Journal of Mathematical Inequalities
Volume 14, Number 3 (2020), 939–950 doi:10.7153/jmi-2020-14-61

SUBORDINATION FOR MEROMORPHIC HARMONIC FUNCTIONS

J. DZIOK

(*Communicated by N. Elezović*)

Abstract. We introduce and study classes of meromorphic harmonic functions defined by subordination. In addition to finding certain analytic criteria, we obtain some topological properties for the defined classes of functions. Some applications of these results are also given.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D. If D is the exterior of the unit disc i.e. $\mathbb{D} := \{z \in \mathbb{C} : |z| > 1\}$, then we say that f is meromorphic harmonic function. Hengartner and Schober [4] showed that meromorphic harmonic, orientation preserving, univalent mapping f, satisfying $f(\infty) = \infty$, must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A\log|z| \tag{1}$$

where

$$h(z) = az + \sum_{n=1}^{\infty} a_n z^{-n} \quad g(z) = bz + \sum_{n=1}^{\infty} b_n z^{-n} \quad (z \in \mathbb{D}),$$

$0 \leqslant |a_0| < |b_0|$, $A \in \mathbb{C}$, and $\overline{f_{\overline{z}}}/f_z$ is analytic and bounded by 1 in $\mathbb{D}$. Let $\Sigma(k)$ denote the class of meromorphic harmonic functions f of the form

$$f(z) = z + \sum_{n=k}^{\infty} \left(a_n z^{-n} + \overline{b_n}\overline{z}^{-n}\right) \quad (z \in \mathbb{D}). \tag{2}$$

and let $\Sigma_{\mathscr{H}}(k)$ denote the class of functions $f \in \Sigma(k)$ which are univalent and orientation preserving in $\mathbb{D}$.

Jahangiri and Silverman [6] investigated the class of meromorphic harmonic starlike functions. A function $f \in \Sigma_{\mathscr{H}}(2)$ is meromorphic harmonic starlike in $\mathbb{D}(r) := \{z \in \mathbb{C} : |z| > r\}$, $r > 1$, if

$$\frac{\partial}{\partial t}\left(\arg f\left(re^{it}\right)\right) \geqslant 0 \ \ (0 \leqslant t \leqslant 2\pi)$$

Mathematics subject classification (2010): 30C45, 30C80.
Keywords and phrases: Meromorphic harmonic functions, subordination, extreme points, starlike functions, convex functions.

or

$$\operatorname{Re}\frac{D_{\mathscr{H}}f(z)}{f(z)} \geqslant 0 \quad (|z| = r),$$

where

$$D_{\mathscr{H}}f(z) := zh'(z) - \overline{zg'(z)} \quad (z \in \mathbb{D}).$$

To obtain a generalization of these functions we use a concept of weak subordination. We say that a function $f \in \Sigma(k)$ is *weakly subordinate* to a function $F \in \Sigma(k)$, and write $f(z) \prec F(z)$ (or simply $f \prec F$) if there exists a complex-valued function ω which maps $\mathbb{D}$ into oneself with $\omega(\infty) = \infty$, such that $f(z) = F(\omega(z)) \quad (z \in \mathbb{D})$. In particular, if F is univalent in $\mathbb{D}$, we have the following equivalence

$$f(z) \prec F(z) \Longleftrightarrow \{f(0) = F(0) \wedge f(\mathbb{D}) \subset F(\mathbb{D})\}.$$

Let $-B \leqslant A < B \leqslant 1,\ 0 \leqslant \alpha < 1$. Motivated by Janowski [7] (see also [3]) we define the following classes of functions.

Let $\Sigma^{*}_{\mathscr{H}}(k;A,B)$ denote the class of functions $f \in \Sigma_{\mathscr{H}}(k)$ such that

$$\frac{D_{\mathscr{H}}f(z)}{f(z)} \prec \frac{A+z}{B+z} \tag{3}$$

and by $\Sigma^{c}_{\mathscr{H}}(k;A,B)$ we denote the class of functions $f \in \Sigma_{\mathscr{H}}(k)$ such that

$$\frac{D_{\mathscr{H}}(D_{\mathscr{H}}f)(z)}{D_{\mathscr{H}}f(z)} \prec \frac{A+z}{B+z}.$$

Finally, let $\Sigma_{\mathscr{H}}(k;A,B)$ denote the class of functions $f \in \Sigma_{\mathscr{H}}(k)$ such that

$$\frac{D_{\mathscr{H}}f(z)}{z} \prec \frac{A+z}{B+z}.$$

We should notice, that the classes $\Sigma^{*}_{\mathscr{H}}(k;\alpha) := \Sigma^{*}_{\mathscr{H}}(k;2\alpha-1,1)$ and $\Sigma^{c}_{\mathscr{H}}(k;\alpha) := \Sigma^{c}_{\mathscr{H}}(k;2\alpha-1,1)$ are investigated by Jahangiri [5] in the case $k = 2$. The classes $\Sigma^{*}_{\mathscr{H}}(k) := \Sigma^{*}_{\mathscr{H}}(k;0)$ and $\Sigma^{c}_{\mathscr{H}}(k) := \Sigma^{c}_{\mathscr{H}}(k;0)$ are the classes of functions $f \in \Sigma_{\mathscr{H}}(k)$ which are starlike in $\mathbb{U}(r)$ or convex in $\mathbb{U}(r)$, respectively, for all $r > 1$ (see [6]). It is clear that

$$\Sigma^{*}_{\mathscr{H}}(k;A,B) \subset \Sigma^{*}_{\mathscr{H}}(k),\ \mathscr{S}^{c}_{\mathscr{H}}(k;A,B) \subset \Sigma^{c}_{\mathscr{H}}(k).$$

In the paper we obtain some necessary and sufficient conditions for the defined classes of functions. Some topological properties and extreme points of the classes are also considered. By using extreme points theory we obtain coefficients estimates, distortion theorems, integral mean inequalities for the classes of functions. Some applications of these results are also given.

2. Analytic criteria

Let $\mathscr{V} \subset \Sigma$. Motivated by Ruscheweyh we define the dual set of $\mathscr{V}$ by

$$\mathscr{V}^* := \left\{ f \in \Sigma_{\mathscr{H}}(k) : \bigwedge_{q \in \mathscr{V}} (f * q)(z) \neq 0 \quad (z \in \mathbb{D}) \right\}.$$

THEOREM 1.

$$\Sigma^*_{\mathscr{H}}(k;A,B) = \left\{ \psi_\xi : |\xi| = 1 \right\}^*,$$

where

$$\psi_\xi(z) := (B-A)z - \frac{(B+A+2\xi)z + (\lambda B + A) - \xi}{(1-z)^2} - (2\xi + B + A)\bar{z} + \frac{(B-A)\bar{z} - (\lambda B - A) + \xi}{(1-\bar{z})^2} \quad (z \in \mathbb{D}). \tag{4}$$

Proof. Let $f \in \Sigma_{\mathscr{H}}(k)$ be of the form (2). Then $f \in \Sigma^*_{\mathscr{H}}(k;A,B)$ if and only if it satisfies (3) or equivalently

$$\frac{D_{\mathscr{H}}(z)}{f(z)} \neq \frac{A+\xi}{B+\xi} \quad (z \in \mathbb{D},\ |\xi| = 1). \tag{5}$$

Since

$$h(z) = h(z) * \left(z + \frac{1}{z-1} \right),\ D_{\mathscr{H}} h(z) = h(z) * \left(z - \frac{z}{(z-1)^2} \right),$$

the above inequality yields

$$\begin{aligned}
&(B+\xi) D_{\mathscr{H}} f(z) - (A+\xi) f(z) \\
&= (B+\xi) D_{\mathscr{H}} h(z) - (A+\xi) h(z) - \left[(B+\xi) \overline{D_{\mathscr{H}} g(z)} + (A+\xi) \overline{h(z)} \right] \\
&= h(z) * \left((B+\xi) \left(z - \frac{z}{(z-1)^2} \right) - (A+\xi) \left(z + \frac{1}{z-1} \right) \right) \\
&\quad - \overline{g(z)} * \left((B+\xi) \left(\bar{z} - \frac{\bar{z}}{(\bar{z}-1)^2} \right) + (A+\xi) \left(\bar{z} + \frac{1}{\bar{z}-1} \right) \right) \\
&= f(z) * \psi_\xi(z) \neq 0 \quad (z \in \mathbb{D},\ |\xi| = 1).
\end{aligned}$$

Thus, $f \in \Sigma^*_{\mathscr{H}}(k;A,B)$ if and only if $f(z) * \psi_\xi(z) \neq 0$ for $z \in \mathbb{D},\ |\xi| = 1$ *i.e.* $\Sigma^*_{\mathscr{H}}(k;A,B) = \left\{ \psi_\xi : |\xi| = 1 \right\}^*$.

Similarly we prove

THEOREM 2.

$$\Sigma^c_{\mathscr{H}}(k;A,B) = \left\{ \psi_\xi : |\xi| = 1 \right\}^*,$$

where

$$\psi_{\xi}(z) := (B-A)z + z\frac{(B+A+2\xi)z - 2\xi + (\lambda B + A)}{(1-z)^3} + (2\xi + B + A)\overline{z} + \overline{z}\frac{(B-A)\overline{z} - (\lambda B - A)}{(1-\overline{z})^3} \quad (z \in \mathbb{D}).$$

THEOREM 3.

$$\Sigma_{\mathscr{H}}(k;A,B) = \left\{\delta_{\xi} : |\xi| = 1\right\}^{*},$$

where

$$\delta_{\xi}(z) := (B-A)z - \frac{B+\xi}{z-1} + (B+\xi)\overline{z} - \frac{B+\xi}{\overline{z}-1} \quad (z \in \mathbb{D}).$$

LEMMA 1. [5] *If a function* $f \in \Sigma(k)$ *of the form* (2) *satisfies the condition*

$$\sum_{n=k}^{\infty} (n|a_n| + n|b_n|) \leqslant 1. \tag{6}$$

then f is orientation preserving and univalent in $\mathbb{D}$.

THEOREM 4. *If a function* $f \in \Sigma(k)$ *of the form* (2) *satisfies the condition*

$$\sum_{n=k}^{\infty} \{(n(1+B)+(1+A))|a_n| + (n(1+B)-(1+A))|b_n|\} \leqslant B-A, \tag{7}$$

then $f \in \Sigma^{*}_{\mathscr{H}}(k;A,B)$.

Proof. Since

$$\frac{n(1+B)+(1+A)}{B-A} \geqslant n, \ \frac{n(1+B)-(1+A)}{B-A} \geqslant n \quad (n \in \mathbb{N}_k), \tag{8}$$

by (7) we get (6). Thus, by Lemma 1 the function f is univalent and orientation preserving in $\mathbb{D}$. Therefore, $f \in \Sigma^{*}_{\mathscr{H}}(k;A,B)$ if and only if there exists a complex-valued function ω, $\omega(\infty) = \infty$, $|\omega(z)| > 1$ $(z \in \mathbb{D})$ such that

$$\frac{D_{\mathscr{H}} f(z)}{f(z)} = \frac{A+\omega(z)}{B+\omega(z)} \quad (z \in \mathbb{D}),$$

or equivalently

$$\left|\frac{D_{\mathscr{H}} f(z) - f(z)}{BD_{\mathscr{H}} f(z) - Af(z)(z)}\right| < 1 \quad (z \in \mathbb{D}). \tag{9}$$

Thus for $z \in \mathbb{D}$ it suffices to show that

$$|D_{\mathscr{H}} f(z) - f(z)| - |BD_{\mathscr{H}} f(z) - f(z)| < 0.$$

Indeed, letting $|z| = r\ (r > 1)$ we have

$$
\begin{aligned}
&|D_{\mathscr{H}} f(z) - f(z)| - |BD_{\mathscr{H}} f(z) - f(z)| \\
=&\left| \sum_{n=k}^{\infty} (n+1) a_n z^{-n} - \sum_{n=k}^{\infty} (n-1) \overline{b_n} \overline{z}^{-n} \right| \\
&- \left| (B-A) z + \sum_{n=k}^{\infty} (Bn+A) a_n z^{-n} + \sum_{n=k}^{\infty} (Bn-A) \overline{b_n} \overline{z}^{-n} \right| \\
\leqslant& \sum_{n=k}^{\infty} (n+1) |a_n| r^{-n} + \sum_{n=k}^{\infty} (n-1) |b_n| r^{-n} - (B-A) r \\
&+ \sum_{n=k}^{\infty} (Bn+A) |a_n| r^{-n} + \sum_{n=k}^{\infty} (Bn-A) |b_n| r^{-n} \\
\leqslant& \frac{1}{r} \sum_{n=k}^{\infty} \{ (n(1+B) + (1+A)) |a_n| + (n(1+B) - (1+A)) |b_n| \} r^{-n+1} - (B-A) < 0,
\end{aligned}
$$

whence $f \in \Sigma^*_{\mathscr{H}}(k; A, B)$.

THEOREM 5. *If a function $f \in \Sigma(k)$ of the form (2) satisfies the condition*

$$
\sum_{n=k}^{\infty} n \{ (n(1+B) + (1+A)) |a_n| + (n(1+B) - (1+A)) |b_n| \} \leqslant B - A, \tag{10}
$$

then $f \in \Sigma^c_{\mathscr{H}}(k; A, B)$.

Let $\mathscr{T}_\eta(k)$ be the class of functions $f = h + \overline{g} \in \Sigma(k)$ so that

$$
f = h + \overline{g} = z + \sum_{n=k}^{\infty} e^{i(1+n)\eta} |a_n| z^{-n} - \sum_{n=k}^{\infty} e^{i(1-n)\eta} |b_n| \overline{z}^{-n} \quad (z \in \mathbb{D}). \tag{11}
$$

and let

$$
\begin{aligned}
&\Sigma^*_\eta(k; A, B) : = \mathscr{T}_\eta(k) \cap \Sigma^*_{\mathscr{H}}(k; A, B), \ \Sigma^c_\eta(k; A, B) := \mathscr{T}_\eta(k) \cap \Sigma^c_{\mathscr{H}}(k; A, B), \\
&\Sigma_\eta(k; A, B) : = \mathscr{T}_\eta(k) \cap \Sigma_{\mathscr{H}}(k; A, B).
\end{aligned}
$$

Next we show that the condition (7) is also necessary for the functions $f \in \Sigma(k)$ to be in the class $\Sigma^*_\eta(k; A, B)$

THEOREM 6. *Let $f \in \mathscr{T}_\eta(k)$ be a function of the form (2). Then $f \in \Sigma^*_\eta(k; A, B)$ if and only if the condition (7) holds true.*

Proof. The "*if*" part follows from Theorem 4. For the "*only − if*" part, assume that $f \in \Sigma^*_\eta(k; A, B)$, then by (9) we have

$$
\left| \frac{\sum_{n=k}^{\infty} \left\{ (n+1) a_n z^{-n} - (n-1) \overline{b_n} \overline{z}^{-n} \right\}}{(B-A) z - \sum_{n=k}^{\infty} \left\{ (Bn+A) a_n z^{-n} - (Bn-A) \overline{b_n} \overline{z}^{-n} \right\}} \right| < 1 \quad (z \in \mathbb{D}).
$$

Therefore, by (11) for $z = re^{i\eta}$ $(r > 1)$, we obtain

$$\frac{\sum_{n=k}^{\infty} \{(n+1)|a_n| + (n-1)|b_n|\} r^{-n-1}}{(B-A) - \sum_{n=k}^{\infty} \{(Bn+A)|a_n| + (Bn-A)|b_n|\} r^{-n-1}} < 1. \tag{12}$$

It is clear that the denominator of the left hand side cannot vanish for $r > 1$. Moreover, it is positive for $r = \infty$, and in consequence for $r > 1$. Thus, by (12) we have

$$\sum_{n=k}^{\infty} \{(n(1+B) + (1+A))|a_n| + (n(1+B) - (1+A))|b_n|\} r^{-n-1} < B - A. \tag{13}$$

The sequence of partial sums $\{S_n\}$ associated with the series of the left hand side of (13) is non-decreasing sequence. Moreover, by (13) it is bounded by $B - A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=k}^{\infty} \{(n(1+B) + (1+A))|a_n| + (n(1+B) - (1+A))|b_n|\} = \lim_{n\to\infty} S_n \leqslant B - A,$$

which yields the assertion (7).

The following result may be proved in much the same way as Theorem 6.

THEOREM 7. *Let $f \in \Sigma(k)$ be a function of the form* (11)*. Then $f \in \Sigma_\eta(k;A,B)$ if and only if*

$$\sum_{n=k}^{\infty} n(|a_n| + |b_n|) \leqslant \frac{B-A}{1+B}.$$

By Theorems 6 and 7 we have the following corollary.

COROLLARY 1. *Let $a = \frac{1+A}{1+B}$ and*

$$\phi(z) = z + \sum_{n=k}^{\infty} \left(\frac{1}{n-a} z^n + \frac{1}{n+a} \overline{z}^n \right) \quad (z \in \mathbb{D}), \tag{14}$$

$$\omega(z) = z + \sum_{n=k}^{\infty} ((n-a)z^n + (n+a)\overline{z}^n) \quad (z \in \mathbb{D}).$$

Then

$$f \in \Sigma_\eta(k;A,B) \Leftrightarrow f * \phi \in \Sigma_\eta^*(k;A,B),$$
$$f \in \Sigma_\eta^*(k;A,B) \Leftrightarrow f * \omega \in \Sigma_\eta(k;A,B).$$

In particular,

$$\Sigma_\eta(k;-1,B) = \Sigma_\eta^*(k;-1,B).$$

3. Extreme points

The Krein-Milman theorem (see [8]) is fundamental in the theory of extreme points. In particular, it implies the following lemma.

LEMMA 2. *Let $\mathscr{F}$ be a non-empty compact convex subclass of the class Σ and $\mathscr{J}:\Sigma\to\mathbb{R}$ be a real-valued, continuous and convex functional on $\mathscr{F}$. Then*

$$\max\{\mathscr{J}(f): f\in\mathscr{F}\}=\max\{\mathscr{J}(f): f\in E\mathscr{F}\},$$

where $E\mathscr{F}$ to denote the set of extreme points of $\mathscr{F}$.

Since Σ is a complete metric space, Montel's theorem implies the following lemma.

LEMMA 3. *A class $\mathscr{F}\subset\Sigma$ is compact if and only if $\mathscr{F}$ is closed and locally uniformly bounded.*

THEOREM 8. *The class $\Sigma_\eta^*(k;A,B)$ is convex and compact subset of Σ.*

Proof. Let $f_1,f_2\in\Sigma_\eta^*(k;A,B)$ be functions of the form

$$f_l(z)=z+\sum_{n=k}^{\infty}\left(a_{l,n}z^{-n}+\overline{b_{l,n}}\overline{z}^{-n}\right)\quad(z\in\mathbb{D},l\in\mathbb{N}),\tag{15}$$

$0\leqslant\gamma\leqslant 1$. Since

$$\gamma f_1(z)+(1-\gamma)f_2(z)=z+\sum_{n=k}^{\infty}\left\{(\gamma a_{1,n}+(1-\gamma)a_{2,n})z^{-n}+\overline{(\gamma b_{1,n}+(1-\gamma)b_{2,n})z^{-n}}\right\},$$

and by Theorem 6 we have

$$\begin{aligned}
&\sum_{n=k}^{\infty}(n(1+B)+(1+A))|\gamma a_{1,n}+(1-\gamma)a_{2,n}|\\
&\quad+\sum_{n=k}^{\infty}(n(1+B)-(1+A))|\gamma b_{1,n}+(1-\gamma)b_{2,n}|\\
&\leqslant\gamma\sum_{n=k}^{\infty}\{(n(1+B)+(1+A))|a_{1,n}|+(n(1+B)-(1+A))|b_{1,n}|\}\\
&\quad+(1-\gamma)\sum_{n=k}^{\infty}\{(n(1+B)+(1+A))|a_{2,n}|+(n(1+B)-(1+A))|b_{2,n}|\}\\
&\leqslant\gamma(B-A)+(1-\gamma)(B-A)=B-A,
\end{aligned}$$

the function $\phi=\gamma f_1+(1-\gamma)f_2$ belongs to the class $\Sigma_\eta^*(k;A,B)$. Hence, the class is convex. Furthermore, for $f\in\Sigma_\eta^*(k;A,B)$, $|z|\geqslant r$, $r>1$, we have

$$\begin{aligned}
|f(z)|\leqslant& r+\sum_{n=k}^{\infty}(|a_n|+|b_n|)r^{-n}\leqslant r+\sum_{n=k}^{\infty}(n(1+B)+(1+A))|a_n|\\
&+\sum_{n=k}^{\infty}(n(1+B)-(1+A))|b_n|\leqslant r+(B-A).
\end{aligned}\tag{16}$$

Thus, we conclude that the class $\Sigma_\eta^*(k;A,B)$ is locally uniformly bounded. By Lemma 3, we only need to show that it is closed *i.e.* if $f_l \in \Sigma_\eta^*(k;A,B)$ $(l \in \mathbb{N})$ and $f_l \to f$, then $f \in \Sigma_\eta^*(k;A,B)$. Let f_l and f be given by (15) and (11), respectively. Using Theorem 6 we have

$$\sum_{n=k}^{\infty} \left((n(1+B)+(1+A)) \left| a_{l,n} \right| + (n(1+B)-(1+A)) \left| b_{l,n} \right| \right) \leqslant B-A \quad (l \in \mathbb{N}). \tag{17}$$

Since $f_l \to f$, we conclude that $a_{l,n} \to a_n$ and $b_{l,n} \to b_n$ as $l \to \infty$ $(n \in \mathbb{N}_{\mathbf{k}})$. The sequence of partial sums $\{S_n\}$ associated with the series of the left hand side of (17) is a non-decreasing sequence. Moreover, it is bounded by $B-A$. Therefore, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=k}^{\infty} \{(n(1+B)+(1+A))|a_n| + (n(1+B)-(1+A))|b_n|\} = \lim_{n\to\infty} S_n \leqslant B-A.$$

This gives the condition (7), and, in consequence, $f \in \Sigma_\eta^*(k;A,B)$, which completes the proof.

THEOREM 9. *The set of all extreme points of the class* $\Sigma_\eta^*(k;A,B)$ *is given by*

$$E\Sigma_\eta^*(k;A,B) = \{h_n : n \in \mathbb{N}_{k-1}\} \cup \{g_n : n \in \mathbb{N}_k\},$$

where

$$h_{k-1}(z) = z, \; h_n(z) = z + e^{i(1+n)\eta} \frac{B-A}{n(1+B)+(1+A)} z^{-n}, \tag{18}$$
$$g_n(z) = z - e^{i(1-n)\eta} \frac{B-A}{n(1+B)-(1+A)} \overline{z}^{-n} \quad (z \in \mathbb{D}).$$

Proof. Suppose that $0 < \gamma < 1$ and

$$g_n = \gamma f_1 + (1-\gamma) f_2,$$

where $f_1, f_2 \in \Sigma_\eta^*(k;A,B)$ are functions of the form (15). Then, by (7) we have $|b_{1,n}| = |b_{2,n}| = \frac{B-A}{n(1+B)-(1+A)}$, and, in consequence, $a_{1,l} = a_{2,l} = 0$ for $l \in \mathbb{N}_k$ and $b_{1,l} = b_{2,l} = 0$ for $l \in \mathbb{N}_k \diagdown \{n\}$. It follows that $g_n = f_1 = f_2$, and consequently $g_n \in E\mathscr{S}_{\mathscr{T}}^*(\eta;k;A,B)$. Similarly, we verify that the functions h_n of the form (18) are the extreme points of the class $\Sigma_\eta^*(k;A,B)$. Now, suppose that a function f belongs to the set $E\Sigma_\eta^*(k;A,B)$ and f is not of the form (18). Then there exists $s \in \mathbb{N}_k$ such that

$$0 < |a_s| < \frac{B-A}{s(1+B)+(1+A)} \quad \text{or} \quad 0 < |b_s| < \frac{B-A}{s(1+B)-(1+A)}.$$

If $0 < |a_s| < \frac{B-A}{s(1+B)+(1+A)}$, then puting

$$\gamma = \frac{s(1+B)+(1+A)}{B-A} |a_s|, \; \varphi = \frac{1}{1-\gamma}(f - \gamma h_s),$$

we have that $0<\gamma<1$, $h_s \neq \varphi$ and

$$f=\gamma h_s+(1-\gamma)\varphi.$$

Thus, $f \notin E\Sigma^*_\eta(k;A,B)$. Similarly, if $0<|b_s|<\frac{B-A}{s(1+B)-(1+A)}$, then puting

$$\gamma=\frac{s(1+B)-(1+A)}{B-A}|b_s|,\ \phi=\frac{1}{1-\gamma}(f-\gamma g_s),$$

we have that $0<\gamma<1$, $g_s \neq \phi$ and

$$f=\gamma g_s+(1-\gamma)\phi.$$

It follows that $f \notin E\Sigma^*_\eta(k;A,B)$, and the proof is completed.

4. Applications

It is clear that if the class

$$\mathscr{F}=\{f_n \in \Sigma:\ n \in \mathbb{N}\},$$

is locally uniformly bounded, then

$$\overline{co}\mathscr{F}=\left\{\sum_{n=1}^{\infty}\gamma_n f_n:\ \sum_{n=1}^{\infty}\gamma_n=1,\ \gamma_n \geqslant 0\ (n \in \mathbb{N})\right\}. \tag{19}$$

Thus, by Theorem 4 we have the following corollary.

COROLLARY 2.

$$\Sigma^*_\eta(k;A,B)=\left\{\sum_{n=k-1}^{\infty}(\gamma_n h_n+\delta_n g_n):\ \sum_{n=k-1}^{\infty}(\gamma_n+\delta_n)=1\ (\delta_{k-1}=0,\gamma_n,\delta_n \geqslant 0)\right\},$$

where h_n, g_n *are defined by* (18).

For each fixed value of $n \in \mathbb{N}_k$, $z \in \mathbb{D}$, the following real-valued functionals are continuous and convex on Σ:

$$\mathscr{J}(f)=a_n,\ \mathscr{J}(f)=b_n,\ \mathscr{J}(f)=|f(z)|\ \mathscr{J}(f)=|D_{\mathscr{H}}f(z)| \quad (f \in \Sigma). \tag{20}$$

Moreover, for $\gamma \geqslant 1$, $r>1$, the real-valued functional

$$\mathscr{J}(f)=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^\gamma d\theta\right)^{1/\gamma} \quad (f \in \Sigma) \tag{21}$$

is also continuous and convex on Σ.

Therefore, by Lemma 2 and Theorem 4 we have the following corollaries.

COROLLARY 3. *Let* $f \in \Sigma_\eta^*(k;A,B)$ *be a function of the form* (2). *Then*

$$|a_n| \leqslant \frac{B-A}{n(1+B)+(1+A)},\ |b_n| \leqslant \frac{B-A}{n(1+B)-(1+A)} \quad (n \in \mathbb{N}_k), \tag{22}$$

The result is sharp. The functions h_n, g_n *of the form* (18) *are the extremal functions.*

COROLLARY 4. *Let* $f \in \Sigma_\eta^*(k;A,B)$, $|z| = r > 1$. *Then*

$$r - \frac{B-A}{k(1+B)-(1+A)} r^{-k} \leqslant |f(z)| \leqslant r + \frac{B-A}{k(1+B)-(1+A)} r^{-k},$$
$$r - k\frac{B-A}{k(1+B)-(1+A)} r^{-k} \leqslant |D_{\mathscr{H}} f(z)| \leqslant r + k\frac{B-A}{k(1+B)-(1+A)} r^{-k}.$$

The result is sharp. The function g_k *of the form* (18) *is the extremal function.*

COROLLARY 5. *Let* $r > 1$, $\gamma \geqslant 1$. *If* $f \in \Sigma_\eta^*(k;A,B)$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^\gamma d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|g_k(re^{i\theta})\right|^\gamma d\theta,$$
$$\frac{1}{2\pi}\int_0^{2\pi} |D_{\mathscr{H}} f(z)|^\gamma d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|D_{\mathscr{H}} g_k(re^{i\theta})\right|^\gamma d\theta,$$

where g_k *is the function defined by* (18).

The following covering result follows from Corollary 4.

COROLLARY 6. *If* $f \in \Sigma_\eta^*(k;A,B)$, *then* $\mathbb{D}(r) \subset f(\mathbb{D})$, *where*

$$r = 1 + \frac{B-A}{k(1+B)-(1+A)}.$$

Using the above results and the relation

$$f \in \Sigma_\eta^c(k;A,B) \Longleftrightarrow D_{\mathscr{H}} f \in \Sigma_\eta^*(k;A,B)$$

we obtain the following corollaries.

COROLLARY 7. *The class* $\Sigma_\eta^c(k;A,B)$ *is convex and compact subset of* Σ. *Moreover,*

$$E\Sigma_\eta^c(k;A,B) = \{h_n :\ n \in \mathbb{N}_{k-1}\} \cup \{g_n :\ n \in \mathbb{N}_k\},$$

and

$$\Sigma_\eta(k;A,B) = \left\{ \sum_{n=k-1}^{\infty} (\gamma_n h_n + \delta_n g_n) :\ (\gamma_n + \delta_n) = 1\ (\delta_{k-1} = 0, \gamma_n, \delta_n \geqslant 0) \right\},$$

where

$$h_{k-1}(z)=z,\ h_n(z)=z+\frac{e^{i(1+n)\eta}(B-A)}{n(n(1+B)+(1+A))}z^{-n},$$
$$g_n(z)=z-\frac{e^{i(1-n)\eta}(B-A)}{n(n(1+B)-(1+A))}\overline{z}^{-n}\quad(z\in\mathbb{D}). \tag{23}$$

COROLLARY 8. *Let* $f\in\Sigma_\eta^c(k;A,B)$ *be a function of the form* (2), $|z|=r>1,\ \gamma\geqslant 1,\ n\in\mathbb{N}$. *Then*

$$|a_n|\leqslant\frac{B-A}{n(n(1+B)-(1+A))},\ |b_n|\leqslant\frac{B-A}{n(n(1+B)-(1+A))},$$
$$r-\frac{B-A}{k(k(1+B)-(1+A))}r^{-k}\leqslant|f(z)|\leqslant r+\frac{B-A}{k(k(1+B)-(1+A))}r^{-k},$$
$$r-\frac{B-A}{k(1+B)-(1+A)}r^{-k}\leqslant|D_{\mathscr{H}}f(z)|\leqslant r+\frac{B-A}{k(1+B)-(1+A)}r^{-k},$$
$$\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^\gamma d\theta\leqslant\frac{1}{2\pi}\int_0^{2\pi}\left|g_k(re^{i\theta})\right|^\gamma d\theta,$$
$$\frac{1}{2\pi}\int_0^{2\pi}\left|D_{\mathscr{H}}f(re^{i\theta})\right|^\gamma d\theta\leqslant\frac{1}{2\pi}\int_0^{2\pi}\left|D_{\mathscr{H}}g_k(re^{i\theta})\right|^\gamma d\theta.$$

The results are sharp. The functions h_n,g_n *of the form* (23) *are the extremal functions.*

COROLLARY 9. *If* $f\in\Sigma_\eta^c(k;A,B)$, *then* $\mathbb{D}(r)\subset f(\mathbb{D})$, *where*

$$r=1+\frac{B-A}{k(k(1+B)-(1+A))}.$$

By using Corollary 1 and the results above we obtain corollaries listed below.

COROLLARY 10. *The class* $\Sigma_\eta(k;A,B)$ *is convex and compact subset of* Σ. *Moreover,*

$$E\Sigma_\eta(k;A,B)=\{h_n:\ n\in\mathbb{N}_{k-1}\}\cup\{g_n:\ n\in\mathbb{N}_k\}$$

and

$$\Sigma_\eta(k;A,B)=\left\{\sum_{n=k-1}^{\infty}(\gamma_nh_n+\delta_ng_n):\ (\gamma_n+\delta_n)=1\ (\delta_{k-1}=0,\gamma_n,\delta_n\geqslant 0)\right\},$$

where $h_{k-1}(z)=z$, *and*

$$h_n(z)=z+\frac{(B-A)e^{i(1+n)\eta}}{(1+B)n}z^{-n},\ g_n(z)=z-\frac{(B-A)e^{i(1-n)\eta}}{(1+B)n}\overline{z}^{-n}\quad(z\in\mathbb{D}). \tag{24}$$

COROLLARY 11. *Let* $f \in \Sigma_\eta(k;A,B)$ *be a function of the form* (11), $|z| = r > 1$, $\gamma \geqslant 1$, $n \in \mathbb{N}$. *Then*

$$|a_n| \leqslant \frac{B-A}{(1+B)n}, \ |b_n| \leqslant \frac{B-A}{(1+B)n},$$

$$r - \frac{B-A}{(1+B)k} r^{-k} \leqslant |f(z)| \leqslant r + \frac{B-A}{(1+B)k} r^{-k},$$

$$r - \frac{B-A}{1+B} r^{-k} \leqslant |D_{\mathscr{H}} f(z)| \leqslant r + \frac{B-A}{1+B} r^{-k},$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^\gamma d\theta \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left| g_k(re^{i\theta}) \right|^\gamma d\theta,$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| D_{\mathscr{H}} f(re^{i\theta}) \right|^\gamma d\theta \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left| D_{\mathscr{H}} g_k(re^{i\theta}) \right|^\gamma d\theta.$$

The results are sharp. The functions h_n, g_n *of the form* (24) *are the extremal functions.*

COROLLARY 12. *If* $f \in \Sigma_\eta(k;A,B)$, *then* $\mathbb{D}(r) \subset f(\mathbb{D})$, *where* $r = 1 + \frac{B-A}{(1+B)k}$.

REMARK 1. By varying the parameters in the defined classes of functions we can obtain new and also well-known results (see for example [1, 2, 5, 6]).

Acknowledgement. The work is supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge, University of Rzeszów.

REFERENCES

[1] O.P. AHUJA, J.M. JAHANGIRI, *Certain meromorphic harmonic functions*, Bull. Malays. Math. Sci. Soc. (2) 25 (2002), 1–10.
[2] I. ALDAWISH, M. DARUS, *On certain class of meromorphic harmonic concave functions*, Tamkang J. Math. 46 (2015), 101–109.
[3] J. DZIOK, *On Janowski harmonic functions*, J. Appl. Anal. 21(2015), 99–107.
[4] W. HENGARTNER, G. SCHOBER, *Univalent Harmonic Functions*, Trans. Amer. Math. Soc. 299 (1987), 1–31.
[5] J.M. JAHANGIRI, *Harmonic meromorphic starlike functions*, Bull. Korean Math. Soc. 37(2000), 291–301.
[6] J.M. JAHANGIRI AND H. SILVERMAN, *Meromorphic univalent harmonic functions with negative coefficients*, Bull. Korean Math. Soc. 36(1999), 763–770.
[7] W. JANOWSKI, *Some extremal problems for certain families of analytic functions I*, Annales Polonici Mathematici. 28(1973), 297–326.
[8] M. KREIN, D. MILMAN, *On the extreme points of regularly convex sets*, Studia Mathematica 9(1940), 133–138.

(Received August 24, 2018)

J. Dziok
Institute of Mathematics
University of Rzeszów
ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland
e-mail: jdziok@ur.edu.pl