

A NECESSARY AND SUFFICIENT CONDITION FOR THE CONVEXITY OF THE GENERALIZED ELLIPTIC INTEGRAL OF THE FIRST KIND WITH RESPECT TO HÖLDER MEAN

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Abstract. In the article, we present the necessary and sufficient conditions such that the generalized elliptic integral and complete p -elliptic integral of the first kind are convex with respect to Hölder mean, which generalize and refine some previously known results.

1. Introduction

Let $|z| < 1$ and $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \dots$. Then the classical Gaussian hypergeometric function $F(a, b; c; z)$ [48, 58, 59, 61, 64, 65, 93] is given by

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (1.1)$$

where $(a)_n = a(a+1)(a+2)(a+3)\cdots(a+n-1) = \Gamma(n+a)/\Gamma(a)$ for $n = 1, 2, 3, \dots$ is the shifted fractional function, $(a)_0 = 1$ for $a \neq 0$ and $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ is the gamma function [31, 47, 68, 83, 91]. $F(a, b; c; z)$ is said to be zero-balanced if $c = a + b$.

It is well known that the Gaussian hypergeometric function has wide applications in mathematics, physics, mechanics and engineering science [1, 6, 29, 39, 40, 45, 46, 50, 71, 87], many elementary and special functions are the particular cases [2, 11, 16, 30, 32, 33, 35, 36, 37, 43, 56, 69] of the Gaussian hypergeometric function. For instance, the Legendre and Chebyshev polynomials $P_n(z)$ and $T_n(z)$ can be expressed in terms of the Gaussian hypergeometric function as follows:

$$P_n(z) = 2^n F\left(-n, n+1; 1; \frac{1+z}{2}\right),$$

$$T_n(z) = (-1)^n F\left(-n, n; \frac{1}{2}; \frac{1+z}{2}\right),$$

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and the complete elliptic integrals $\mathcal{K}(r)$ [22, 44, 63, 72, 76, 82, 84, 85] and $\mathcal{E}(r)$ [19, 20, 21, 23, 24, 57, 73, 81] ($0 < r < 1$) of the first and second kinds can be written as

$$\mathcal{K}(r) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right),$$

$$\mathcal{E}(r) = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}} dt = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right),$$

respectively.

For $a, r \in (0, 1)$, Borweins [16] introduced the generalized elliptic integrals $\mathcal{K}_a(r)$ and $\mathcal{E}_a(r)$ of first and second kinds by use of the Gaussian hypergeometric function as follows:

$$\begin{cases} \mathcal{K}_a = \mathcal{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1; r^2), \\ \mathcal{E}_a = \mathcal{E}_a(r) = \frac{\pi}{2} F(a-1, 1-a; 1; r^2). \end{cases} \tag{1.2}$$

It is well-known that the function $r \rightarrow \mathcal{K}_a(r)$ is strictly increasing and the function $r \rightarrow \mathcal{E}_a(r)$ is strictly decreasing on the interval $(0, 1)$ for all $a \in (0, 1)$. Their limiting values are $\mathcal{K}_a(0) = \mathcal{E}_a(0) = \pi/2$, $\mathcal{K}_a(1) = +\infty$, $\mathcal{E}_a(1) = \sin(\pi a)/[2(1-a)]$. The generalized elliptic integrals can be used to study the theory of Ramanujan’s generalized modular equations and the computation of π . For example, Anderson, Qiu, Vamanamurthy and Vuorinen [10] employed the generalized elliptic integrals to investigate many special functions in generalized Ramanujan’s modular equation, established a lot of monotonicity and convexity properties, identities and inequalities involving \mathcal{K}_a , \mathcal{E}_a and the solution of the generalized modular equation, found that \mathcal{K}_a and \mathcal{E}_a satisfy the following differential equations:

$$\frac{d\mathcal{K}_a}{dr} = 2(1-a) \frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{r r'^2}, \quad \frac{d\mathcal{E}_a}{dr} = 2(a-1) \frac{\mathcal{K}_a - \mathcal{E}_a}{r},$$

$$\frac{d}{dr}(\mathcal{K}_a - \mathcal{E}_a) = 2(1-a) \frac{r \mathcal{E}_a}{r'^2}, \quad \frac{d}{dr}(\mathcal{E}_a - r'^2 \mathcal{K}_a) = 2ar \mathcal{K}_a,$$

where and in what follows we denote $r' = \sqrt{1-r^2}$ for $r \in (0, 1)$. For more results on the generalized elliptic integral, we recommend the literature [66, 67, 70, 74, 77, 80] to the readers.

We now recall the definition of convexity (concavity) of a function with respect to Hölder mean.

Let $\alpha, \beta \in \mathbb{R}$ and f be a continuous function defined on the interval $I \subset (0, \infty)$ such that $f(I) \subset (0, \infty)$. Then f is said to be $H_{\alpha, \beta}$ -convex (concave) on I if the inequality

$$f(H_\alpha(x, y)) \leq (\geq) H_\beta(f(x), f(y)) \tag{1.3}$$

holds for all $x, y \in I$. f is said to be strictly $H_{\alpha, \beta}$ -convex (concave) if inequality (1.3) is strict except for $x = y$. Here $H_\alpha(x, y)$ is the the classical Hölder mean [18, 25, 26, 27, 28, 41, 49, 75, 78, 79] of order $\alpha \in \mathbb{R}$ of two positive numbers x and y which defined by $H_\alpha(x, y) = [(x^\alpha + y^\alpha)/2]^{1/\alpha}$ ($\alpha \neq 0$), and $H_0(x, y) = \sqrt{xy}$. Furthermore, it is worth pointing out that $H_{1,1}$ -convexity (concavity) reduces to the usual convexity

(concavity), $H_{1,0}$ -convexity (concavity) means the log-convexity (log-concavity) and $H_{\alpha,0}$ -convexity (concavity) is the so-called geometrical convexity (concavity).

Recently, the $H_{\alpha,\beta}$ -convexity (concavity) properties for the special functions in geometric function theory have attracted the attention many researchers [14, 15, 38, 92]. In particular, many generalizations and variants for the convexity (concavity) can be found in the literature [3, 4, 5, 7, 8, 17, 42, 51, 86, 88, 94]. Anderson, Vamanamurthy and Vuorinen [12], Baricz [13], and Zhang, Wang and Chu [90] investigate the $H_{\alpha,\beta}$ -convexity of the zero-balanced Gaussian hypergeometric function $F(a, b; a + b; x)$ ($a, b > 0$) for some α and β . They proved that that the complete elliptic integral $\mathcal{K}(r)$ of the first kind is strictly $H_{\alpha,\beta}$ -convex on $(0, 1)$ if $(\alpha, \beta) \in \{(\alpha, \beta) | \alpha \leq 2, \beta \geq 0\}$. Later, the best possible region in $\alpha\beta$ -plane for which $\mathcal{K}(r)$ is strictly $H_{\alpha,\beta}$ -convex was given by Wang, Chu, Qiu and Jiang [62] as follows.

THEOREM 1.1. (See [62, Theorem 1.1]) *The complete elliptic integral $\mathcal{K}(r)$ of the first kind is strictly $H_{\alpha,\beta}$ -convex on $(0, 1)$ if and only if*

$$(\beta, \alpha) \in \{(\beta, \alpha) | \alpha \leq C(\beta)\},$$

where $C(\beta) = \inf_{r \in (0,1)} \{(\beta - 1)(\mathcal{E} - r^2 \mathcal{K}) / (r^2 \mathcal{K}) + r^2(2\mathcal{E} - r^2 \mathcal{K}) / [r^2(\mathcal{E} - r^2 \mathcal{K})]\}$ is a continuous function with $C(\beta) = 2$ for all $\beta \geq -7/2$, and $C(\beta) < 2$ for all $\beta < -7/2$. There are no values of α and β for which $\mathcal{K}(r)$ is $H_{\alpha,\beta}$ -concave on $(0, 1)$.

Zhou, Qiu and Wang [95] found the interval $I(a)$ such that the generalized elliptic integral $\mathcal{K}_a(r)$ of the first kind is $H_{\alpha,\alpha}$ -convex on $(0, 1)$ if and only if $\alpha \in I(a)$.

The main purpose of this paper is to extend Theorem 1.1 to the case of generalized elliptic integral $\mathcal{K}_a(r)$ of the first kind. Our main result is the following Theorem 1.2, which refine the result obtained by Zhou, Qiu and Wang [95]. Besides, as a corollary of Theorem 1.2, a necessary and sufficient condition such that the complete p -elliptic integral of the first kind is $H_{\alpha,\beta}$ -convex on $(0, 1)$ is also derived.

THEOREM 1.2. *Let $\alpha, \beta \in \mathbb{R}$ and $a \in (0, 1)$. Then the generalized elliptic integral $\mathcal{K}_a(r)$ of the first kind is $H_{\alpha,\beta}$ -convex on $(0, 1)$, namely the inequality*

$$\mathcal{K}_a(H_\alpha(x, y)) \leq H_\beta(\mathcal{K}_a(x), \mathcal{K}_a(y))$$

holds for all $x, y \in (0, 1)$ if and only if

$$(\beta, \alpha) \in D = \{(\beta, \alpha) | \alpha \leq L(\beta)\},$$

where

$$L(\beta) = \inf_{r \in (0,1)} \frac{2(1-a)(\beta-1)(\mathcal{E}_a - r^2 \mathcal{K}_a) / (r^2 \mathcal{K}_a) + r^2 [2\mathcal{E}_a - 2(1-a)r^2 \mathcal{K}_a]}{r^2 (\mathcal{E}_a - r^2 \mathcal{K}_a)}$$

is a continuous function with $L(\beta) = 2$ for all $\beta \geq -(a^2 - a + 2) / [2a(1 - a)]$, and $L(\beta) < 2$ for all $\beta < -(a^2 - a + 2) / [2a(1 - a)]$. Moreover, there exists no values of α and β such that $\mathcal{K}_a(r)$ is $H_{\alpha,\beta}$ -concave on $(0, 1)$.

REMARK 1.3. Figure 1 (See Appendix 3 in Section 5) shows the visualized regions D and their corresponding boundary curves $\alpha = L(\beta)$ in Theorem 1.2 when $a = 1/8, 1/4, 3/8$ and $1/2$.

2. Lemmas

In order to prove our main result we need several lemmas, which we present in this section.

LEMMA 2.1. (See [11, Theorem 1.25]) *Let $a, b \in \mathbb{R}$ with $a < b$, $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , and $g'(x) \neq 0$ on (a, b) . Then both the functions $[f(x) - f(a)]/[g(x) - g(a)]$ and $[f(x) - f(b)]/[g(x) - g(b)]$ are increasing (decreasing) on (a, b) if the function $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) . If the function $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

The following Lemma 2.2 follows directly from [10, Lemma 5.2(1), (3) and (4), and Lemma 5.4(1)].

LEMMA 2.2. *Let $a \in (0, 1)$. Then the following statements are true:*

- (1) *The function $r \mapsto (\mathcal{K}_a - \mathcal{E}_a)/(r^2 \mathcal{K}_a)$ is strictly increasing from $(0, 1)$ onto $(1 - a, 1)$.*
- (2) *The function $r \mapsto (\mathcal{E}_a - r'^2 \mathcal{K}_a)/r^2$ is strictly increasing from $(0, 1)$ onto $(\frac{\pi a}{2}, \frac{\sin(\pi a)}{2(1-a)})$.*
- (3) *The function $r \mapsto r'^c \mathcal{K}_a$ is strictly decreasing from $(0, 1)$ onto $(0, \pi/2)$ if and only if $c \geq 2a(1 - a)$. Moreover, $\sqrt{r'} \mathcal{K}_a$ is strictly decreasing on $(0, 1]$.*
- (4) *The function $r \mapsto (\mathcal{E}_a - r'^2 \mathcal{K}_a)/(r^2 \mathcal{K}_a)$ is strictly decreasing from $(0, 1)$ onto $(0, a)$.*

LEMMA 2.3. *Let $a \in (0, 1)$, $c \in \mathbb{R}$ and*

$$f_c(r) = r^c \frac{[a(\mathcal{K}_a - \mathcal{E}_a) - (1 - a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)]}{r^4}.$$

Then one has:

- (1) *$f_c(r)$ is strictly decreasing from $(0, 1)$ onto $(0, a^2(1 - a)\pi/4)$ if and only if $c \geq 2(a + 1)(2 - a)/3$.*
- (2) *$f_c(r)$ is strictly increasing from $(0, 1)$ onto $(a^2(1 - a)\pi/4, +\infty)$ if and only if $c \leq 0$.*
- (3) *If $0 < c < 2(a + 1)(2 - a)/3$, then there exists $\xi = \xi(c) \in (0, 1)$ such that $f_c(r)$ is strictly increasing on $(0, \xi)$ and strictly decreasing on $(\xi, 1)$.*

Proof. It follows from (1.1) and (1.2) that

$$\begin{aligned} & a(\mathcal{K}_a - \mathcal{E}_a) - (1 - a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \\ &= \frac{\pi}{2} a(1 - a) r^2 \sum_{n=0}^{\infty} \frac{(a)_n (2 - a)_n}{(2)_n} \frac{r^{2n}}{n!} - \frac{\pi}{2} a(1 - a) r^2 \sum_{n=0}^{\infty} \frac{(a)_n (1 - a)_n}{(2)_n} \frac{r^{2n}}{n!} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\pi}{2} a(1-a)r^2 \sum_{n=1}^{\infty} \frac{(a)_n(2-a)_{n-1}}{(2)_n} \frac{r^{2n}}{(n-1)!} \\
 &= \frac{\pi}{2} a(1-a)r^4 \sum_{n=0}^{\infty} \frac{(a)_{n+1}(2-a)_n}{(2)_{n+1}} \frac{r^{2n}}{n!} = \frac{\pi}{4} a^2(1-a)r^4 \sum_{n=0}^{\infty} \frac{(a+1)_n(2-a)_n}{(3)_n} \frac{r^{2n}}{n!} \\
 &= \frac{\pi}{4} a^2(1-a)r^4 F(a+1, 2-a; 3; r^2).
 \end{aligned}$$

Thus

$$f_c(r) = \frac{\pi}{4} a^2(1-a)r^c F(a+1, 2-a; 3; r^2). \tag{2.1}$$

Differentiating f_c leads to

$$\begin{aligned}
 f'_c(r) &= \frac{\pi}{4} a^2(1-a)cr^{c-1} \left(-\frac{r}{r'}\right) F(a+1, 2-a; 3; r^2) \\
 &\quad + \frac{\pi}{4} a^2(1-a)r^c \frac{(a+1)(2-a)}{3} F(a+2, 3-a; 4; r^2) 2r \\
 &= \frac{\pi}{4} a^2(1-a)rr^{c-2} F(a+1, 2-a; 3; r^2) \\
 &\quad \times \left[\frac{2(a+1)(2-a)}{3} (1-r^2) \frac{F(a+2, 3-a; 4; r^2)}{F(a+1, 2-a; 3; r^2)} - c \right] \\
 &= \frac{\pi}{4} a^2(1-a)rr^{c-2} F(a+1, 2-a; 3; r^2) \\
 &\quad \times \left[\frac{2(a+1)(2-a)}{3} \frac{F(a+1, 2-a; 4; r^2)}{F(a+1, 2-a; 3; r^2)} - c \right].
 \end{aligned} \tag{2.2}$$

Let

$$\tilde{f}(r) = \frac{F(a+1, 2-a; 4; r^2)}{F(a+1, 2-a; 3; r^2)}.$$

Then [9, Lemma 2] leads to the conclusion that $\tilde{f}(r)$ is strictly decreasing on $(0, 1)$. Clearly $\tilde{f}(0^+) = 1$ and $\tilde{f}(1^-) = 0$. From (2.2) we know that $f'_c(r) < 0$ for all $r \in (0, 1)$ if and only if $c \geq 2(a+1)(2-a)/3$, $f'_c(r) > 0$ for all $r \in (0, 1)$ if and only if $c \leq 0$, and there exists $\xi \in (0, 1)$ such that $f'_c(r) > 0$ for $r \in (0, \xi)$ and $f'_c(r) < 0$ for $r \in (\xi, 1)$. Consequently, the assertions of monotonicity properties in Lemma 2.3 follows. Moreover, $f_c(0^+) = \pi a^2(1-a)/4$, and by (2.1) we have $f_c(1^-) = 0$ for $c > 0$ and $f_c(1^-) = +\infty$ for $c \leq 0$. \square

LEMMA 2.4. Let $a \in (0, 1)$. Then the function

$$f(r) = \frac{(1-a)\mathcal{E}_a(\mathcal{E}_a - r'^2 \mathcal{H}_a) - ar'^2 \mathcal{H}_a(\mathcal{H}_a - \mathcal{E}_a)}{r^4}$$

is strictly increasing from $(0, 1)$ onto $(\pi^2 a^2(1-a)/8, \sin^2(\pi a)/[4(1-a)])$.

Proof. Let $f_1(r) = (1-a)\mathcal{E}_a(\mathcal{E}_a - r'^2 \mathcal{H}_a) - ar'^2 \mathcal{H}_a(\mathcal{H}_a - \mathcal{E}_a)$ and $f_2(r) = r^4$. Then $f(r) = f_1(r)/f_2(r)$, $f_1(0) = f_2(0) = 0$,

$$f'_1(r) = (1-a) \left[2ar \mathcal{H}_a \mathcal{E}_a - 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{H}_a) \frac{\mathcal{H}_a - \mathcal{E}_a}{r} \right]$$

$$\begin{aligned}
 & -a \left[2r \mathcal{K}_a (\mathcal{E}_a - \mathcal{K}_a) + 2(1-a)r^2 \frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{rr'^2} (\mathcal{K}_a - \mathcal{E}_a) + 2(1-a)r'^2 \mathcal{K}_a \frac{r\mathcal{E}_a}{r'^2} \right] \\
 & = -2(1-a) \frac{(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r} + 2ar \mathcal{K}_a (\mathcal{K}_a - \mathcal{E}_a) \\
 & = 2(\mathcal{K}_a - \mathcal{E}_a) \frac{ar^2 \mathcal{K}_a - (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r}
 \end{aligned}$$

and

$$\frac{f'_1(r)}{f'_2(r)} = \frac{\mathcal{K}_a - \mathcal{E}_a}{2r^2} \times \left(\frac{a(\mathcal{K}_a - \mathcal{E}_a) - (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r^2} + a \frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{r^2} \right). \tag{2.3}$$

Lemma 2.2(1), (2) and Lemma 2.3(2) imply that $f'_1(r)/f'_2(r)$ is strictly increasing on $(0, 1)$. By application of Lemma 2.1, we can conclude that the function $f(r)$ is strictly increasing on $(0, 1)$. Clearly $f(1^-) = \sin^2(\pi a)/[2(1-a)]$, and by l'Hôpital's rule and Lemma 2.2(1) we have

$$\lim_{r \rightarrow 0^+} f(r) = \lim_{r \rightarrow 0^+} \frac{f'_1(r)}{f'_2(r)} = \frac{1}{2} \cdot \frac{\pi}{2} (1-a) \cdot \frac{\pi}{2} a^2 = \frac{\pi^2}{8} a^2 (1-a). \quad \square$$

LEMMA 2.5. *Let $a \in (0, 1)$. Then the function $g(r) = (\mathcal{E}_a - r'^2 \mathcal{K}_a)/(r^2 \sqrt{\mathcal{K}_a})$ is strictly decreasing from $(0, 1)$ onto $(0, \sqrt{\pi/2a})$.*

Proof. Logarithmic differentiating g leads to

$$\begin{aligned}
 \frac{g'(r)}{g(r)} &= \frac{2ar \mathcal{K}_a}{\mathcal{E}_a - r'^2 \mathcal{K}_a} - \frac{2}{r} - \frac{(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{rr'^2 \mathcal{K}_a} \\
 &= \frac{2ar^2 r'^2 \mathcal{K}_a^2 - 2r'^2 \mathcal{K}_a (\mathcal{E}_a - r'^2 \mathcal{K}_a) - (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)^2}{rr'^2 \mathcal{K}_a (\mathcal{E}_a - r'^2 \mathcal{K}_a)} \\
 &= \frac{2r'^2 \mathcal{K}_a \left[a(\mathcal{K}_a - \mathcal{E}_a) - (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \right] - (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)^2}{rr'^2 \mathcal{K}_a (\mathcal{E}_a - r'^2 \mathcal{K}_a)} \\
 &= \frac{r^3}{r'^2 \mathcal{K}_a (\mathcal{E}_a - r'^2 \mathcal{K}_a)} g_1(r),
 \end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
 g_1(r) &= 2r'^2 \mathcal{K}_a \left[\frac{a(\mathcal{K}_a - \mathcal{E}_a) - (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r^4} \right] - (1-a) \left(\frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{r^2} \right)^2 \\
 &= 2r'^{2(1-2a)/3} \left[r^{2a(1-a)} \mathcal{K}_a \right] r'^{2(a+1)(2-a)/3} \left[\frac{a(\mathcal{K}_a - \mathcal{E}_a) - (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r^4} \right] \\
 &\quad - (1-a) \left(\frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{r^2} \right)^2.
 \end{aligned}$$

It follows from Lemma 2.2(2), (3) and Lemma 2.4 that $g_1(r)$ is strictly decreasing on $(0, 1)$. Note that

$$\lim_{r \rightarrow 0^+} g_1(r) = \pi \cdot \frac{a^2(1-a)\pi}{4} - (1-a) \left(\frac{\pi a}{2} \right)^2 = 0.$$

Thus $g_1(r) < g_1(0) = 0$ for all $r \in (0, 1)$. Then from (2.4) we know that $g'(r) < 0$ for all $r \in (0, 1)$, so that $\frac{g(r)}{r}$ is strictly decreasing on $(0, 1)$. Moreover, by Lemma 2.2(3) one has $g(0^+) = \sqrt{\pi/2a}$, and $g(1^-) = 0$. \square

LEMMA 2.6. *Let $a \in (0, 1)$. Then the function*

$$h(r) = \frac{ar^2 \mathcal{K}_a \mathcal{E}_a + (1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r^2 \mathcal{K}_a)}{r^2 \mathcal{K}_a}$$

is strictly increasing from $(0, 1)$ onto $(\pi a/2, \sin(\pi a)/[2(1-a)])$.

Proof. Let $h_1(r) = ar^2 \mathcal{K}_a \mathcal{E}_a + (1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r^2 \mathcal{K}_a)$ and $h_2(r) = r^2 \mathcal{K}_a$. Then simple computations gives

$$\begin{aligned} h_1'(r) &= 2ar \mathcal{K}_a \mathcal{E}_a + 2a(1-a)r \mathcal{E}_a \frac{\mathcal{E}_a - r^2 \mathcal{K}_a}{r^2} & (2.5) \\ &+ 2a(1-a)r^2 \mathcal{K}_a \left(\frac{\mathcal{E}_a - \mathcal{K}_a}{r} \right) \\ &+ 2(1-a)^2 \frac{r \mathcal{E}_a}{r^2} (\mathcal{E}_a - r^2 \mathcal{K}_a) + (1-a)(\mathcal{K}_a - \mathcal{E}_a) \cdot 2ar \mathcal{K}_a \\ &= 2ar \mathcal{K}_a \mathcal{E}_a + 2a(1-a)r \mathcal{E}_a \frac{\mathcal{E}_a - r^2 \mathcal{K}_a}{r^2} + 2(1-a)^2 \frac{r \mathcal{E}_a}{r^2} (\mathcal{E}_a - r^2 \mathcal{K}_a) \\ &= \frac{2r \mathcal{E}_a}{r^2} \left[ar^2 \mathcal{K}_a + (1-a)(\mathcal{E}_a - r^2 \mathcal{K}_a) \right], \\ h_2'(r) &= 2r \mathcal{K}_a + 2(1-a)r \frac{\mathcal{E}_a - r^2 \mathcal{K}_a}{r^2} = \frac{2r}{r^2} \left[r^2 \mathcal{K}_a + (1-a)(\mathcal{E}_a - r^2 \mathcal{K}_a) \right]. \end{aligned}$$

Thus

$$\begin{aligned} h'(r) &= \frac{h_1'(r)h_2(r) - h_1(r)h_2'(r)}{h_2(r)^2} = \frac{2\mathcal{E}_a \left[ar^2 \mathcal{K}_a + (1-a)(\mathcal{E}_a - r^2 \mathcal{K}_a) \right] r^2 \mathcal{K}_a}{r^2 r^3 \mathcal{K}_a^2} \\ &- \frac{2 \left[ar^2 \mathcal{K}_a \mathcal{E}_a + (1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r^2 \mathcal{K}_a) \right] \left[r^2 \mathcal{K}_a + (1-a)(\mathcal{E}_a - r^2 \mathcal{K}_a) \right]}{r^2 r^3 \mathcal{K}_a^2}, \end{aligned}$$

that is

$$\begin{aligned} \frac{r^2 r^3 \mathcal{K}_a^2}{2} h'(r) &= ar^2 r^2 \mathcal{K}_a^2 \mathcal{E}_a + (1-a)r^2 \mathcal{K}_a \mathcal{E}_a (\mathcal{E}_a - r^2 \mathcal{K}_a) \\ &- ar^2 r^2 \mathcal{K}_a^2 \mathcal{E}_a - a(1-a)r^2 \mathcal{K}_a \mathcal{E}_a (\mathcal{E}_a - r^2 \mathcal{K}_a) \\ &- (1-a)r^2 \mathcal{K}_a (\mathcal{K}_a - \mathcal{E}_a) (\mathcal{E}_a - r^2 \mathcal{K}_a) - (1-a)^2 (\mathcal{K}_a - \mathcal{E}_a) (\mathcal{E}_a - r^2 \mathcal{K}_a)^2 \\ &= (1-a)(\mathcal{E}_a - r^2 \mathcal{K}_a) \left[r^2 \mathcal{K}_a \mathcal{E}_a - ar^2 \mathcal{K}_a \mathcal{E}_a \right. \\ &\quad \left. - r^2 \mathcal{K}_a (\mathcal{K}_a - \mathcal{E}_a) - (1-a)(\mathcal{K}_a - \mathcal{E}_a) (\mathcal{E}_a - r^2 \mathcal{K}_a) \right] \\ &= (1-a)(\mathcal{E}_a - r^2 \mathcal{K}_a) \left[(1-a)\mathcal{E}_a (\mathcal{E}_a - r^2 \mathcal{K}_a) - ar^2 \mathcal{K}_a (\mathcal{K}_a - \mathcal{E}_a) \right]. \end{aligned}$$

Lemma 2.4 shows that $h'(r) > 0$ for all $r \in (0, 1)$, so that $h(r)$ is strictly increasing on $(0, 1)$. Moreover, by Lemma 2.2(1) and (2) we get

$$\begin{aligned} \lim_{r \rightarrow 0^+} h(r) &= \lim_{r \rightarrow 0^+} \left[a\mathcal{E}_a + (1-a) \frac{\mathcal{K}_a - \mathcal{E}_a}{r^2 \mathcal{K}_a} (\mathcal{E}_a - r'^2 \mathcal{K}_a) \right] = a\mathcal{E}_a(0) = \frac{\pi}{2}a, \\ \lim_{r \rightarrow 1^-} h(r) &= \lim_{r \rightarrow 1^-} \left[a\mathcal{E}_a + (1-a) \frac{\mathcal{K}_a - \mathcal{E}_a}{r^2 \mathcal{K}_a} (\mathcal{E}_a - r'^2 \mathcal{K}_a) \right] = \mathcal{E}_a(1) = \frac{\sin(\pi a)}{2(1-a)}. \quad \square \end{aligned}$$

LEMMA 2.7. *Let $a \in (0, 1)$. Then the function*

$$\begin{aligned} F(r) &= 4(1-a^2) \left(\frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{r^2} \right) \\ &\quad + 4a\mathcal{E}_a \frac{a(\mathcal{E}_a - r'^2 \mathcal{K}_a) - (1-a)r'^2(\mathcal{K}_a - \mathcal{E}_a) + (1-2a)r^2r'^2 \mathcal{K}_a}{r^2(\mathcal{E}_a - r'^2 \mathcal{K}_a)} \end{aligned}$$

is strictly increasing from $(0, 1)$ onto $(\pi a(1+a)(2-a), 2\sin(\pi a)/(1-a))$.

Proof. It follows from

$$\begin{aligned} \frac{d[(\mathcal{E}_a - r'^2 \mathcal{K}_a)/r^2]}{dr} &= \frac{2ar\mathcal{K}_a \cdot r^2 - (\mathcal{E}_a - r'^2 \mathcal{K}_a) \cdot 2r}{r^4} \\ &= \frac{2a(\mathcal{K}_a - \mathcal{E}_a) - 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r^3}, \end{aligned}$$

$$\frac{d[r^2(\mathcal{E}_a - r'^2 \mathcal{K}_a)]}{dr} = 2r(\mathcal{E}_a - r'^2 \mathcal{K}_a) + 2ar^3 \mathcal{K}_a = 2r(\mathcal{E}_a - r'^2 \mathcal{K}_a + ar^2 \mathcal{K}_a)$$

and

$$\begin{aligned} &\frac{d[a(\mathcal{E}_a - r'^2 \mathcal{K}_a) - (1-a)r'^2(\mathcal{K}_a - \mathcal{E}_a) + (1-2a)r^2r'^2 \mathcal{K}_a]}{dr} \\ &= 2a^2r\mathcal{K}_a + 2(1-a)r(\mathcal{K}_a - \mathcal{E}_a) - 2(1-a)^2r\mathcal{E}_a + 2(1-2a)rr'^2 \mathcal{K}_a - 2(1-2a)r^3 \mathcal{K}_a \\ &\quad + 2(1-a)(1-2a)r(\mathcal{E}_a - r'^2 \mathcal{K}_a) \\ &= [2a^2 + 2(1-a) + 2(1-2a)r'^2 - 2(1-2a)r^2 - 2(1-a)(1-2a)r'^2] r\mathcal{K}_a \\ &\quad + 2(1-a)[-1 - (1-a) + 1 - 2a]r\mathcal{E}_a \\ &= (1+a) [2a + 2(1-2a)r'^2] r\mathcal{K}_a - 2(1-a^2)r\mathcal{E}_a \\ &= (1+a)r [2ar^2 \mathcal{K}_a - 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)] \end{aligned}$$

that

$$\begin{aligned} F'(r) &= 4(1-a^2) \frac{2a(\mathcal{K}_a - \mathcal{E}_a) - 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r^3} - \left[\frac{8a(1-a)(\mathcal{K}_a - \mathcal{E}_a)}{r} \right] \\ &\quad \times \frac{a(\mathcal{E}_a - r'^2 \mathcal{K}_a) - (1-a)r'^2(\mathcal{K}_a - \mathcal{E}_a) + (1-2a)r^2r'^2 \mathcal{K}_a}{r^2(\mathcal{E}_a - r'^2 \mathcal{K}_a)} \\ &\quad + 4a\mathcal{E}_a \frac{(1+a)r [2ar^2 \mathcal{K}_a - 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)] r^2(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r^4(\mathcal{E}_a - r'^2 \mathcal{K}_a)^2} - 4a\mathcal{E}_a \end{aligned}$$

$$\begin{aligned} & \times \frac{[a(\mathcal{E}_a - r'^2 \mathcal{K}_a) - (1-a)r^2(\mathcal{K}_a - \mathcal{E}_a) + (1-2a)r^2r'^2 \mathcal{K}_a]2r(\mathcal{E}_a - r'^2 \mathcal{K}_a + ar^2 \mathcal{K}_a)}{r^4(\mathcal{E}_a - r'^2 \mathcal{K}_a)^2}, \\ & r^3(\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 F'(r) \tag{2.6} \\ & = 4(1-a^2)(\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 \left[2a(\mathcal{K}_a - \mathcal{E}_a) - 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \right] \\ & \quad - 8a(1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \\ & \quad \times \left[a(\mathcal{E}_a - r'^2 \mathcal{K}_a) - (1-a)r'^2(\mathcal{K}_a - \mathcal{E}_a) + (1-2a)r^2r'^2 \mathcal{K}_a \right] \\ & \quad + 4a(1+a)r^2 \mathcal{E}_a \left[2ar^2 \mathcal{K}_a - 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \right] (\mathcal{E}_a - r'^2 \mathcal{K}_a) \\ & \quad - 8a \mathcal{E}_a [a(\mathcal{E}_a - r'^2 \mathcal{K}_a) - (1-a)r'^2(\mathcal{K}_a - \mathcal{E}_a) + (1-2a)r^2r'^2 \mathcal{K}_a] \\ & \quad \times (\mathcal{E}_a - r'^2 \mathcal{K}_a + ar^2 \mathcal{K}_a) =: F_1(r). \end{aligned}$$

A tedious computation (See Appendix 1 in Section 5 for checking the following identity of $F_1(r)$ by use of Mathematical Software *Maple 13*) gives

$$\begin{aligned} F_1(r) & = 8(\mathcal{E}_a - r'^2 \mathcal{K}_a + ar'^2 \mathcal{K}_a) \tag{2.7} \\ & \quad \times \left[a(1-a)r^2(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) + a^2r^4 \mathcal{K}_a \mathcal{E}_a - (\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 \right]. \end{aligned}$$

Thus

$$\begin{aligned} & \frac{F_1(r)}{8r^4 \mathcal{K}_a(\mathcal{E}_a - r'^2 \mathcal{K}_a + ar^2 \mathcal{K}_a)} \tag{2.8} \\ & = a \left[\frac{ar^2 \mathcal{K}_a \mathcal{E}_a + (1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r^2 \mathcal{K}_a} \right] - \left(\frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{r^2 \sqrt{\mathcal{K}_a}} \right)^2 =: F_2(r). \end{aligned}$$

It follows from Lemmas 2.5 and 2.6 that $F_2(r)$ is strictly increasing from $(0, 1)$ onto $(0, a \sin(\pi a) / [2(1-a)])$, so that $F'(r) > 0$ for all $r \in (0, 1)$ by (2.6)-(2.8). Therefore, $F(r)$ is strictly increasing on $(0, 1)$. Moreover, by Lemma 2.2(1) and l'Hôpital's rule we get

$$\begin{aligned} \lim_{r \rightarrow 0^+} F(r) & = 4(1-a^2) \times \frac{\pi}{2} a + 4a \times \frac{\pi}{2} \lim_{r \rightarrow 0^+} \frac{(1+a)r [2ar^2 \mathcal{K}_a - 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)]}{2r(\mathcal{E}_a - r'^2 \mathcal{K}_a + ar^2 \mathcal{K}_a)} \\ & = 2\pi a(1-a^2) + 2\pi a \frac{a(1+a)}{2} = \pi a(1+a)(2-a), \\ \lim_{r \rightarrow 1^-} F(r) & = 4(1-a^2) \mathcal{E}_a(1) + 4a^2 \mathcal{E}_a(1) = 4\mathcal{E}_a(1) = \frac{2 \sin(\pi a)}{1-a}. \quad \square \end{aligned}$$

LEMMA 2.8. *Let $a \in (0, 1)$. Then the function*

$$G(r) = \frac{(\mathcal{E}_a - r'^2 \mathcal{K}_a)[2ar^2 \mathcal{K}_a \mathcal{E}_a + 2(1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)]}{r^4 \mathcal{K}_a^2}$$

is strictly decreasing from $(0, 1)$ onto $(0, \pi a^2)$.

Proof. It follows from (2.5) that

$$\begin{aligned}
 r^8 \mathcal{K}_a^4 G'(r) &= 2ar^5 \mathcal{K}_a^3 [2ar^2 \mathcal{K}_a \mathcal{E}_a + 2(1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)] \\
 &\quad + r^4 \mathcal{K}_a^2 (\mathcal{E}_a - r'^2 \mathcal{K}_a) \frac{4r \mathcal{E}_a}{r'^2} \left[ar'^2 \mathcal{K}_a + (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \right] \\
 &\quad - (\mathcal{E}_a - r'^2 \mathcal{K}_a) [2ar^2 \mathcal{K}_a \mathcal{E}_a + 2(1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)] r^3 \mathcal{K}_a \\
 &\quad \times \left[4\mathcal{K}_a + 4(1-a) \frac{(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r'^2} \right],
 \end{aligned}$$

which is equivalent to

$$\begin{aligned}
 &r^5 r'^2 \mathcal{K}_a^3 G'(r) \tag{2.9} \\
 &= 2ar^2 r'^2 \mathcal{K}_a^2 [2ar^2 \mathcal{K}_a \mathcal{E}_a + 2(1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)] \\
 &\quad + 4r^2 \mathcal{K}_a \mathcal{E}_a (\mathcal{E}_a - r'^2 \mathcal{K}_a) \left[ar'^2 \mathcal{K}_a + (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \right] \\
 &\quad - (\mathcal{E}_a - r'^2 \mathcal{K}_a) \left[2ar^2 \mathcal{K}_a \mathcal{E}_a + 2(1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \right] \\
 &\quad \times \left[4r'^2 \mathcal{K}_a + 4(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \right] =: G_1(r).
 \end{aligned}$$

A tedious computation (See Appendix 2 in Section 5 for checking the following identity of $G_1(r)$ by use of Mathematical Software *Maple 13*) shows that

$$\begin{aligned}
 G_1(r) &= -4 \left[(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \left((1+a)(\mathcal{K}_a - \mathcal{E}_a) - (1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \right) \right. \tag{2.10} \\
 &\quad \left. + ar^2 \mathcal{K}_a \left(a\mathcal{E}_a + (1-a)r'^2 \mathcal{K}_a \right) \right] \left[(1-a)\mathcal{E}_a(\mathcal{E}_a - r'^2 \mathcal{K}_a) - ar'^2 \mathcal{K}_a(\mathcal{K}_a - \mathcal{E}_a) \right].
 \end{aligned}$$

It follows from Lemmas 2.3 and 2.4 that $G_1(r)$ is negative for all $r \in (0, 1)$. Hence (2.9) and (2.10) lead to the conclusion that $G(r)$ is strictly decreasing on $(0, 1)$. Clearly, $G(1^-) = 0$, and from Lemma 2.2(1) we get

$$\begin{aligned}
 \lim_{r \rightarrow 0} G(r) &= \lim_{r \rightarrow 0} \left(\frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{r^2} \right) \frac{1}{\mathcal{K}_a(r)^2} \left[2a\mathcal{K}_a \mathcal{E}_a + 2(1-a) \frac{(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r^2} (\mathcal{K}_a - \mathcal{E}_a) \right] \\
 &= \frac{\pi}{2} a \cdot \frac{4}{\pi^2} \cdot 2a \frac{\pi^2}{4} = \pi a^2. \quad \square
 \end{aligned}$$

From Lemmas 2.7 and 2.8, we get Corollary 2.9 immediately.

COROLLARY 2.9. *Let $a, r \in (0, 1)$, and $P(r)$ and $Q(r)$ be defined by*

$$\begin{aligned}
 P(r) &= r^2 \mathcal{K}_a^2 \left[4(1-a^2)(\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 \right. \\
 &\quad \left. + 4a\mathcal{E}_a \left(a(\mathcal{E}_a - r'^2 \mathcal{K}_a) - (1-a)r'^2(\mathcal{K}_a - \mathcal{E}_a) + (1-2a)r^2 r'^2 \mathcal{K}_a \right) \right]
 \end{aligned}$$

and

$$Q(r) = (\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 \left[2ar^2 \mathcal{K}_a \mathcal{E}_a + 2(1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) \right]$$

Then the function

$$H(r) = \frac{P(r)}{Q(r)}$$

is strictly increasing from $(0, 1)$ onto $((1+a)(2-a)/a, +\infty)$ if $a \in (0, 1)$.

LEMMA 2.10. Let $\beta \in \mathbb{R}$, $a \in (0, 1)$,

$$I_\beta(r) = (\beta - 1) \frac{2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r'^2 \mathcal{K}_a} + \frac{r^2 [2\mathcal{E}_a - 2(1-a)r'^2 \mathcal{K}_a]}{r'^2 (\mathcal{E}_a - r'^2 \mathcal{K}_a)}$$

and $L(\beta) = \inf_{r \in (0,1)} I_\beta(r)$. Then the following statements are true:

- (1) $I_\beta(r)$ is strictly increasing from $(0, 1)$ onto $(2, +\infty)$ if and only if $\beta \geq -(a^2 - a + 2)/[2a(1 - a)]$, in which case $L(\beta) = 2$;
- (2) If $\beta < -(a^2 - a + 2)/[2a(1 - a)]$, then there exists $\eta = \eta(\beta) \in (0, 1)$ such that $I_\beta(r)$ is strictly decreasing on $(0, \eta)$ and strictly increasing on $(\eta, 1)$. Moreover, in this case, the range of $I_\beta(r)$ is $(L(\beta), +\infty)$ and $L(\beta) < 2$.

Proof. Let $f(r) = (\mathcal{E}_a - r'^2 \mathcal{K}_a)/(r'^2 \mathcal{K}_a)$ and

$$g(r) = \frac{r^2 [2\mathcal{E}_a - 2(1-a)r'^2 \mathcal{K}_a]}{r'^2 (\mathcal{E}_a - r'^2 \mathcal{K}_a)}.$$

Then simple computations lead to

$$\begin{aligned} f'(r) &= \frac{2ar\mathcal{K}_a r'^2 \mathcal{K}_a - (\mathcal{E}_a - r'^2 \mathcal{K}_a)[-2r\mathcal{K}_a + 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)/r]}{r'^4 \mathcal{K}_a^2} \\ &= \frac{2ar^2 r'^2 \mathcal{K}_a^2 - (\mathcal{E}_a - r'^2 \mathcal{K}_a)[-2ar^2 \mathcal{K}_a - 2(1-a)r^2 \mathcal{K}_a + 2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)]}{r r'^4 \mathcal{K}_a^2} \\ &= \frac{2ar^2 \mathcal{K}_a \mathcal{E}_a + 2(1-a)(\mathcal{K}_a - \mathcal{E}_a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r r'^4 \mathcal{K}_a^2} \end{aligned}$$

and

$$\begin{aligned} r'^4 (\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 g'(r) &= 2r [2\mathcal{E}_a - 2(1-a)r'^2 \mathcal{K}_a] r'^2 (\mathcal{E}_a - r'^2 \mathcal{K}_a) \\ &\quad + r^2 [4a(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)/r] r'^2 (\mathcal{E}_a - r'^2 \mathcal{K}_a) \\ &\quad - r^2 [2\mathcal{E}_a - 2(1-a)r'^2 \mathcal{K}_a] [-2r(\mathcal{E}_a - r'^2 \mathcal{K}_a) + 2arr'^2 \mathcal{K}_a] \\ &= 2r [2\mathcal{E}_a - 2(1-a)r'^2 \mathcal{K}_a] (\mathcal{E}_a - r'^2 \mathcal{K}_a) \\ &\quad + 4a(1-a)rr'^2 (\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 - 2ar^3 r'^2 \mathcal{K}_a [2\mathcal{E}_a - 2(1-a)r'^2 \mathcal{K}_a] \\ &= 2r [2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a) + 2a\mathcal{E}_a] (\mathcal{E}_a - r'^2 \mathcal{K}_a) \\ &\quad + 4a(1-a)r (\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 - 4a(1-a)r^3 (\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 \\ &\quad + 4a(1-a)r^3 r'^4 \mathcal{K}_a^2 - 4ar^3 r'^2 \mathcal{K}_a \mathcal{E}_a \\ &= 4(1-a^2)r (\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 + 4ar\mathcal{E}_a (\mathcal{E}_a - r'^2 \mathcal{K}_a) \\ &\quad - 4a(1-a)r^3 (\mathcal{E}_a^2 - 2r'^2 \mathcal{K}_a \mathcal{E}_a) - 4ar^3 r'^2 \mathcal{K}_a \mathcal{E}_a \\ &= 4(1-a^2)r (\mathcal{E}_a - r'^2 \mathcal{K}_a)^2 + 4ar\mathcal{E}_a [a(\mathcal{E}_a - r'^2 \mathcal{K}_a) \end{aligned}$$

$$-(1-a)r^2(\mathcal{K}_a - \mathcal{E}_a) + (1-2a)r^2r'^2 \mathcal{K}_a].$$

Differentiating I_β yields

$$I'_\beta(r) = 2(1-a)(\beta-1)f'(r) + g'(r) = 2(1-a)f'(r) \left[\beta - \left(1 - \frac{H(r)}{2(1-a)} \right) \right], \tag{2.11}$$

where $H(r)$ is defined in Corollary 2.9 which is strictly increasing on $(0, 1)$ with the range $((1+a)(2-a)/a, +\infty)$. Therefore, it follows from (2.11) that $I'_\beta(r) > 0$ for all $r \in (0, 1)$ and $I_\beta(r)$ is strictly increasing on $(0, 1)$ if and only if

$$\beta \geq \sup_{r \in (0,1)} \frac{1-H(r)}{2(1-a)} = -(a^2-a+2)/[2a(1-a)],$$

and $I_\beta(r)$ is piecewise monotone if $\beta < -(a^2-a+2)/[2a(1-a)]$. The limiting values are

$$I_\beta(0^+) = \lim_{r \rightarrow 0^+} \frac{2\mathcal{E}_a - 2(1-a)r'^2 \mathcal{K}_a}{(\mathcal{E}_a - r'^2 \mathcal{K}_a)/r^2} = \frac{\pi a}{\pi a/2} = 2$$

and

$$I_\beta(1^-) = \lim_{r \rightarrow 1^-} \frac{2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r'^2 \mathcal{K}_a} \left[\beta - 1 + \frac{r^2 \mathcal{K}_a [2\mathcal{E}_a - 2(1-a)r'^2 \mathcal{K}_a]}{2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)^2} \right] = +\infty.$$

This completes the proof of Lemma 2.10. \square

LEMMA 2.11. *Let $a \in (0, 1)$, $\alpha, \beta \in \mathbb{R}$, $L(\beta)$ be defined in Lemma 2.10 and*

$$J_{\alpha,\beta}(r) = \mathcal{K}_a(r)^{\beta-1} \frac{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r)}{r^\alpha r'^2}.$$

Then we have the following conclusions:

- (1) If $\beta \geq -(a^2-a+2)/[2a(1-a)]$, then $J_{\alpha,\beta}(r)$ is strictly increasing on $(0, 1)$ if and only if $\alpha \leq 2$, and there exists $\lambda = \lambda(\alpha, \beta) \in (0, 1)$ such that $J_{\alpha,\beta}(r)$ is strictly decreasing on $(0, \lambda)$ and strictly increasing on $(\lambda, 1)$ if $\alpha > 2$;
- (2) If $\beta < -(a^2-a+2)/[2a(1-a)]$, then $J_{\alpha,\beta}(r)$ is strictly increasing on $(0, 1)$ if and only if $\alpha \leq L(\beta)$, and $J_{\alpha,\beta}(r)$ is piecewise monotone if $\alpha > L(\beta)$.

Proof. Logarithmic differentiation $J_{\alpha,\beta}(r)$ gives

$$\frac{J'_{\alpha,\beta}(r)}{J_{\alpha,\beta}(r)} = (\beta-1) \frac{2(1-a)(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{r r'^2 \mathcal{K}_a} + \frac{2ar \mathcal{K}_a}{\mathcal{E}_a - r'^2 \mathcal{K}_a} - \frac{\alpha}{r} + \frac{2r}{r'^2} = \frac{1}{r} [I_\beta(r) - \alpha]. \tag{2.12}$$

We divide the proof into four cases.

Case I $\{(\beta, \alpha) | \beta \geq -(a^2-a+2)/[2a(1-a)], \alpha \leq 2\}$. Then it follows from (2.12) and Lemma 2.10(1) that $J'_{\alpha,\beta}(r) > 0$ for $r \in (0, 1)$, so that $J_{\alpha,\beta}(r)$ is strictly increasing on $(0, 1)$.

Case II $\{(\beta, \alpha) | \beta \geq -(a^2-a+2)/[2a(1-a)], \alpha > 2\}$. Then (2.12) and Lemma 2.10(1) imply that there exists $\lambda = \lambda(\alpha, \beta) \in (0, 1)$ such that $J'_{\alpha,\beta}(r) < 0$ for $r \in (0, \lambda)$

and $J'_{\alpha,\beta}(r) > 0$ for $r \in (\lambda, 1)$. Hence the piecewise monotonicity property of $J_{\alpha,\beta}(r)$ on $(0, 1)$ follows.

Case III $\{(\beta, \alpha) | \beta < -(a^2 - a + 2)/[2a(1 - a)], \alpha \leq L(\beta)\}$. Then it follows from (2.12) and Lemma 2.10(2) together with a similar argument of Case I that $J'_{\alpha,\beta}(r) > 0$ for $r \in (0, 1)$, so that $J_{\alpha,\beta}(r)$ is strictly increasing on $(0, 1)$.

Case IV $\{(\beta, \alpha) | \beta < -(a^2 - a + 2)/[2a(1 - a)], \alpha > L(\beta)\}$. Then by (2.12) and Lemma 2.10(2), the conclusion in this case can be derived analogously. \square

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $0 < x \leq y < 1$, $s = H_\alpha(x, y)$, D_1, D_2, D_3, D_4, D_5 and D_6 be defined by

$$\begin{aligned} D_1 &= \{(\beta, \alpha) | \beta \geq -(a^2 - a + 2)/[2a(1 - a)], \alpha \leq 2, \beta \neq 0\}, \\ D_2 &= \{(\beta, \alpha) | \beta \geq -(a^2 - a + 2)/[2a(1 - a)], \alpha > 2, \beta \neq 0\}, \\ D_3 &= \{(\beta, \alpha) | \beta < -(a^2 - a + 2)/[2a(1 - a)], \alpha \leq L(\beta), \beta \neq 0\}, \\ D_4 &= \{(\beta, \alpha) | \beta < -(a^2 - a + 2)/[2a(1 - a)], \alpha > L(\beta), \beta \neq 0\}, \\ D_5 &= \{(\beta, \alpha) | \beta = 0, \alpha \leq 2\} \end{aligned}$$

and

$$D_6 = \{(\beta, \alpha) | \beta = 0, \alpha > 2\},$$

respectively. Then we clearly see that $D = D_1 \cup D_3 \cup D_5$, $\partial s / \partial x = (x/s)^{\alpha-1}/2$, and $s > x$ if $y > x$.

We divide the proof into two cases of $\beta \neq 0$ and $\beta = 0$.

Case 1 $\beta \neq 0$. Let

$$F(x, y) = \mathcal{K}_\alpha(H_\alpha(x, y))^\beta - \frac{\mathcal{K}_\alpha(x)^\beta + \mathcal{K}_\alpha(y)^\beta}{2}. \tag{3.1}$$

Then differentiating $F(x, y)$ with respect to x we get

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\beta}{2} \mathcal{K}_\alpha(s)^{\beta-1} 2(1-a) \frac{\mathcal{E}_a(s) - s'^2 \mathcal{K}_a(s)}{ss'^2} \left(\frac{x}{s}\right)^{\alpha-1} \\ &\quad - \frac{\beta}{2} \mathcal{K}_\alpha(x)^{\beta-1} 2(1-a) \frac{\mathcal{E}_a(x) - x'^2 \mathcal{K}_a(x)}{xx'^2} \\ &= (1-a)\beta x^{\alpha-1} \left[\mathcal{K}_\alpha(s)^{\beta-1} \frac{\mathcal{E}_a(s) - s'^2 \mathcal{K}_a(s)}{s^\alpha s'^2} - \mathcal{K}_\alpha(x)^{\beta-1} \frac{\mathcal{E}_a(x) - x'^2 \mathcal{K}_a(x)}{x^\alpha x'^2} \right]. \end{aligned} \tag{3.2}$$

We divide the proof of Case 1 into four subcases.

Subcase 1.1 $(\beta, \alpha) \in D_1$. Then it follows from Lemma 2.11(1) and (3.2) that $\partial F / \partial x < 0$ if $\beta < 0$, and $\partial F / \partial x > 0$ if $\beta > 0$, so that $F(x, y) > F(y, y) = 0$ if $\beta < 0$, and $F(x, y) < F(y, y) = 0$ if $\beta > 0$. As a consequence, the inequality

$$\mathcal{K}_\alpha(H_\alpha(x, y)) \leq H_\beta(\mathcal{K}_\alpha(x), \mathcal{K}_\alpha(y)) \tag{3.3}$$

holds for all $x, y \in (0, 1)$. Furthermore, it is easy to check that inequality (3.3) becomes equality if and only if $x = y$.

Therefore, $\mathcal{K}_a(r)$ is strictly $H_{\alpha,\beta}$ -convex for $(\alpha, \beta) \in D_1$.

Subcase 1.2 $(\beta, \alpha) \in D_2$. Then it follows from (3.1), (3.2) and Lemma 2.11(1) together with the similar argument of Subcase 1.1 that there exist $\lambda \in (0, 1)$, $x_0, y_0 \in (0, \lambda)$ and $x_1, y_1 \in (\lambda, 1)$ such that the inequalities $\mathcal{K}_a(H_\alpha(x_0, y_0)) \geq H_\beta(\mathcal{K}_a(x_0), \mathcal{K}_a(y_0))$ and $\mathcal{K}_a(H_\alpha(x_1, y_1)) \leq H_\beta(\mathcal{K}_a(x_1), \mathcal{K}_a(y_1))$ hold. Therefore, $\mathcal{K}_a(r)$ is neither $H_{\alpha,\beta}$ -concave nor $H_{\alpha,\beta}$ -convex in the whole interval $(0, 1)$.

Subcase 1.3 $(\beta, \alpha) \in D_3$. Then the $H_{\alpha,\beta}$ -convexity of $\mathcal{K}_a(r)$ on $(0, 1)$ follows directly from (3.1), (3.2) and Lemma 2.11(2) together with the similar argument in the proof of Subcase 1.1.

Subcase 1.4 $(\beta, \alpha) \in D_4$. Then it follows from (3.1), (3.2) and Lemma 2.11(2) together with the same procedure as in the proof of Subcase 1.2 that $\mathcal{K}_a(r)$ is neither $H_{\alpha,\beta}$ -concave nor $H_{\alpha,\beta}$ -convex on the whole interval $(0, 1)$.

Case 2 $\beta = 0$. Let

$$G(x, y) = \frac{[\mathcal{K}_a(H_\alpha(x, y))]^2}{\mathcal{K}_a(x)\mathcal{K}_a(y)}. \tag{3.4}$$

Then logarithmical differentiation of $G(x, y)$ with respect to x yields

$$\frac{1}{G(x, y)} \frac{\partial G}{\partial x} = 2(1 - a)x^{\alpha-1} \left[\frac{\mathcal{E}_a(s) - s'^2 \mathcal{K}_a(s)}{s^\alpha s'^2 \mathcal{K}_a(s)} - \frac{\mathcal{E}_a(x) - x'^2 \mathcal{K}_a(x)}{x^\alpha x'^2 \mathcal{K}_a(x)} \right]. \tag{3.5}$$

We divide the proof of Case 2 into two subcases.

Subcase 2.1 $(\beta, \alpha) \in D_5$. Then equation (3.5) and Lemma 2.11(1) lead to the conclusion that $\partial G / \partial x > 0$, so that $G(x, y) \leq G(y, y) = 1$. Consequently,

$$\mathcal{K}_a(H_\alpha(x, y)) \leq H_0(\mathcal{K}_a(x), \mathcal{K}_a(y)),$$

with equality if and only if $x = y$.

Therefore, $\mathcal{K}_a(r)$ is strictly $H_{\alpha,\beta}$ -convex on $(0, 1)$ for $(\alpha, \beta) \in D_5$.

Subcase 2.2 $(\beta, \alpha) \in D_6$. Then making use of (3.4), (3.5) and Lemma 2.11(1) we know that $\mathcal{K}_a(r)$ is neither $H_{\alpha,\beta}$ -concave nor $H_{\alpha,\beta}$ -convex on the whole interval $(0, 1)$. \square

REMARK 3.1. Until now, the readers easily find that the proof of Theorem 1.2 depends on the monotonicity properties for the function $J_{\alpha,\beta}(r)$ in Lemma 2.11, or rather the monotonicity of $H(r)$ in Corollary 2.9. What we must emphasize here is that the proof of Corollary 2.9 is not just a simple generalization of that in [62]. Actually, our obtained results in section 2, including Lemmas 2.3-2.8 are new even for the particular case of $a = 1/2$.

4. Convexity of the complete p -elliptic integral of the first kind with respect to Höler mean

For $p \in (1, \infty)$ and $k \in (0, 1)$, the complete p -elliptic integrals $\mathbf{K}_p(k)$ and $\mathbf{E}_p(k)$ [34, 60] of the first and second kinds are defined by

$$\mathbf{K}_p(k) = \int_0^1 \frac{dt}{(1-t^p)^{1/p}(1-k^p t^p)^{1-1/p}} = \frac{\pi_p}{2} F\left(1 - \frac{1}{p}, \frac{1}{p}; 1; k^p\right) \tag{4.1}$$

and

$$E_p(k) = \int_0^1 \left(\frac{1 - k^p t^p}{1 - t^p} \right)^{\frac{1}{p}} dt = \frac{\pi_p}{2} F \left(-\frac{1}{p}, \frac{1}{p}; 1; k^p \right), \tag{4.2}$$

respectively, where $\pi_p = 2\pi/[p \sin \pi/p]$ is the generalized circumference ratio with $\pi_2 = \pi$.

In the past five years, the complete p -elliptic integrals and generalized circumference ratio have been studied by Takeuchi [52, 53, 54, 55] and Zhang [89] from different points of view. In particular, a lot of new analytical properties and inequalities involving the complete p -elliptic integrals and generalized circumference ratio, such as Legendre-type relations and monotonicity theorems of $\mathbf{K}_p(k)$ and $E_p(k)$, asymptotical inequalities of π_p and computational formulas for π_3 and π_4 have been obtained.

Let $a = 1 - 1/p$ and $r = k^{p/2}$. Then from (1.2), (4.1) and (4.2) one has

$$\begin{aligned} \mathbf{K}_p(k) &= \frac{\pi_p}{\pi} \mathcal{K}_a(r), & E_p(k) &= \frac{\pi_p}{\pi} \mathcal{E}_a(r), \\ \mathbf{K}_p(\sqrt[p]{1 - k^p}) &= \frac{\pi_p}{\pi} \mathcal{K}_a(r'), & E_p(\sqrt[p]{1 - k^p}) &= \frac{\pi_p}{\pi} \mathcal{E}_a(r'). \end{aligned}$$

From Theorem 1.2, the analogous convexity properties of the complete p -elliptic integral $\mathbf{K}_p(k)$ of the first kind with respect to Hölder mean can also be derived.

THEOREM 4.1. *Let $p \in (1, \infty)$. Then the complete p -elliptic integral $\mathbf{K}_p(k)$ of the first kind is $H_{\alpha, \beta}$ -convex on $(0, 1)$, namely the inequality*

$$\mathbf{K}_p(H_\alpha(x, y)) \leq H_\beta(\mathbf{K}_p(x), \mathbf{K}_p(y))$$

holds for all $x, y \in (0, 1)$ if and only if

$$(\beta, \alpha) \in \tilde{D} = \{(\beta, \alpha) | \alpha \leq \tilde{L}(\beta)\},$$

where

$$\tilde{L}(\beta) = \inf_{k \in (0, 1)} \left\{ (\beta - 1) \frac{E_p(k) - (1 - k^p)\mathbf{K}_p(k)}{(1 - k^p)\mathbf{K}_p(k)} + \frac{k^p [pE_p(k) - (1 - k^p)\mathbf{K}_p(k)]}{(1 - k^p)[E_p(k) - (1 - k^p)\mathbf{K}_p(k)]} \right\}$$

is a continuous function with $\tilde{L}(\beta) = p$ for all $\beta \geq -(2p^2 - p + 1)/[2(p - 1)]$, and $\tilde{L}(\beta) < p$ for all $\beta < -(2p^2 - p + 1)/[2(p - 1)]$. Moreover, there exists no values of α and β such that $\mathbf{K}_p(k)$ is $H_{\alpha, \beta}$ -concave on $(0, 1)$.

Proof. Let $\alpha, \beta \in \mathbb{R}$ be the parameters such that the inequality

$$\mathbf{K}_p(H_\alpha(x, y)) \leq (\geq) H_\beta(\mathbf{K}_p(x), \mathbf{K}_p(y))$$

holds for all $x, y \in (0, 1)$. Then we clearly see that

$$\mathbf{K}_p \left(H_\alpha(x^{2/p}, y^{2/p}) \right) \leq (\geq) H_\beta(\mathbf{K}_p(x^{2/p}), \mathbf{K}_p(y^{2/p}))$$

for $p \in (1, \infty)$.

Let $a = 1 - 1/p$. Then multiplying both sides of the above inequality by π/π_p leads to

$$\mathcal{K}_a \left(H_{\frac{2\alpha}{p}}(x, y) \right) \leq (\geq) H_\beta(\mathcal{K}_a(x), \mathcal{K}_a(y)).$$

By Theorem 1.2 we know that $\mathbf{K}_p(k)$ is $H_{\alpha, \beta}$ -convex on $(0, 1)$ if and only if $\alpha \leq pL(\beta)/2$, and there are no values α and β such that $\mathbf{K}_p(k)$ is $H_{\alpha, \beta}$ -concave on $(0, 1)$.

Taking $r = k^{p/2}$ and $a = 1 - 1/p$, it is obvious to see that $-(a^2 - a + 2)/[2a(1 - a)] = -(2p^2 - p + 1)/[2(p - 1)]$ and

$$\begin{aligned} L(\beta) &= \inf_{r \in (0,1)} \left\{ (\beta - 1) \frac{2(1-a)(\mathcal{E}_a - r^{j^2} \mathcal{H}_a)}{r^{j^2} \mathcal{H}_a} + \frac{r^2[2\mathcal{E}_a - 2(1-a)r^{j^2} \mathcal{H}_a]}{r^{j^2}(\mathcal{E}_a - r^{j^2} \mathcal{H}_a)} \right\} \\ &= \inf_{k \in (0,1)} \left\{ (\beta - 1) \frac{2 \mathbf{E}_p(k) - (1 - k^p) \mathbf{K}_p(k)}{p(1 - k^p) \mathbf{K}_p(k)} + \frac{2k^p[p\mathbf{E}_p(k) - (1 - k^p)\mathbf{K}_p(k)]}{p(1 - k^p)[\mathbf{E}_p(k) - (1 - k^p)\mathbf{K}_p(k)]} \right\} \\ &= \frac{2}{p} \tilde{L}(\beta). \end{aligned}$$

Therefore, Theorem 4.1 follows. \square

5. Appendixes

5.1. Appendix 1

We check that equation (2.6) is equivalent to (2.7) by Mathematical Software *Maple 13* as follows:

```
>f:=4*(1-a^2)*(E-(1-r^2)*K)^2*(2*a*(K-E)
-2*(1-a)*(E-(1-r^2)*K))-8*a*(1-a)*(K-E)
*(E-(1-r^2)*K)*(a*(E-(1-r^2)*K)-(1-a)
*(1-r^2)*(K-E)+(1-2*a)*r^2*(1-r^2)*K)
+4*a*(1+a)*r^2*E*(2*a*r^2*K-2*(1-a)
*(E-(1-r^2)*K))*(E-(1-r^2)*K)
-8*a*E*(a*(E-(1-r^2)*K)-(1-a)*(1-r^2)
*(K-E)+(1-2*a)*r^2*(1-r^2)*K)
*(E-(1-r^2)*K+a*r^2*K):
>g:=8*(E-(1-r^2)*K+a*(1-r^2)*K)
*(a*(1-a)*r^2*(K-E)*(E-(1-r^2)*K)
+a^2*r^4*K*E-(E-(1-r^2)*K)^2):
>simplify(f-g);
0
```

5.2. Appendix 2

We check that equation (2.9) is equivalent to (2.10) by Mathematical Software *Maple 13* as follows:

```
>f:=2*a*r^2*(1-r^2)*K^2*(2*a*r^2*K*E
+2*(1-a)*(K-E)*(E-(1-r^2)*K))
+4*r^2*K*E*(E-(1-r^2)*K)
*(a*(1-r^2)*K+(1-a)*(E-(1-r^2)*K))
-(E-(1-r^2)*K)*(2*a*r^2*K*E+2*(1-a)
*(K-E)*(E-(1-r^2)*K)*(4*(1-r^2)*K
+4*(1-a)*(E-(1-r^2)*K))):
```



```

>g:=-4*((1-a)*(E-(1-r^2)*K)
*((1+a)*(K-E)-(1-a)*(E-(1-r^2)*K))
+a*r^2*K*(a*E+(1-a)*(1-r^2)*K))
*((1-a)*E*(E-(1-r^2)*K)-a*(1-r^2)*K*(K-E)):
>simplify(f-g);
0
    
```

5.3. Appendix 3

Figure 1 shows the regions D in Theorem 1.2 with the special parameters $a = 1/8, 1/4, 3/8$ and $1/2$, and their corresponding boundary curves $\alpha = L(\beta)$.

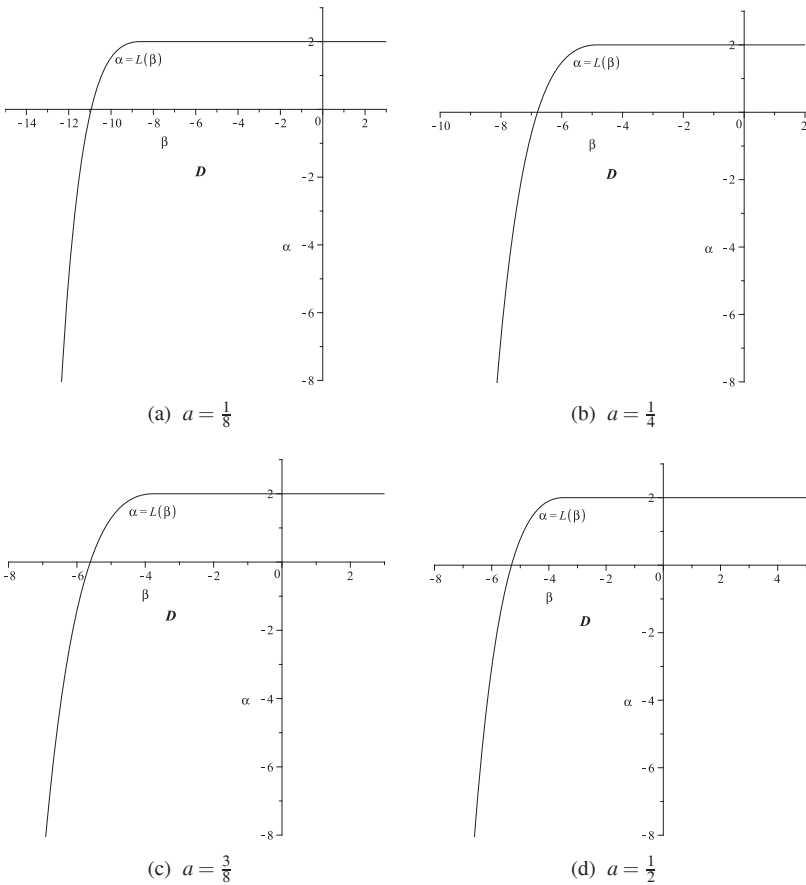


Figure 1: The regions D with different parameters $a = 1/8, 1/4, 3/8, 1/2$ and their corresponding boundary curves $\alpha = L(\beta)$

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