2}\right)dx \tag{33}$$

and

$$2^{s-1}\left[\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{c_2}{12}(d-c)^2\right]\leqslant\frac{1}{d-c}\int_c^d\Psi\left(\frac{a+b}{2},y\right)dy. \tag{34}$$

Adding $\frac{c_2}{12}(d-c)^2$ and $\frac{c_1}{12}(b-a)^2$ to both sides of (33) and (34) respectively and then multiplying both inequalities by 2^{s-1}, we obtain the first inequality of (30).

By using the right side of Hermite-Hadamard type inequality given in [5], to obtain the last inequality

$$\frac{1}{b-a}\int_a^b\Psi(x,c)dx\leqslant\frac{\Psi(a,c)+\Psi(b,c)}{s+1}-\frac{c_1}{6}(b-a)^2,$$

$$\frac{1}{b-a}\int_a^b\Psi(x,d)dx\leqslant\frac{\Psi(a,d)+\Psi(b,d)}{s+1}-\frac{c_1}{6}(b-a)^2,$$

$$\frac{1}{d-c}\int_c^d\Psi(a,y)dy\leqslant\frac{\Psi(a,c)+\Psi(a,d)}{s+1}-\frac{c_2}{6}(d-c)^2,$$

and

$$\frac{1}{d-c}\int_c^d\Psi(b,y)dy\leqslant\frac{\Psi(b,c)+\Psi(b,d)}{s+1}-\frac{c_2}{6}(d-c)^2,$$

adding all these inequalities and then multiplying both sides by $\frac{1}{s+1}$. Adding $\frac{c_1}{12}(b-a)^2$ and $\frac{c_2}{12}(d-c)^2$ to the both sides of obtained inequality, we deduce the last inequality of (30). □

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