Journal of Mathematical Inequalities
Volume 14, Number 3 (2020), 923–938 doi:10.7153/jmi-2020-14-60

COMPLETE MOMENT CONVERGENCE FOR (α,β)–MIXING RANDOM VARIABLES AND ITS APPLICATION

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(*Communicated by X. Wang*)

Abstract. In this paper, the complete moment convergence for weighted sums of (α,β)-mixing random variables is investigated. The result improves and extends the corresponding one of Wu et al. (2017). As a corollary, the complete convergence for weighted sums of (α,β)-mixing random variables is obtained, which is applied to establish the complete consistency for the P-C estimator in a nonparametric regression model.

1. Introduction

Up to now, the research on the convergence is still an important topic in probability limit theory. Recently, the complete moment convergence received more and more attention of scholars since it is much stronger than other types of convergence such as convergence in probability, almost sure convergence, L_r convergence (or namely, mean convergence), and complete convergence. The concept of complete moment convergence was first introduced by Chow (1988) as follows:

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables and $a_n > 0, b_n > 0, q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty, \text{ for all } \varepsilon > 0,$$

then $\{X_n, n \geqslant 1\}$ is said to be complete moment convergence.

It is easy to check that the complete moment convergence can derive the complete convergence, the concept of which was proposed by Hsu and Robbins (1947) as follows:

A sequence $\{X_n, n \geqslant 1\}$ of random variables converges completely to a constant C if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty.$$

By the Borel-Cantelli lemma, the inequality above implies that $X_n \to C$ almost surely (a.s., for short).

Mathematics subject classification (2010): 60F15, 62G20.
Keywords and phrases: (α,β)-mixing random variables, complete moment convergence, P-C estimator, nonparametric regression model, complete consistency.
Supported by the Key Research Project of Chaohu University (XLZ-201904, XLZ-201903).
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The complete convergence is a powerful tool not only in establishing the strong law of large numbers, but also characterizing the convergence rate of partial sums of random variables. Therefore, this topic was studied continuously by many scholars in the past decades. We refer the readers to Erdös (1949), Baum and Katz (1965), Chow (1973), Bai and Cheng (2000), Liang and Jing (2005), Sung (2013), Wang et al. (2014), Shen (2016), Chen and Sung (2018), among others. For more details about the complete moment convergence, we refer the readers to Wang and Hu (2014), Wu et al. (2014), Liang et al. (2010), Shen et al. (2016), Wu and Wang (2018) among others.

Recently, Wu et al. (2017) obtained the following result on complete moment convergence for weighted sums of ρ^*-mixing random variables, which improves the corresponding result of Sung (2010).

THEOREM A. *Let $r>0$, $\gamma>1/2$ and $\gamma p \geqslant 1$. Let $\{X, X_n, n\geqslant 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX=0$ if $p\vee r\geqslant 1$. Assume that $\{a_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ is an array of constants satisfying $\sum_{i=1}^n |a_{ni}|^q \ll n$ for some $q>p\vee r$. Then*

$$\begin{cases} E|X|^p<\infty, & if\ r<p,\\ E|X|^p\log|X|<\infty, & if\ r=p,\\ E|X|^r<\infty, & if\ r>p,\end{cases}$$

implies that

$$\sum_{n=1}^{\infty} n^{\gamma p-2-\alpha r} E\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}X_i\right|-\varepsilon n^{\gamma}\right)_{+}^{r}<\infty,\ \ for\ all\ \ \varepsilon>0.$$

In this paper, we will further study the complete moment convergence for weighted sums of (α,β)-mixing random variables under the condition of stochastic domination. Therefore, in what follows, we need to recall the concept of (α,β)-mixing random variables and stochastic domination.

The concept of (α,β)-mixing random variables was first introduced by Bradley (1985) as follows.

Let $\{X_n, n\geqslant 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega,\mathscr{F},P)$. Denote $S_n=\sum_{i=1}^n X_i, n\geqslant 1$, and $S_0=0$. Let n and m be positive integers. Write $\mathscr{F}_n^m=\sigma(X_i, n\leqslant i\leqslant m)$. Given σ-algebras $\mathscr{A}$ and $\mathscr{B}$ in $\mathscr{F}$, let

$$\lambda(\mathscr{A},\mathscr{B})=\sup_{X\in L_{1/\alpha}(\mathscr{A}),Y\in L_{1/\beta}(\mathscr{B})}\frac{|EXY-EXEY|}{\|X\|_{1/\alpha}\|Y\|_{1/\beta}},$$

where $0<\alpha,\beta<1, \alpha+\beta=1$, and $\|X\|_p=(E|X|^p)^{1/p}$. Define the (α,β)-mixing coefficients by

$$\lambda(n)=\sup_{k\geqslant 1}\lambda(\mathscr{F}_1^k,\mathscr{F}_{k+n}^{\infty}),\quad n\geqslant 0.$$

DEFINITION 1.1. A sequence $\{X_n, n \geqslant 1\}$ of random variable is said to be (α,β)-mixing if $\lambda(n) \downarrow 0$ as $n \to \infty$.

Since the concept of (α,β)-mixing was introduced by Bradley (1985), many limit theorems were established. For more results, one can refer to Shao (1989), Cai (1991), Lu and Lin (1997), Shen et al. (2011), Gao (2016), Yu (2016), Samura et al. (2019), and so on.

The concept of stochastic domination below will be used in the paper.

DEFINITION 1.2. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X, if there exists a positive constant C such that

$$P(|X_n| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$ and $n \geqslant 1$.

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X, if there exists a positive constant C such that

$$P(|X_{ni}| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$, $i \geqslant 1$ and $n \geqslant 1$.

This paper mainly investigates the complete moment convergence for double indexed weighted sums of (α,β)-mixing random variables with stochastic domination. As an application, we further study the complete consistency for the P-C estimator in a nonparametric regression model and get some new results.

This paper is organized as follows. Some preliminary lemmas are provided in Section 2. Main results and their proofs are stated in Section 3. An application to nonparametric regression models is presented in Section 4. Throughout this paper, C represents some positive constant whose value may vary in different places. Let $\log x = \ln\max(x,e)$, and $I(A)$ be the indicator function of the set A. Denote $x_+ = xI(x \geqslant 0)$. $a \ll b$ means that there exists some positive constant c such that $a \leqslant cb$. $a \vee b$ stands for $\max(a,b)$ and $a \wedge b$ means $\min(a,b)$.

2. Preliminary lemmas

To prove the main results of the paper, we need the following important lemmas. The first lemma is essential in proving our main results, which can be seen in Wu et al. (2017).

LEMMA 2.1. *Let $\{Y_i, 1 \leqslant i \leqslant n\}$ and $\{Z_i, 1 \leqslant i \leqslant n\}$ be two sequences of random variables. Then for any $q > r > 0$, $\varepsilon > 0$, and $a > 0$, the following inequality holds:*

$$E\left(\left|\sum_{i=1}^{n}(Y_i + Z_i)\right| - \varepsilon a\right)_+^r \leqslant C_r\left(\varepsilon^{-q} + \frac{r}{q-r}\right)a^{r-q}E\left|\sum_{i=1}^{n}Y_i\right|^q + C_rE\left|\sum_{i=1}^{n}Z_i\right|^r,$$

where $C_r = 1$ if $0 < r \leqslant 1$ or $C_r = 2^{r-1}$ if $r > 1$.

The next lemma concerns the Rosenthal type inequality and Marcinkiewicz-
-Zygmund type inequality for (α,β)-mixing random variables. The first inequality

comes from Yu (2016) while the second inequality can be obtained by the first one and the method used in Chen et al. (2014).

LEMMA 2.2. *Let* $\{X_i, i \geqslant 1\}$ *be a sequence of* (α,β)*-mixing random variables with* $EX_i = 0$, $E|X_i|^p < \infty$ *for some* $p \geqslant 1$ *and* $\sum_{n=1}^{\infty}(\lambda(n))^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}} < \infty$, *where* $0 < \alpha,\beta < 1$ *and* $\alpha+\beta = 1$. *Assume that* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *is an array of real numbers. Then there exists a positive constant* C *depending only on* α, β *and* $\lambda(\cdot)$ *such that if* $p \geqslant 2$,

$$E\left|\sum_{i=1}^{n} a_{ni}X_i\right|^p \leqslant C\left\{\sum_{i=1}^{n}|a_{ni}|^p E|X_i|^p + \left(\sum_{i=1}^{n} a_{ni}^2 EX_i^2\right)^{p/2}\right\},$$

and if $1 \leqslant p < 2$,

$$E\left|\sum_{i=1}^{n} a_{ni}X_i\right|^p \leqslant C\sum_{i=1}^{n}|a_{ni}|^p E|X_i|^p.$$

The following two lemmas can be seen in Chen and Sung (2018).

LEMMA 2.3. *Let* $\alpha > 0$, $s > 0$, $p > 0$ *and* X *be a random variable. Assume that* $\sum_{i=1}^{n}|a_{ni}|^q \ll n$ *for some* $q > p \vee s$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2}\sum_{i=1}^{n} E|a_{ni}X|^s I(|a_{ni}X| > n^{\alpha}) \ll \begin{cases} E|X|^p, & if\ s < p,\\ E|X|^p\log|X|, & if\ s = p,\\ E|X|^s, & if\ s > p.\end{cases}$$

LEMMA 2.4. *Let* $\alpha > 0$, $p > 0$ *and* X *be a random variable. Assume that* $\sum_{i=1}^{n}|a_{ni}|^q \ll n$ *for some* $q > p$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\sum_{i=1}^{n} E|a_{ni}X|^q I(|a_{ni}X| \leqslant n^{\alpha}) \ll E|X|^p.$$

By using the definition of stochastic domination and integration by parts, one can easily establish the following important property of stochastic domination, which can also be found in Wu (2006).

LEMMA 2.5. *Let* $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ *be an array of random variables which is stochastically dominated by a random variable* X. *For any* $\alpha > 0$ *and* $b > 0$, *the following two statements hold:*

$$E|X_{ni}|^{\alpha} I(|X_{ni}| \leqslant b) \leqslant C_1\left[E|X|^{\alpha} I(|X| \leqslant b) + b^{\alpha}P(|X| > b)\right],$$
$$E|X_{ni}|^{\alpha} I(|X_{ni}| > b) \leqslant C_2 E|X|^{\alpha} I(|X| > b),$$

where C_1 *and* C_2 *are positive constants. Thus,* $E|X_{ni}|^{\alpha} \leqslant CE|X|^{\alpha}$, *where* C *is a positive constant.*

3. Main results

Our main results and proofs are presented as follows.

THEOREM 3.1. *Let $r>0$, $\gamma>1/2$ and $\gamma(p\vee r)\geqslant 1$. Let $\{X_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ be an array of rowwise (α,β)-mixing random variables stochastically dominated by a random variable X with $\sum_{n=1}^{\infty}(\lambda(n))^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}<\infty$, where $0<\alpha,\beta<1$ and $\alpha+\beta=1$. Assume further that $EX_{ni}=0$ if $p\vee r\geqslant 1$. Let $\{a_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ be an array of constants satisfying $\sum_{i=1}^{n}|a_{ni}|^q\ll n$ for some $q>p\vee r$. Then*

$$\begin{cases} E|X|^p<\infty, & if\ r<p,\\ E|X|^p\log|X|<\infty, & if\ r=p,\\ E|X|^r<\infty, & if\ r>p, \end{cases} \tag{1}$$

implies that

$$\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r}E\left(\left|\sum_{i=1}^{n}a_{ni}X_{ni}\right|-\varepsilon n^{\gamma}\right)_{+}^{r}<\infty. \tag{2}$$

Proof. We may assume without loss of generality that $\sum_{i=1}^{n}|a_{ni}|^q\leqslant n$. It follows from Hölder's inequality that $\sum_{i=1}^{n}|a_{ni}|^s\leqslant n$ for any $0<s<q$. For fixed $n\geqslant 1$, denote for $1\leqslant i\leqslant n$ that

$$Y_{ni}=a_{ni}X_{ni}I(|a_{ni}X_{ni}|\leqslant n^{\gamma}),$$
$$Z_{ni}=a_{ni}X_{ni}-Y_{ni}=a_{ni}X_{ni}I(|a_{ni}X_{ni}|>n^{\gamma}).$$

Now we will consider the following three cases.

Case 1. $0<p\vee r<1$.

Take $\theta=q\wedge 1$. It follows from C_r inequality and Lemmas 2.3-2.5 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r}E\left(\left|\sum_{i=1}^{n}a_{ni}X_{ni}\right|-\varepsilon n^{\gamma}\right)_{+}^{r}\\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\theta-2}E\left|\sum_{i=1}^{n}Y_{ni}\right|^{\theta}+\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r}E\left|\sum_{i=1}^{n}Z_{ni}\right|^{r}\\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\theta-2}\sum_{i=1}^{n}E|a_{ni}X_{ni}|^{\theta}I(|a_{ni}X_{ni}|\leqslant n^{\gamma})+\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r}\sum_{i=1}^{n}E|a_{ni}X_{ni}|^{r}I(|a_{ni}X_{ni}|>n^{\gamma})\\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\theta-2}\sum_{i=1}^{n}E|a_{ni}X|^{\theta}I(|a_{ni}X|\leqslant n^{\gamma})+\sum_{n=1}^{\infty} n^{\gamma p-2}\sum_{i=1}^{n}P(|a_{ni}X|>n^{\gamma})\\
&\quad+\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r}\sum_{i=1}^{n}E|a_{ni}X|^{r}I(|a_{ni}X|>n^{\gamma})
\end{aligned}$$

$$\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\theta-2} \sum_{i=1}^{n} E|a_{ni}X|^{\theta} I(|a_{ni}X| \leqslant n^{\gamma}) + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} \sum_{i=1}^{n} E|a_{ni}X|^{r} I(|a_{ni}X| > n^{\gamma})$$
$$< \infty.$$

Case 2. $1 \leqslant p \vee r < 2$.

It follows from (1) that $E|X|^{p\vee r} < \infty$. Moreover, by $\sum_{i=1}^{n} |a_{ni}|^{q} \leqslant n$ we can see that $\sum_{i=1}^{n} |a_{ni}|^{p\vee r} \leqslant n$ and $\max_{1\leqslant i\leqslant n} |a_{ni}| \leqslant n^{1/q}$. Therefore, we have by $EX_{ni} = 0$, Lemma 2.5, and the Dominated Convergence Theorem that

$$\begin{aligned}
n^{-\gamma}\left|\sum_{i=1}^{n} EY_{ni}\right| &= n^{-\gamma}\left|\sum_{i=1}^{n} EZ_{ni}\right| \\
&\leqslant n^{-\gamma}\sum_{i=1}^{n} E|a_{ni}X_{ni}| I(|a_{ni}X_{ni}| > n^{\gamma}) \\
&\leqslant Cn^{-\gamma}\sum_{i=1}^{n} E|a_{ni}X| I(|a_{ni}X| > n^{\gamma}) \\
&\leqslant Cn^{-\gamma(p\vee r)}\sum_{i=1}^{n} E|a_{ni}X|^{p\vee r} I(|a_{ni}X| > n^{\gamma}) \\
&\leqslant Cn^{1-\gamma(p\vee r)} E|X|^{p\vee r} I(|X| > n^{\gamma-1/q}) \\
&\leqslant CE|X|^{p\vee r} I(|X| > n^{\gamma-1/q}) \to 0, \text{ as } n\to\infty. \qquad (3)
\end{aligned}$$

Hence, $|\sum_{i=1}^{n} EY_{ni}| \leqslant \varepsilon n^{\gamma}/2$ for all n large enough. Take $\vartheta = q \wedge 2$. Similar to the proof of Case 1, we have by Lemmas 2.1-2.5, C_r-inequality and Jensen's inequality that if $0 < r < 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon n^{\gamma}\right)_{+}^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left(\left|\sum_{i=1}^{n} (Y_{ni} - EY_{ni} + Z_{ni})\right| - \varepsilon n^{\gamma}/2\right)_{+}^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\vartheta-2} E\left|\sum_{i=1}^{n} (Y_{ni} - EY_{ni})\right|^{\vartheta} + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left|\sum_{i=1}^{n} Z_{ni}\right|^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\vartheta-2} \sum_{i=1}^{n} E|a_{ni}X|^{\vartheta} I(|a_{ni}X| \leqslant n^{\gamma}) + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} \sum_{i=1}^{n} E|a_{ni}X|^{r} I(|a_{ni}X| > n^{\gamma}) \\
&< \infty,
\end{aligned}$$

and if $1 \leqslant r < 2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon n^{\gamma}\right)_{+}^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\vartheta-2} E\left|\sum_{i=1}^{n} (Y_{ni} - EY_{ni})\right|^{\vartheta} + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left|\sum_{i=1}^{n} (Z_{ni} - EZ_{ni})\right|^{r}
\end{aligned}$$

$$\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\vartheta-2} \sum_{i=1}^{n} E|a_{ni}X|^{\vartheta} I(|a_{ni}X| \leqslant n^{\gamma}) + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} \sum_{i=1}^{n} E|a_{ni}X|^{r} I(|a_{ni}X| > n^{\gamma})$$
$$< \infty.$$

Case 3. $p \vee r \geqslant 2$.

Note that $EY_{ni}^2 \leqslant EX_{ni}^2 \leqslant CEX^2$ and in the case we always have $EX^2 < \infty$. Choose $\mu > q \vee (\gamma p-1)/(\gamma-1/2)$ such that $\gamma p - \gamma\mu - 2 + \mu/2 < -1$. Hence, we have by Lemmas 2.1-2.5, (3), C_r-inequality and Jensen's inequality that if $0 < r < 1$,

$$\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\gamma}\right)_+^r$$
$$\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2} E\left|\sum_{i=1}^{n}(Y_{ni}-EY_{ni})\right|^{\mu} + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left|\sum_{i=1}^{n} Z_{ni}\right|^{r}$$
$$\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2}\left\{\sum_{i=1}^{n} E|Y_{ni}|^{\mu} + \left(\sum_{i=1}^{n} EY_{ni}^2\right)^{\mu/2}\right\} + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} \sum_{i=1}^{n} E|a_{ni}X|^{r} I(|a_{ni}X| > n^{\gamma})$$
$$\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2} \sum_{i=1}^{n} E|a_{ni}X|^{\mu} I(|a_{ni}X| \leqslant n^{\gamma}) + \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2+\mu/2}(EX^2)^{\mu/2}$$
$$+ \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} \sum_{i=1}^{n} E|a_{ni}X|^{r} I(|a_{ni}X| > n^{\gamma}) < \infty,$$

if $1 \leqslant r \leqslant 2$,

$$\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\gamma}\right)_+^r$$
$$\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2} E\left|\sum_{i=1}^{n}(Y_{ni}-EY_{ni})\right|^{\mu} + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left|\sum_{i=1}^{n}(Z_{ni}-EZ_{ni})\right|^{r}$$
$$\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2}\left\{\sum_{i=1}^{n} E|Y_{ni}|^{\mu} + \left(\sum_{i=1}^{n} EY_{ni}^2\right)^{\mu/2}\right\} + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} \sum_{i=1}^{n} E|a_{ni}X|^{r} I(|a_{ni}X| > n^{\gamma})$$
$$\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2} \sum_{i=1}^{n} E|a_{ni}X|^{\vartheta} I(|a_{ni}X| \leqslant n^{\gamma}) + \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2+\mu/2}(EX^2)^{\mu/2}$$
$$+ \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} \sum_{i=1}^{n} E|a_{ni}X|^{r} I(|a_{ni}X| > n^{\gamma}) < \infty,$$

and if $r > 2$,

$$\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\gamma}\right)_+^r$$
$$\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2} E\left|\sum_{i=1}^{n}(Y_{ni}-EY_{ni})\right|^{\mu} + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left|\sum_{i=1}^{n}(Z_{ni}-EZ_{ni})\right|^{r}$$

$$\begin{aligned}
&\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2}\left\{\sum_{i=1}^{n} E|Y_{ni}|^{\mu}+\left(\sum_{i=1}^{n} EY_{ni}^2\right)^{\mu/2}\right\}\\
&\quad+\sum_{n=1}^{\infty} n^{\gamma p-\gamma r-2}\left\{\sum_{i=1}^{n} E|Z_{ni}|^{r}+\left(\sum_{i=1}^{n} EZ_{ni}^2\right)^{r/2}\right\}\\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2}\sum_{i=1}^{n} E|a_{ni}X|^{\vartheta} I(|a_{ni}X|\leqslant n^{\gamma})+\sum_{n=1}^{\infty} n^{\gamma p-\gamma\mu-2+\mu/2}(EX^2)^{\mu/2}\\
&\quad+\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r}\sum_{i=1}^{n} E|a_{ni}X|^{r} I(|a_{ni}X|>n^{\gamma})+\sum_{n=1}^{\infty} n^{\gamma p-\gamma r-2}\left(\sum_{i=1}^{n} EZ_{ni}^2\right)^{r/2}\\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-\gamma r-2}\left(\sum_{i=1}^{n} EZ_{ni}^2\right)^{r/2}.
\end{aligned}$$

Therefore, we only need to deal with $\sum_{n=1}^{\infty} n^{\gamma p-\gamma r-2}\left(\sum_{i=1}^{n} EZ_{ni}^2\right)^{r/2}$ when $r>2$. Actually, if $p\geqslant 2$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\gamma p-\gamma r-2}\left(\sum_{i=1}^{n} EZ_{ni}^2\right)^{r/2}\\
&\leqslant \sum_{n=1}^{\infty} n^{\gamma p-\gamma rp/2-2}\left(\sum_{i=1}^{n} E|a_{ni}X_{ni}|^{p} I(|a_{ni}X_{ni}|>n^{\gamma})\right)^{r/2}\\
&\leqslant \sum_{n=1}^{\infty} n^{\gamma p-\gamma rp/2-2}\left(\sum_{i=1}^{n} |a_{ni}|^{p} E|X|^{p}\right)^{r/2}\\
&\leqslant \sum_{n=1}^{\infty} n^{-1+(\gamma p-1)(1-r/2)}(E|X|^{p})^{r/2}<\infty
\end{aligned}$$

since $-1+(\gamma p-1)(1-r/2)<-1$ and $E|X|^{p}<\infty$. If $0<p<2$, we also have by $\gamma p-2-(\gamma r-1)r/2<-1+\gamma p-\gamma r<-1$ that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\gamma p-\gamma r-2}\left(\sum_{i=1}^{n} EZ_{ni}^2\right)^{r/2}\\
&\leqslant \sum_{n=1}^{\infty} n^{\gamma r-\gamma r^2/2-2}\left(\sum_{i=1}^{n} E|a_{ni}X_{ni}|^{r} I(|a_{ni}X_{ni}|>n^{\alpha})\right)^{r/2}\\
&\leqslant \sum_{n=1}^{\infty} n^{\gamma p-\gamma r^2/2-2}\left(\sum_{i=1}^{n} |a_{ni}|^{r} E|X|^{r}\right)^{r/2}\\
&\leqslant \sum_{n=1}^{\infty} n^{\gamma p-2-(\gamma r-1)r/2}(E|X|^{r})^{r/2}<\infty.
\end{aligned}$$

Combining the aforementioned three cases, we can complete the proof of (2). This completes the proof of the theorem. □

REMARK 3.1. Comparing Theorem 3.1 with Theorem A, we not only extend their result from ρ^*-mixing random variables to (α,β)-mixing random variables, but also improve the single indexed variables with identical distribution to double indexed variables with stochastic domination. If we consider the maximum weighted sums, the moment conditions would be slightly stronger since the moment inequalities for (α,β)-mixing random variables are inferior to that of ρ^*-mixing random variables. In addition, the proof is simpler than that of Theorem A.

By Theorem 3.1, we can obtain the following result on complete convergence.

THEOREM 3.2. *Let $\gamma > 1/2$ and $\gamma p \geqslant 1$. Let $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of rowwise (α,β)-mixing random variables stochastically dominated by a random variable X with $\sum_{n=1}^{\infty}(\lambda(n))^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}} < \infty$, where $0 < \alpha,\beta < 1$ and $\alpha+\beta = 1$. Assume further that $EX_{ni} = 0$ if $p \geqslant 1$. Let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying $\sum_{i=1}^{n}|a_{ni}|^q \ll n$ for some $q > p$. Then $E|X|^p < \infty$ implies that*

$$\sum_{n=1}^{\infty} n^{\gamma p-2} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon n^{\gamma}\right) < \infty. \tag{4}$$

Proof. We only need to show that (2) implies (4) with $r < p$. Actually, it can be easily obtained that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\gamma}\right)_+^r \\
&= \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} \int_0^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\gamma} > t^{1/r}\right) dt \\
&\geqslant \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma r} \int_0^{\varepsilon^r n^{\gamma r}} P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\gamma} > t^{1/r}\right) dt \\
&\geqslant \varepsilon^r \sum_{n=1}^{\infty} n^{\gamma p-2} P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| > 2\varepsilon n^{\gamma}\right),
\end{aligned}$$

which yields (4) immediately by the arbitrariness of $\varepsilon > 0$. □

4. An application to the nonparametric regression model

4.1. Complete consistency for the P-C estimator

Consider the following nonparametric regression model:

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni}, \ \ i = 1,2,\cdots,n, \ n \geqslant 1, \tag{1}$$

where g is an unknown function defined on the interval $[0,1]$, and $\varepsilon_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1$ are random errors with zero mean. Without loss of generality, we assume that $0 = x_0 \leqslant x_{n1} \leqslant \cdots \leqslant x_{n,n-1} \leqslant x_{nn} = 1$. Then, Priestley-Chao (P-C, for short) estimator of $g(x)$ is given by

$$g_n(x) = \sum_{i=1}^{n} Y_{ni} \frac{x_{ni} - x_{n,i-1}}{h_n} K\left(\frac{x - x_{ni}}{h_n}\right), \tag{2}$$

where $K(u)$ is a measurable function and $0 < h_n \to 0$ as $n \to \infty$.

Priestley and Chao (1972) first proposed the estimator (2) and established the weak consistency for the estimator based on independent and identically distributed (i.i.d., for short) samples; Benedetti (1977) further studied the strong consistency and asymptotic normality for the estimator based on i.i.d. samples; Yang and Wang (1999) improved and generalized the strong consistency from i.i.d. samples to NA samples without identical distribution. Wu et al. (2020) obtained the rates of strong consistency, complete consistency, and the mean consistency for the estimator based on END random samples.

Let $\delta_n = \max_{1 \leqslant i \leqslant n}(x_{ni} - x_{n,i-1})$, the following assumptions are indispensable.

(A_1) $g(x)$ satisfies the Lipschitz condition of order $\alpha (>0)$ on $[0,1]$;

(A_2) (i) $K(\cdot)$ satisfies the Lipschitz condition of order $\beta (>0)$; (ii) $K(\cdot)$ is bounded in $\mathbb{R}^1$; (iii) $\int_{-\infty}^{+\infty} K(u)du = 1$; (iv) $\int_{-\infty}^{+\infty} |K(u)|du < \infty$;

(A_3) $h_n \to 0$, $\delta_n \to 0$ and $h_n^{-1}\{(\delta_n/h_n)^{\beta} + \delta_n^{\alpha}\} \to 0$ as $n \to \infty$.

REMARK 4.1. $(A_1) - (A_3)$ are basic assumptions and have been adopted in Yang and Wang (1999), Wu et al. (2020) and so on.

THEOREM 4.1. *In model (1), assume that $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of rowwise (α,β)-mixing random errors stochastically dominated by a random variable ε with $E\varepsilon_{ni} = 0$, $E|\varepsilon|^p < \infty$ for some $2 < p < 4$ and $\sum_{n=1}^{\infty} (\lambda(n))^{\frac{1}{2\alpha} \wedge \frac{1}{2\beta}} < \infty$, where $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$. Suppose that conditions $(A_1) - (A_3)$ hold, and*

$$\sum_{i=1}^{n} \left(\frac{x_{ni} - x_{n,i-1}}{h_n}\right)^q \ll n^{1-2q/p} \text{ for some } q > p. \tag{3}$$

Then for any $x \in (0,1)$,

$$g_n(x) \to g(x) \text{ completely.}$$

Proof. Note that for any $x \in (0,1)$,

$$|g_n(x) - g(x)| \leqslant |g_n(x) - Eg_n(x)| + |Eg_n(x) - g(x)|. \tag{4}$$

Similar to the proof of Lemma 3 in Yang and Wang (1999), we obtain by $(A_1) - (A_3)$ that

$$|Eg_n(x) - g(x)| \to 0 \text{ as } n \to \infty,$$

which together with (4) derives that, to finish the proof, we only need to show

$$g_n(x)-Eg_n(x)=\sum_{i=1}^{n}\frac{x_{ni}-x_{n,i-1}}{h_n}K\left(\frac{x-x_{ni}}{h_n}\right)\varepsilon_{ni}\to 0 \text{ completely},$$

or equivalently, for all $\varepsilon>0$,

$$\sum_{n=1}^{\infty}P\left(\left|\sum_{i=1}^{n}\frac{x_{ni}-x_{n,i-1}}{h_n}K\left(\frac{x-x_{ni}}{h_n}\right)\varepsilon_{ni}\right|>\varepsilon\right)<\infty. \tag{5}$$

Apply Theorem 3.2 with $\gamma=2/p$, and $X_{ni}=\varepsilon_{ni}$, $a_{ni}=n^{2/p}\frac{x_{ni}-x_{n,i-1}}{h_n}K\left(\frac{x-x_{ni}}{h_n}\right)$. In what follows, we will verify the condition $\sum_{i=1}^{n}|a_{ni}|^q\ll n$ for some $q>p$ in Theorem 3.2. Actually, it is easy to obtain by the boundedness of $K(\cdot)$ and (3) that

$$\begin{aligned}\sum_{i=1}^{n}|a_{ni}|^q &= n^{2q/p}\sum_{i=1}^{n}\left(\frac{x_{ni}-x_{n,i-1}}{h_n}\right)^q\left|K\left(\frac{x-x_{ni}}{h_n}\right)\right|^q\\ &\ll n^{2q/p}\sum_{i=1}^{n}\left(\frac{x_{ni}-x_{n,i-1}}{h_n}\right)^q\ll n.\end{aligned}$$

Hence, the desired result (4) follows from Theorem 3.2 immediately. The proof is completed. □

COROLLARY 4.1. *In model (1), assume that $\{\varepsilon_{ni},1\leqslant i\leqslant n,n\geqslant 1\}$ is an array of rowwise (α,β)-mixing random errors stochastically dominated by a random variable ε with $E\varepsilon_{ni}=0$, $E|\varepsilon|^p<\infty$ for some $2<p<4$ and $\sum_{n=1}^{\infty}(\lambda(n))^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}<\infty$, where $0<\alpha,\beta<1$ and $\alpha+\beta=1$. Suppose that conditions $(A_1)-(A_3)$ hold, and $\delta_n/h_n\ll n^{-2/p}$, then for any $x\in(0,1)$,*

$$g_n(x)\to g(x) \textit{ completely}.$$

Proof. In view of Theorem 4.1, we only need to check that (3) holds. In fact, it can be easily obtained that

$$\sum_{i=1}^{n}\left(\frac{x_{ni}-x_{n,i-1}}{h_n}\right)^q\leqslant\sum_{i=1}^{n}\left(\frac{\delta_n}{h_n}\right)^q\ll n^{1-2q/p}.$$

Therefore, Corollary 4.1 follows directly from Theorem 4.1. □

4.2. Numerical analysis

In this section, we will carry out a simulation to study the numerical performance of the complete consistency for the P-C estimator $g_n(x)$ based on (α,β)-mixing samples.

For fixed positive integer m, let $e_i \overset{i.i.d.}{\sim} N(0,\sigma^2)$, where $\sigma^2 = 1/(m+1)$. Let $\varepsilon_{ni} = \sum_{j=0}^{m} e_{n,i+j}$ for each $1 \leqslant i \leqslant n$, then it is easy to see that $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of rowwse (α,β)-mixing random variables with $\varepsilon_{ni} \sim N(0,1)$. Let $m=5$, $h_n = n^{-1/3}$, and $x_{ni} = i/n$ for all $1 \leqslant i \leqslant n$. Taking the sample size n as $n = 50, 100, 200, 400$, respectively, we use R software to compute $g_n(x)$ for 1000 times to obtain the plots with $g(x) = \sin 2\pi x$, $g(x) = e^{-2x}$, and $g(x) = x - \cos^2 x$. The results are presented in Figures 1-3. We also calculate the mean bias and the MSE of $g_n(x)$ under different sample sizes and functions. These results are shown in Table 1.

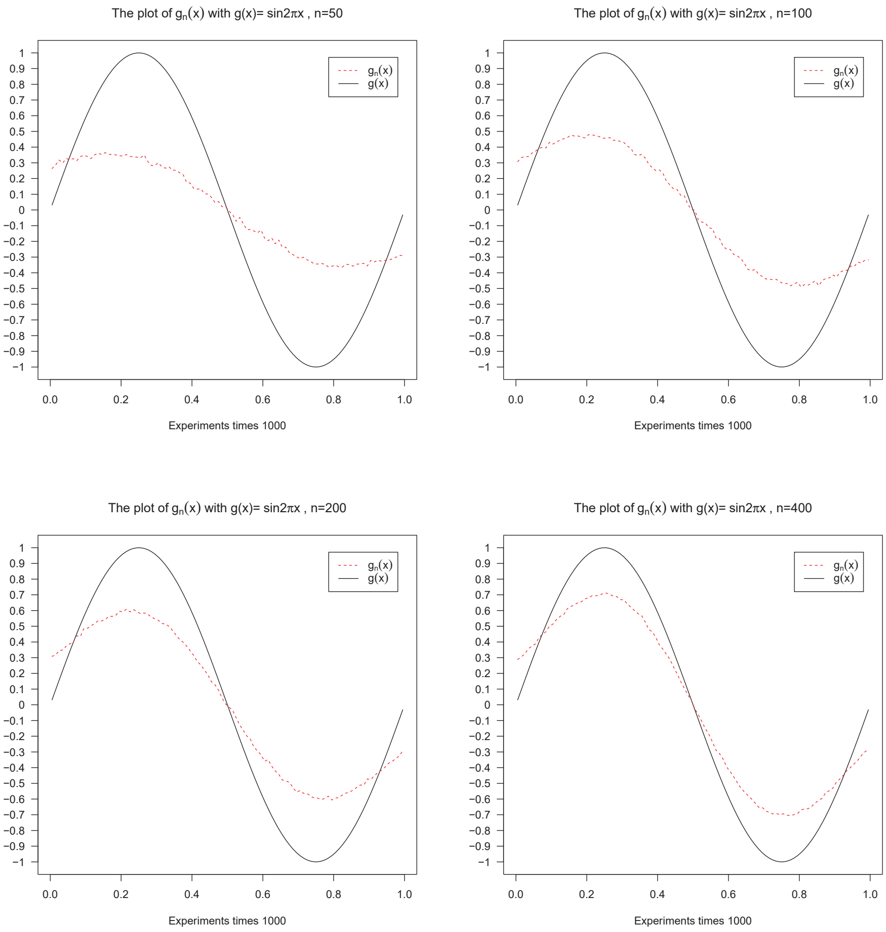

Figure 1: The plots of $g_n(x)$ with $g(x) = \sin 2\pi x$.

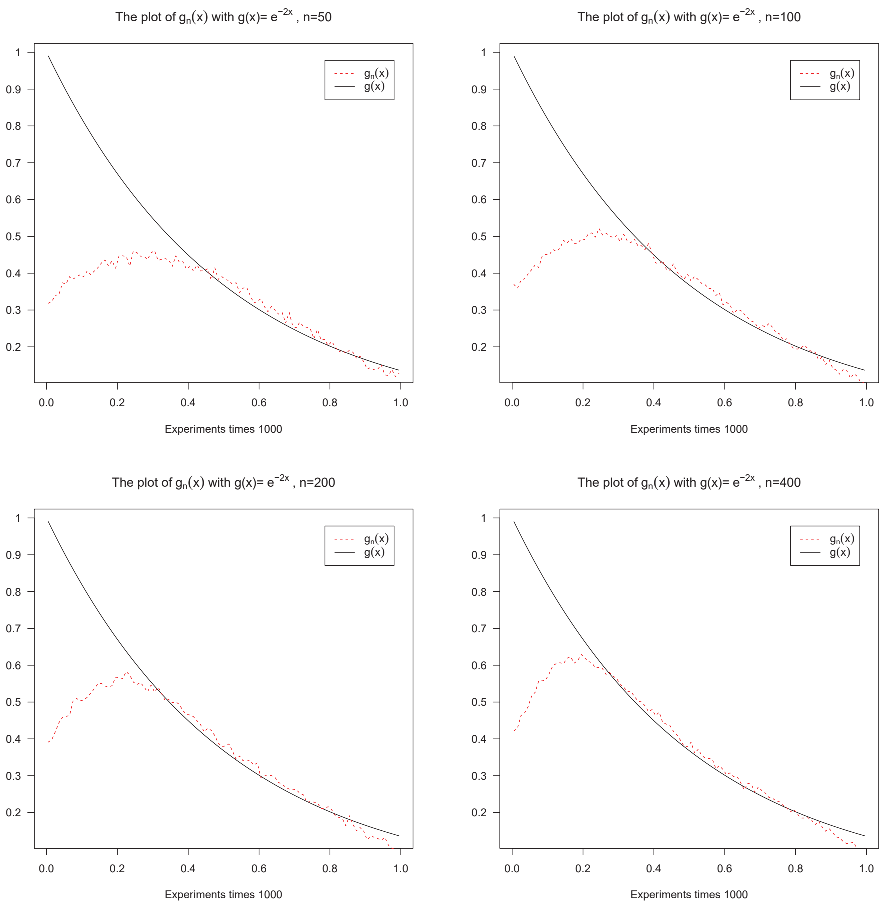

Figure 2: The plots of $g_n(x)$ with $g(x) = e^{-2x}$.

Table 1: The Bias and the MSE of the estimator

$g(x)$	Differences	n=50	n=100	n=200	n=400
$\sin 2x$	Bias	0.0012	-0.0002	0.0007	-0.00005
	MSE	0.279	0.1817	0.1093	0.0613
e^{-2x}	Bias	-0.1069	-0.0859	-0.00685	-0.054
	MSE	0.1152	0.0819	0.0573	0.0398
$x - \cos^2 x$	Bias	0.0548	0.0379	0.0269	0.001903851
	MSE	0.1663	0.1129	0.0768	0.007267938

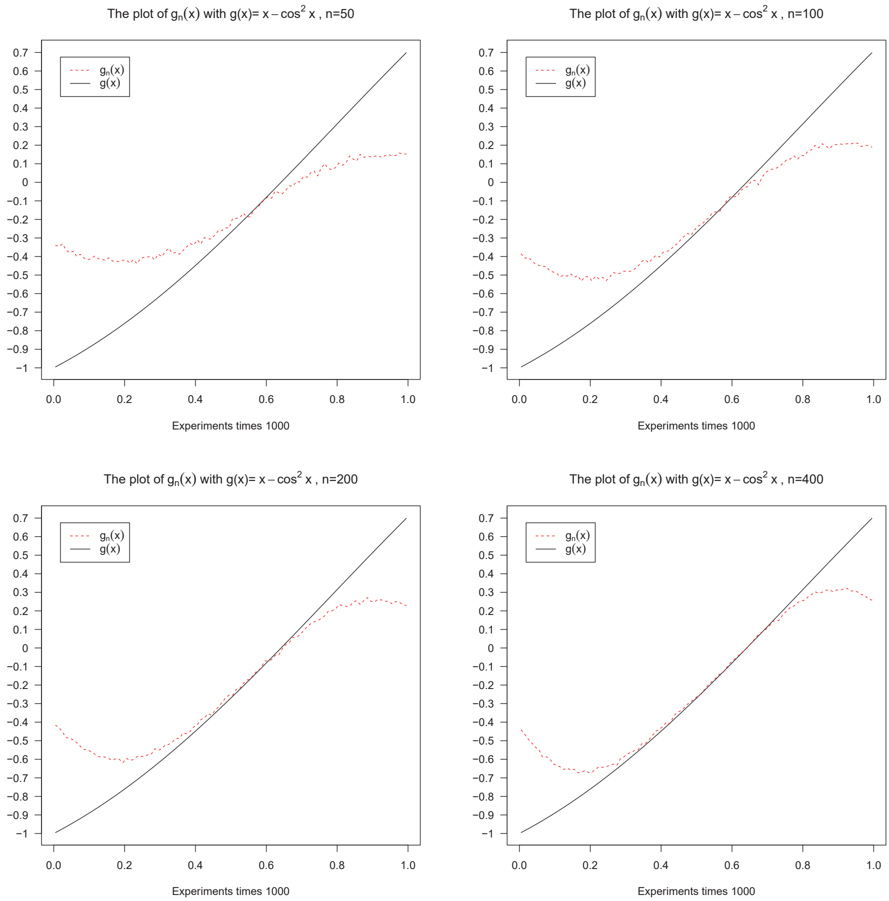

Figure 3: The plots of $g_n(x)$ with $g(x) = x - \cos^2 x$.

Figures 1-3 are the plots of $g_n(x)$ with $g(x) = \sin 2\pi x$, $g(x) = e^{-2x}$, and $g(x) = x - \cos^2 x$, respectively. We can see that under the three functions, the estimator $g_n(x)$ fits better and better to the true function as the sample size n increases. Specifically, one can see in Table 1 that the mean bias fluctuates to zero and the MSE of $g_n(x)$ decreases markedly as n increases. We are convinced that the estimator will fit better to the true function if n becomes larger. These results mainly agree with the theoretical results.

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(Received April 29, 2020)

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