Journal of Mathematical Inequalities
Volume 14, Number 3 (2020), 867–874
doi:10.7153/jmi-2020-14-56

ON HERMITE–HADAMARD TYPE INEQUALITIES FOR F–CONVEX FUNCTIONS

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(*Communicated by A. Witkowski*)

Abstract. In this paper we give two different Hermite-Hadamard type inequalities for F-convex functions. As special cases of it we get known and new Hermite-Hadamard type inequalities for different concepts of convexity.

1. Introduction

In this paper by I we denote a nonempty and open interval of $\mathbb{R}$. It is well known that for a convex function $f: I \to \mathbb{R}$ the following inequality is true

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(u)du \leqslant \frac{f(x)+f(y)}{2} \tag{1}$$

for all $x,y \in I$ $(x \neq y)$. This is the classical Hermite-Hadamard inequality [7] (see also [9] for interesting historical remarks). This inequality constitutes a crucial element of convex analysis and it has a vast literature concerning its generalizations, refinements, applications and concepts of convexity (cf. e.g. [5, 8, 12] with the references therein). One of the concepts of convex functions was introduced by Polyak [16]. Namely, a function $f: I \to \mathbb{R}$ is called *strongly convex with modulus* $c > 0$ if

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2$$

for all $x,y \in I$ and $t \in [0,1]$. Since strong convexity is an essential strengthening of convexity (cf. [14]), we can expect a better estimation of the integral mean for strongly convex functions than (1). In [10] the authors proved that for a strongly convex function with modulus $c > 0$, $f: I \to \mathbb{R}$, the following Hermite-Hadamard type inequality is true

$$f\left(\frac{x+y}{2}\right) + \frac{c}{12}(x-y)^2 \leqslant \frac{1}{y-x}\int_x^y f(u)du \leqslant \frac{f(x)+f(y)}{2} - \frac{c}{6}(x-y)^2$$

for all $x,y \in I$ $(x \neq y)$. Notice that by following the proof of this result, the assumption "$c > 0$" is not essential – we can assume that $c \in \mathbb{R}$. In [1], as a generalization of strongly convex functions, we can find the concept of F-convex functions. We adopt the following definition.

Mathematics subject classification (2010): 26D15, 26A51, 39B62.
Keywords and phrases: F-convex functions, generalized convexity, Hermite-Hadamard type inequalities.

Paper JMI-14-56

DEFINITION 1. Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function. A function $f:I\to\mathbb{R}$ is called *F-convex* if

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y)-t(1-t)F(x-y) \tag{2}$$

for all $x,y\in I$ and $t\in[0,1]$.

Such functions were defined in the context of strongly convex functions. But note that from F-convexity we can also obtain another concepts of convexity:

- for $F(x)=-cx^2$ we get the definition of *c-convex functions* introduced by J.P. Vial (see [18]);
- for $F(x)=-c|x|$ with $c>0$ we get *approximate convex functions* introduced by H.V. Ngai, D.T. Luc and M. Théra (see [11]);
- for $F(x)=-c|x|^p$ with $c>0$ and $p>0$ we get *approximately convex functions of order p* introduced by K. Nikodem and Zs. Páles (see [13]);
- for $F(x)=-|x|\omega(|x|)$ with nondecreasing, upper-semicontinuous function $\omega:[0,+\infty)\to[0,+\infty)$ such that $\omega(0)=0$ we obtain the definition of *semiconvex functions* introduced by G. Alberti, L. Ambrosio and P. Cannarsa (see [4]).

2. Useful tools

From the result presented in [2] (Theorem 1 and its proof) we conclude:

THEOREM 1. *Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function. If an F-convex function $f:I\to\mathbb{R}$ is one-sided differentiable at a point $x_0\in I$ and $f'_-(x_0)\leqslant f'_+(x_0)$, then the following inequality is true*

$$f(x)\geqslant F(x-x_0)+a(x-x_0)+f(x_0),\quad x\in I,$$

where a is an arbitrary number such that $f'_-(x_0)\leqslant a\leqslant f'_+(x_0)$.

In paper [3] we find the following results:

THEOREM 2. *Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function. If a function $f:I\to\mathbb{R}$ is F-convex, then it is continuous.*

THEOREM 3. *Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function. If*

$$\liminf_{x\to 0}\frac{F(x)}{x^2}>-\infty,$$

then every F-convex function $f:I\to\mathbb{R}$ has one-sided derivatives at each point $x\in I$ and $f'_-(x)\leqslant f'_+(x)$.

THEOREM 4. *Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function. If*

$$-\infty<\liminf_{x\to 0}\frac{F(x)}{x^2}<+\infty,$$

then every F-convex function $f:I\to\mathbb{R}$ is c-convex (in the sense of Vial), where $c=\liminf_{x\to 0}\frac{F(x)}{x^2}$.

3. Main results

We start with three lemmas. The first one sets an upper bound of integral mean for an F-convex function. Further lemmas give two different lower bounds of integral mean for an F-convex function, respectively.

LEMMA 1. *Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function. If a function $f:I\to\mathbb{R}$ is F-convex, then*

$$\frac{1}{y-x}\int_x^y f(u)du\leqslant\frac{f(x)+f(y)}{2}-\frac{1}{6}F(x-y)\tag{3}$$

for all $x,y\in I$ $(x\neq y)$.

Proof. From Theorem 2 each F-convex function $f:I\to\mathbb{R}$ must be continuous; thus it is also integrable. Now, integrating side-by-side inequality (2) with respect to t over interval $[0,1]$ we conclude inequality (3).

LEMMA 2. *Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function integrable on each compact subinterval of $(-\alpha,\alpha)$, where $\alpha=\frac{\sup I-\inf I}{2}$. If an F-convex function $f:I\to\mathbb{R}$ is one-sided differentiable and $f_-\leqslant f_+$, then we have the inequality*

$$f\left(\frac{x+y}{2}\right)+\frac{1}{y-x}\int_x^y F\left(u-\frac{x+y}{2}\right)du\leqslant\frac{1}{y-x}\int_x^y f(u)du\tag{4}$$

for all $x,y\in I$ $(x\neq y)$.

Proof. Fix $x,y\in I$ $(x\neq y)$. From Theorem 1 we conclude the inequality

$$\begin{aligned}&f\left(\frac{x+y}{2}\right)+a\left(tx+(1-t)y-\frac{x+y}{2}\right)+F\left(tx+(1-t)y-\frac{x+y}{2}\right)\\&\leqslant f\left(tx+(1-t)y\right),\quad t\in[0,1].\end{aligned}$$

Integrating this inequality side-by-side with respect to t over interval $[0,1]$ we obtain (4).

REMARK 1. Using Theorem 3 we can replace the assumption "$f:I\to\mathbb{R}$ is one-sided differentiable and $f_-\leqslant f_+$" in Theorem 2 by "$\liminf_{x\to 0}\frac{F(x)}{x^2}>-\infty$".

REMARK 2. For an even function F and its primitive function G such that $G(0) = 0$ inequality (4) takes the form

$$f\left(\frac{x+y}{2}\right) + \frac{2}{y-x}G\left(\frac{y-x}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(u)du$$

for all $x,y \in I$ $(x \neq y)$.

Without Theorem 1, but applying methods from paper [6], we get a lower bound of the integral mean other than (4).

LEMMA 3. *Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function integrable on each compact subinterval of $(-\alpha,\alpha)$, where $\alpha = \sup I - \inf I$. If $f: I \to \mathbb{R}$ is an F-convex function, then*

$$f\left(\frac{x+y}{2}\right) + \frac{1}{4(y-x)}\int_x^y F(x+y-2u)\,du \leqslant \frac{1}{y-x}\int_x^y f(u)du \tag{5}$$

for all $x,y \in I$ $(x \neq y)$.

Proof. From F-convexity of a function f and the identity

$$\frac{x+y}{2} = \frac{tx+(1-t)y+(1-t)x+ty}{2}$$

we get

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &= f\left(\frac{tx+(1-t)y+(1-t)x+ty}{2}\right) \\ &\leqslant \frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2} - \frac{1}{4}F((2t-1)(x-y)) \end{aligned}$$

for all $x,y \in I$ and $t \in [0,1]$. Thus

$$f\left(\frac{x+y}{2}\right) + \frac{1}{4}F((2t-1)(x-y)) \leqslant \frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2} \tag{6}$$

for all $x,y \in I$ and $t \in [0,1]$. Fixing different $x,y \in I$ and integrating side-by-side with respect to t over interval $[0,1]$ inequality (6) we obtain

$$f\left(\frac{x+y}{2}\right) + \frac{1}{4}\int_0^1 F((2t-1)(x-y))\,dt \leqslant \frac{1}{2}\int_0^1 \left(f(tx+(1-t)y)+f((1-t)x+ty)\right)dt.$$

Which with substitutions "$(1-t)x+ty = u$" for the integrals

$$\int_0^1 F((2t-1)(x-y))\,dt, \qquad \int_0^1 f((1-t)x+ty)\,dt$$

and "$tx+(1-t)y = u$" for the integral

$$\int_0^1 f(tx+(1-t)y)\,dt$$

gives

$$f\left(\frac{x+y}{2}\right)+\frac{1}{4(y-x)}\int_x^y F(x+y-2u)\,du \leqslant \frac{1}{y-x}\int_x^y f(u)du.$$

REMARK 3. For an even function F and its primitive function G such that $G(0)=0$ inequality (5) takes the form

$$f\left(\frac{x+y}{2}\right)+\frac{G(y-x)}{4(y-x)} \leqslant \frac{1}{y-x}\int_x^y f(u)du$$

for all $x,y\in I$ $(x\neq y)$.

The presented lemmas result in two theorems of Hermite-Hadamard type inequalities for F-convex functions.

THEOREM 5. *Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function integrable on each compact subinterval of $(-\alpha,\alpha)$, where $\alpha=\frac{\sup I-\inf I}{2}$. If an F-convex function $f:I\to\mathbb{R}$ is one-sided differentiable and $f_-\leqslant f_+$, then we have the inequality*

$$\begin{aligned} f\left(\frac{x+y}{2}\right)+\frac{1}{y-x}\int_x^y F\left(u-\frac{x+y}{2}\right)du &\leqslant \frac{1}{y-x}\int_x^y f(u)du \\ &\leqslant \frac{f(x)+f(y)}{2}-\frac{1}{6}F(x-y) \end{aligned} \tag{7}$$

for all $x,y\in I$ $(x\neq y)$.

THEOREM 6. *Let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function integrable on each compact subinterval of $(-\alpha,\alpha)$, where $\alpha=\sup I-\inf I$. If $f:I\to\mathbb{R}$ is an F-convex function, then*

$$\begin{aligned} f\left(\frac{x+y}{2}\right)+\frac{1}{4(y-x)}\int_x^y F(2u-x-y)\,du &\leqslant \frac{1}{y-x}\int_x^y f(u)du \\ &\leqslant \frac{f(x)+f(y)}{2}-\frac{1}{6}F(x-y) \end{aligned} \tag{8}$$

for all $x,y\in I$ $(x\neq y)$.

Notice that for the zero function F we get the classical Hermite-Hadamard inequality, for $F(x)=cx^2$ with $c>0$ we get a Hermite-Hadamard type inequality for strongly convex functions, and inequalities (7) and (8) are the same. In general, one of them could be better than the other – it depends on the function F. More precisely, it depends on the expressions:

$$\frac{1}{y-x}\int_x^y F\left(u-\frac{x+y}{2}\right)du$$

and

$$\frac{1}{4(y-x)}\int_x^y F(2u-x-y)\,du.$$

In particular, for power functions $F(x) = c|x|^p$ with $c \in \mathbb{R}$ and $p > 0$ they take the forms:

$$\frac{1}{y-x}\int_x^y F\left(u - \frac{x+y}{2}\right)du = \frac{c}{4(p+1)}|y-x|^p$$

and

$$\frac{1}{4(y-x)}\int_x^y F(2u-x-y)\,du = \frac{c}{2^p(p+1)}|y-x|^p.$$

Which means that for functions $F(x) = c|x|^p$ with $c < 0$ and $p < 2$ inequality (8) is stronger than inequality (7); for $c > 0$ and $p < 2$ inequality (8) seems to be weaker than inequality (7) – but in this case, there are no F-convex function (see [3]); if $c < 0$ and $p > 2$ inequality (8) also seems to be weaker than inequality (7) – but in this case each F-convex functions must be convex (see [3]) and we have the classical Hermite-Hadamard inequality which in such case is stronger than inequality (7); for $c \in \mathbb{R}$ and $p = 2$ (also $c = 0$ and $p > 0$) inequalities are equivalent. So, for power function $F(x) = cx^p$ with $c \in \mathbb{R}$ and $p > 0$ inequality (8) is better.

Observe that for $H(z) = \frac{1}{2z}\int_{-z}^{z} F(t)dt$ the integrals obtained on the left sides of (7) and (8) take the forms

$$\frac{1}{y-x}\int_x^y F\left(u - \frac{x+y}{2}\right)du = H\left(\frac{y-x}{2}\right)$$

and

$$\frac{1}{4(y-x)}\int_x^y F(2u-x-y)\,du = \frac{1}{4}H(y-x),$$

which may be better for further analysis of Hermite-Hadamard type inequalities.

In [3] the author proved that an F-convex function $f : I \to \mathbb{R}$ is c-convex (in the sense of Vial) as long as

$$-\infty < \liminf_{x\to 0}\frac{F(x)}{x^2} < \infty, \tag{9}$$

moreover, the postulated real number c in the definition of c-convex functions is equal "$-\liminf_{x\to 0}\frac{F(x)}{x^2}$". Thus, we conclude that if (9) holds, then an F-convex function is also with $c(\cdot)^2$-convex with $c = \liminf_{x\to 0}\frac{F(x)}{x^2}$. Therefore, we get (from Theorem 5 or Theorem 6) the following corollary.

COROLLARY 1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If a function $f : I \to \mathbb{R}$ is F-convex and $-\infty < \liminf_{x\to 0}\frac{F(x)}{x^2} < \infty$, then*

$$f\left(\frac{x+y}{2}\right) + \frac{c}{12}(x-y)^2 \leqslant \frac{1}{y-x}\int_x^y f(u)du \leqslant \frac{f(x)+f(y)}{2} - \frac{c}{6}(x-y)^2 \tag{10}$$

for all $x, y \in I$ $(x \neq y)$ and $c = \liminf_{x\to 0}\frac{F(x)}{x^2}$.

Notice that inequality (10) was obtained for strongly convex functions with modulus $c > 0$ by N. Merentes and K. Nikodem in [10]. Their result is also derived from the above corollary – it is enough to take $F(x) = cx^2$.

Having regard to considerations for inequalities (7) and (8) and power functions F we conclude that for approximately convex functions of order p (in the sense of Nikodem and Páles) inequality (8) is better than (7). Therefore, from Theorem 6 we get the following Hermite-Hadamard type inequality for approximately convex functions of order p.

COROLLARY 2. *Let $c > 0$ and $p > 0$. If $f : I \to \mathbb{R}$ is an approximately convex functions of order p i.e.*

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + ct(1-t)|x-y|^p$$

for all $x, y \in I$ and $t \in [0,1]$, then

$$f\left(\frac{x+y}{2}\right) - \frac{c}{4(p+1)}|y-x|^p \leqslant \frac{1}{y-x}\int_x^y f(u)du \leqslant \frac{f(x)+f(y)}{2} + \frac{c}{6}|x-y|^p \qquad (11)$$

for all $x, y \in I$ ($x \neq y$).

Comparing Corollary 1 and Corollary 2, we conclude that for approximately convex functions of orders $p > 2$ we have a stronger inequality than inequality (11), namely the classical Hermite-Hadamard inequality – which in this case is stronger than inequality (11).

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(Received November 4, 2019)

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