Journal of
Mathematical
Inequalities
Volume 14, Number 4 (2020), 1007–1037 doi:10.7153/jmi-2020-14-66

STABILITY INEQUALITIES INVOLVING GRAVITY NORM AND TEMPERATURE

JIAJIN WEN, TIANYONG HAN AND JUN YUAN*

(*Communicated by Q.-H. Ma*)

Abstract. In the centered surround system $S^{(2)}\{P,\Gamma\}$, where Γ is an ellipse and its eccentricity $\mathrm{e}\in[0,1)$, and P is one of the foci of Γ, we establish the following stability inequalities:

$$\sqrt{\frac{4\pi}{15}}\times\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}\leqslant\widetilde{\|\mathbf{F}\|}\leqslant\sqrt{2}\times\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}},$$

where $\widetilde{\|\mathbf{F}\|}$ is the coefficient of variation of the gravity norm $\|\mathbf{F}\|$ and the coefficient $\sqrt{2}$ of $\mathrm{e}/\sqrt{1-\mathrm{e}^2}$ is the best constant. We also demonstrate the applications of the inequalities in the temperature change research, and obtain an approximate temperature coefficient of variation formula and an approximate temperature mean variance formula as follows:

$$\widetilde{T}\approx 1.164752397618432\cdots\times\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}$$

and

$$\overline{\mathrm{Var}}T\approx 1.2897992775023233\cdots\times\frac{\mathrm{e}}{(1-\mathrm{e}^2)^{3/2}}\times R_\Gamma^{-2}.$$

1. Introduction

Stability is an essential attribute of any random variable [1, 2, 3, 4, 5, 6, 7, 8]. The variance [2, 3, 4] and the coefficient of variation [5, 6] are important stability features of a random variable, their research and applications are important topics in mathematics and physics, especially the theory of satellite.

It is well known that there are eight planets in the solar system, i.e., Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus and Neptune, which are the moons of the Sun. In space science, we are concerned with the stability of the radiation energy [9, 10] of the Sun since which will directly affect our daily life. Since the radiation energy is related to the gravity [11] of the Sun and the temperature on a planet is dependent on the radiation energy, so we are special concerned with the stability of the *gravity norm* of the Sun.

In space science, we need to know that the law of temperature change on a planet, especially the *temperature coefficient of variation*. Unfortunately, it is very difficult

Mathematics subject classification (2010): 26D15, 26E60, 51K05, 52A40, 62J10.
Keywords and phrases: Centered surround system, gravity norm, coefficient of variation, Newton formula, Euler formula.

* Corresponding author.

Paper JMI-14-66

that to measure the temperature coefficient of variation on a planet. Therefore, it is of theoretical significance that to study the temperature coefficient of variation on a planet by means of the mathematics.

At present, the research of the climate change on the Earth is a hot topic in the world. Since the rain and the air humidity are related to the temperature and our daily life is dependent on the rain and the air humidity, it is of application value that to study the temperature coefficient of variation on the Earth.

In this paper, our motivation is to study the stability of the temperature on a planet. To this end, we first introduce the basic concepts on the stability and the surround system, as well as we illustrate the background and the significance of these concepts in space science. Next, we establish several identities and inequalities involving the centered surround system $S^{(2)}\{P,\Gamma\}$ [12, 13, 14, 15, 16, 17]. Next, we prove a *stability inequalities* in the centered surround system $S^{(2)}\{P,\Gamma\}$. Finally, we demonstrate the applications of our results in the temperature change research, and obtain an *approximate temperature coefficient of variation formula* and an *approximate temperature mean variance formula*.

In recent years, Jiajin Wen etc. systematically studied the theory of surround system and obtained some results which have the application value, see [12, 13, 14, 15, 16, 17]. One of the theoretical significance of this paper is that use the computer to deal with some complex computational geometry problems, and the another is to establish the geometric and physics theories on satellite motion. The application value of this paper is to analyze the stability of the temperature on a planet, especially the Earth. A large number of mathematics, physics and computer theories are used in this paper, especially the computational geometry, numerical analysis and the inequality theories, and the series [18] is the crucial one.

2. Basic concepts and main result

Let $\Omega \subset \mathbb{R}^m$ be a measurable set where $\mathbb{R} \triangleq (-\infty,\infty)$, and let $X \in \Omega$ be a continuous random variable and its probability density function $p : \Omega \to (0,\infty)$ be integral, as well as let the function $\varphi : \Omega \to (0,\infty)$ be integral. Then the functionals

$$\mathrm{E}\varphi \triangleq \int_\Omega p\varphi,\ \mathrm{Var}\varphi \triangleq \mathrm{E}\varphi^2 - (\mathrm{E}\varphi)^2, \overline{\mathrm{Var}}\varphi \triangleq \sqrt{\mathrm{Var}\varphi} \text{ and } \widetilde{\varphi} \triangleq \frac{\overline{\mathrm{Var}}\varphi}{\mathrm{E}\varphi} \tag{1}$$

are the *mathematical expectation, variance, mean variance* and the *coefficient of variation* of the random variable $\varphi(X)$, respectively [3, 4, 5, 6, 18].

The coefficient of variation $\widetilde{\varphi}$ is an important stability feature of the random variable $\varphi(X)$. If the mean variance $\overline{\mathrm{Var}}\varphi$ is very small and the mathematical expectation $\mathrm{E}\varphi$ is very large, then the coefficient of variation $\widetilde{\varphi}$ is very small. Conversely, if the coefficient of variation $\widetilde{\varphi}$ is very small, then the mean variance $\overline{\mathrm{Var}}\varphi$ is very small or the mathematical expectation $\mathrm{E}\varphi$ is very large. This is the significance of the coefficient of variation $\widetilde{\varphi}$ in the analysis of variance.

The theory of satellite is important in space science. In [12, 13, 14, 15, 16, 17, 19], the authors systematically studied the theory of satellite and obtained some results which have the application background. But in space science, the centered surround system $S^{(2)}\{P,\Gamma\}$ [12, 13, 14, 15, 16, 17] has its special properties, that is, where the Γ is an ellipse and P is one of the *foci* of the ellipse [19]. Therefore, it is necessary for us to do further research on this centered surround system.

Let the particle $A\in\mathbb{R}^2$ be regarded as the Earth, and let its motion trajectory be the ellipse

$$\Gamma \triangleq \left\{x\mathbf{i}+y\mathbf{j}\in\mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ x,y\in\mathbb{R},\ a\geqslant b>0\right\}, \tag{2}$$

where and in the future $\mathbf{0}=(0,0)$, $\mathbf{i}=(1,0)$, $\mathbf{j}=(0,1)$, and let the particle $P\triangleq -c\mathbf{i}$, where $c=\sqrt{a^2-b^2}\geqslant 0$, be regarded as the Sun which is a *focus* of the ellipse Γ, and the *eccentricity* of the ellipse Γ is $\mathrm{e}\triangleq c/a\in[0,1)$. Then the set $S^{(2)}\{P,\Gamma\}\triangleq\{P,\Gamma\}$ is a centered surround system [12, 13, 14, 15, 16, 17].

Let the masses of the Earth A and the Sun P be $m>0$ and $M>0$, respectively. Then, according to the law of universal gravitation, the gravity of the Earth A to the Sun P is

$$\mathbf{F}=\frac{GmM\,(A-P)}{\|A-P\|^3}, \tag{3}$$

where G is the gravitational constant in the solar system. Without loss of generality, here we assume that $GmM=1$.

We say that

$$\|\mathbf{F}\|=\frac{1}{\|A-P\|^2} \tag{4}$$

is the *gravity norm* of the gravity $\mathbf{F}$. When the Earth A traverse one cycle along its orbit Γ, the mean of the gravity norm $\|\mathbf{F}\|$ is

$$\overline{\|\mathbf{F}\|}=\frac{1}{|\Gamma|}\oint_\Gamma\frac{1}{\|A-P\|^2}, \tag{5}$$

where the $|\Gamma|\triangleq\oint_\Gamma \mathrm{d}s$ is the length of the ellipse Γ.

In [3, 15, 20], the authors extended the classic gravity $\mathbf{F}$ and defined the λ-*gravity* as follows:

$$\mathbf{F}_\lambda=\frac{A-P}{\|A-P\|^{\lambda+1}}, \tag{6}$$

where $\lambda\in(0,\infty)$, $\mathbf{F}_2=\mathbf{F}$, and the $\|\mathbf{F}_\lambda\|=\|A-P\|^{-\lambda}$ is the λ-*gravity norm* of the λ-gravity $\mathbf{F}_\lambda$. They also defined the *mean* λ-*gravity norm* of the λ-gravity $\mathbf{F}_\lambda$ as follows:

$$\overline{\|\mathbf{F}_\lambda\|}\triangleq\frac{1}{|\Gamma|}\oint_\Gamma\frac{1}{\|A-P\|^\lambda}, \tag{7}$$

where

$$\overline{\|\mathbf{F}\|}\triangleq\overline{\|\mathbf{F}_2\|}=\frac{1}{|\Gamma|}\oint_\Gamma\frac{1}{\|A-P\|^2} \tag{8}$$

is the *mean gravity norm* of the gravity $\mathbf{F}$.

In the solar system, the gravity of the particle A to the another particle P is $\mathbf{F}$, while for other galaxy in the universe, the gravity may be $\mathbf{F}_\lambda$ where $\lambda \in (0,2) \cup (2,+\infty)$. For example, in the black hole of the universe, we conjecture that the gravity is $\mathbf{F}_\lambda$ [15], where $\lambda \in (0,2)$ is very small. This is the significance of the λ-gravity $\mathbf{F}_\lambda$ in space science.

In [3], the authors defined the planet system in an abstract Euclidean space [21, 22] and the λ-gravity function in the planet system, and obtained some results which have the application background. In [15], the authors obtained the following *isoperimetric inequality* [23, 24]:

$$\overline{\|\mathbf{F}_\lambda\|} \geqslant \left(\frac{2\pi}{|\Gamma|}\right)^\lambda, \ \forall\, \lambda \geqslant 2, \tag{9}$$

where the Γ is a smooth and convex Jordan closed curve in $\mathbb{R}^2$ [25, 26]. Equality in (9) holds if and only if Γ is a circle and P is the center of the circle.

In [16], the authors improved and expanded the inequality (9), and obtained the following isoperimetric inequalities:

$$\left(1+\frac{5}{2\pi}\times\frac{\mathrm{e}^2}{1-\mathrm{e}^2}\right)\left(\frac{2\pi}{|\Gamma|}\right)^2 \leqslant \overline{\|\mathbf{F}\|} \leqslant \left(1+\frac{16-\pi}{4\pi}\times\frac{\mathrm{e}^2}{1-\mathrm{e}^2}\right)\left(\frac{2\pi}{|\Gamma|}\right)^2, \tag{10}$$

where Γ is an ellipse and P is one of the foci of the ellipse, and $\mathrm{e} \in [0,1)$. Equalities in (10) hold if and only if Γ is a circle and P is the center of the circle.

In [17], the authors proved that : For the centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$, we have the following isoperimetric inequalities:

$$\frac{\exp\left(\frac{1}{|\Gamma|}\oint_\Gamma \log \overline{r}_P\right)}{\exp\left(\frac{1}{|\Gamma|}\oint_\Gamma \log r_P\right)} \geqslant \frac{1}{2}\left[\sec\frac{l\pi}{|\Gamma|}+\cot\frac{l\pi}{|\Gamma|}\log\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right] \tag{11}$$

and

$$\frac{\left(\frac{1}{|\Gamma|}\oint_\Gamma \overline{r}_P^2\right)^{1/2}}{\frac{1}{|\Gamma|}\oint_\Gamma r_P} \geqslant \frac{1}{2}\left[\sec\frac{l\pi}{|\Gamma|}+\cot\frac{l\pi}{|\Gamma|}\log\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right] \text{ when } 0<\angle APA_+ \leqslant \tau, \tag{12}$$

where $\tau = 2.49342812654089\ldots$, and $\tau/2$ is the unique real root of the following equation:

$$\frac{\mathrm{d}^2[\sec\theta+\cot\theta\log(\tan\theta+\sec\theta)]}{\mathrm{d}\theta^2}=0,\ \theta\in\left(0,\frac{\pi}{2}\right), \tag{13}$$

and Γ is a smooth and convex Jordan closed curve in $\mathbb{R}^2$. Equalities in (11) and (12) hold if and only if Γ is a circle and P is the center of the circle.

In the centered surround system $S^{(2)}\{P,\Gamma\}$, we may also think that the $\|\mathbf{F}\|$ as a random variable, which follows a uniform distribution, that is , the probability density function of the $\|\mathbf{F}\|$ is $p:\Gamma\to(0,\infty)$, $p(A)=1/|\Gamma|$. Then, by (1), the mathematical expectation $\mathrm{E}\|\mathbf{F}\|$, variance $\mathrm{Var}\|\mathbf{F}\|$, mean variance $\overline{\mathrm{Var}}\|\mathbf{F}\|$ and the coefficient of variation $\widetilde{\|\mathbf{F}\|}$ of the random variable $\|\mathbf{F}\|$ are

$$\mathrm{E}\|\mathbf{F}\| = \overline{\|\mathbf{F}\|},\ \mathrm{Var}\|\mathbf{F}\| = \overline{\|\mathbf{F}_4\|} - \left(\overline{\|\mathbf{F}\|}\right)^2,\ \overline{\mathrm{Var}}\|\mathbf{F}\| = \sqrt{\mathrm{Var}\|\mathbf{F}\|} \text{ and } \widetilde{\|\mathbf{F}\|} = \frac{\overline{\mathrm{Var}}\|\mathbf{F}\|}{\mathrm{E}\|\mathbf{F}\|}, \tag{14}$$

respectively. This is the significance of the λ-gravity $\mathbf{F}_\lambda$ in the analysis of variance.

In general, the coefficient of variation $\widetilde{\|\mathbf{F}\|}$ can not be expressed by the elementary functions since which involves the elliptic integral [27]. In order to facilitate the applications, in this paper, we will find the sharp lower and upper bounds of the coefficient of variation $\widetilde{\|\mathbf{F}\|}$.

Our main result is as follows.

THEOREM 2.1. (Stability inequalities) *Let* $S^{(2)}\{P,\Gamma\}$ *be a centered surround system, where* Γ *is an ellipse and its eccentricity* $\mathrm{e} \in [0,1)$, *and* P *is one of the foci of* Γ. *Then we have the following inequalities:*

$$\sqrt{\frac{4\pi}{15}} \times \frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}} \leqslant \widetilde{\|\mathbf{F}\|} \leqslant \sqrt{2} \times \frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}, \tag{15}$$

where the coefficient $\sqrt{2}$ *of* $\mathrm{e}/\sqrt{1-\mathrm{e}^2}$ *in* (15) *is the best constant.*

In Section 5, we will display the applications of Theorem 2.1 in space science.

3. Preliminaries

In order to prove Theorem 2.1, we need to introduce some preparatory knowledge as follows.

For the ellipse Γ in (2), we say that

$$R_\Gamma \triangleq \sqrt{\frac{a|\Gamma|}{2\pi}} \tag{16}$$

is a *quasi-radius* of the ellipse Γ.

We remark here that, by the isoperimetric inequalities [15, 23, 24]:

$$\pi ab = \mathrm{Area}D(\Gamma) \leqslant \frac{|\Gamma|^2}{4\pi}$$

and the p-mean inequality [15, 20, 28, 29, 30, 31, 32, 33]:

$$\begin{aligned}\frac{|\Gamma|}{2\pi} &= \frac{1}{2\pi}\int_0^{2\pi} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,\mathrm{d}\theta \\ &\leqslant \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\left(a^2\sin^2\theta + b^2\cos^2\theta\right)\mathrm{d}\theta} = \sqrt{\frac{a^2+b^2}{2}},\end{aligned}$$

we have

$$b \leqslant \sqrt{a\sqrt{ab}} \leqslant R_\Gamma \leqslant \sqrt{a\sqrt{\frac{a^2+b^2}{2}}} \leqslant a. \tag{17}$$

Equations in (17) hold if and on if $a = b$. Therefore, the quasi-radius R_Γ is a mean of the positive real a and b.

In order to prove Theorems 2.1, we need to establish several identities and inequalities as follows.

According to the theory of mathematical analysis, we have the following Lemmas 3.1 and 3.2.

LEMMA 3.1. (See Lemma 1 in [16]) *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with the period $T > 0$. Then we have*

$$\int_t^{T+t} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x,\ \forall t \in \mathbb{R}. \tag{18}$$

LEMMA 3.2. (See Lemma 2 in [16]) *For any continuous function $g : [0,1] \to \mathbb{R}$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} g\left(\cos^2\theta\right)\mathrm{d}\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} g\left(\cos^2\theta\right)\mathrm{d}\theta. \tag{19}$$

LEMMA 3.3. (Mean gravity norm formula) *Under the hypotheses in Theorem 2.1, we have*

$$\overline{\|\mathbf{F}\|} = R_\Gamma^{-2}\left[\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1+\mathrm{e}^2\cos^2\theta}{\left(1-\mathrm{e}^2\cos^2\theta\right)^{3/2}}\mathrm{d}\theta\right]. \tag{20}$$

Proof. Since

$$\Gamma = \{x\mathbf{i} + y\mathbf{j} : x = a\cos\theta,\ y = b\sin\theta,\ \theta \in [0,2\pi]\},$$

we have

$$\begin{aligned}
\|A-P\| &= \sqrt{(a\cos\theta + c)^2 + (b\sin\theta)^2} \\
&= \sqrt{a^2\cos^2\theta + 2ac\cos\theta + c^2 + b^2\sin^2\theta} \\
&= \sqrt{a^2\cos^2\theta + 2ac\cos\theta + c^2 + (a^2-c^2)\sin^2\theta} \\
&= \sqrt{a^2 + 2ac\cos\theta + c^2\cos^2\theta} = a + c\cos\theta \\
&= a(1+\mathrm{e}\cos\theta),
\end{aligned}$$

that is,

$$\|A-P\| = a(1+\mathrm{e}\cos\theta). \tag{21}$$

Since

$$\frac{\mathrm{d}s}{\mathrm{d}\theta}=\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}=a\sqrt{1-\mathrm{e}^2\cos^2\theta}, \tag{22}$$

by (21) and (22), we have

$$\begin{aligned}\overline{\|\mathbf{F}\|}&=\frac{1}{|\Gamma|}\oint_\Gamma\frac{1}{\|A-P\|^2}\triangleq\frac{1}{|\Gamma|}\oint_\Gamma\frac{1}{\|A-P\|^2}\mathrm{d}s\\&=\frac{1}{|\Gamma|}\int_0^{2\pi}\frac{1}{a^2(1+\mathrm{e}\cos\theta)^2}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta\\&=\frac{1}{|\Gamma|}\int_0^{2\pi}\frac{a\sqrt{1-\mathrm{e}^2\cos^2\theta}}{a^2(1+\mathrm{e}\cos\theta)^2}\mathrm{d}\theta\\&=\frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{\sqrt{1-\mathrm{e}\cos\theta}}{(1+\mathrm{e}\cos\theta)^{3/2}}\mathrm{d}\theta,\end{aligned}$$

that is,

$$\overline{\|\mathbf{F}\|}=\frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{\sqrt{1-\mathrm{e}\cos\theta}}{(1+\mathrm{e}\cos\theta)^{3/2}}\mathrm{d}\theta. \tag{23}$$

In (23), set $\theta=\vartheta-\pi$. Since the cosine function $\cos x$ is a periodic function with the period 2π, by Lemma 3.1, we have

$$\begin{aligned}\overline{\|\mathbf{F}\|}&=\frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{\sqrt{1-\mathrm{e}\cos\theta}}{(1+\mathrm{e}\cos\theta)^{3/2}}\mathrm{d}\theta\\&=\frac{1}{a|\Gamma|}\int_\pi^{2\pi+\pi}\frac{\sqrt{1-\mathrm{e}\cos(\vartheta-\pi)}}{[1+\mathrm{e}\cos(\vartheta-\pi)]^{3/2}}\mathrm{d}(\vartheta-\pi)\\&=\frac{1}{a|\Gamma|}\int_\pi^{2\pi+\pi}\frac{\sqrt{1+\mathrm{e}\cos\vartheta}}{(1-\mathrm{e}\cos\vartheta)^{3/2}}\mathrm{d}\vartheta\\&=\frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{\sqrt{1+\mathrm{e}\cos\vartheta}}{(1-\mathrm{e}\cos\vartheta)^{3/2}}\mathrm{d}\vartheta\\&=\frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{\sqrt{1+\mathrm{e}\cos\theta}}{(1-\mathrm{e}\cos\theta)^{3/2}}\mathrm{d}\theta,\end{aligned}$$

that is,

$$\overline{\|\mathbf{F}\|}=\frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{\sqrt{1+\mathrm{e}\cos\theta}}{(1-\mathrm{e}\cos\theta)^{3/2}}\mathrm{d}\theta. \tag{24}$$

By (23) and (24), we get

$$
\begin{aligned}
\overline{\|\mathbf{F}\|} &= \frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{1}{2}\left[\frac{\sqrt{1-\mathrm{e}\cos\theta}}{(1+\mathrm{e}\cos\theta)^{3/2}}+\frac{\sqrt{1+\mathrm{e}\cos\theta}}{(1-\mathrm{e}\cos\theta)^{3/2}}\right]\mathrm{d}\theta\\
&= \frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{1}{2}\frac{(1-\mathrm{e}\cos\theta)^2+(1+\mathrm{e}\cos\theta)^2}{(1+\mathrm{e}\cos\theta)^{3/2}(1-\mathrm{e}\cos\theta)^{3/2}}\mathrm{d}\theta\\
&= \frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{1+\mathrm{e}^2\cos^2\theta}{(1-\mathrm{e}^2\cos^2\theta)^{3/2}}\mathrm{d}\theta,
\end{aligned}
$$

that is,

$$
\overline{\|\mathbf{F}\|} = \frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{1+\mathrm{e}^2\cos^2\theta}{(1-\mathrm{e}^2\cos^2\theta)^{3/2}}\mathrm{d}\theta. \tag{25}
$$

On the other hand, by (16) and (22), we have

$$
|\Gamma| = a\int_0^{2\pi}\sqrt{1-\mathrm{e}^2\cos^2\theta}\,\mathrm{d}\theta \Leftrightarrow \left(\frac{2\pi}{|\Gamma|}\right)^2\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{1-\mathrm{e}^2\cos^2\theta}\,\mathrm{d}\theta\right) = R_\Gamma^{-2}. \tag{26}
$$

According to (25), (26) and Lemma 3.2, we get

$$
\begin{aligned}
\overline{\|\mathbf{F}\|} &= \frac{1}{a|\Gamma|}\int_0^{2\pi}\frac{1+\mathrm{e}^2\cos^2\theta}{(1-\mathrm{e}^2\cos^2\theta)^{3/2}}\mathrm{d}\theta\\
&= R_\Gamma^{-2}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{1+\mathrm{e}^2\cos^2\theta}{(1-\mathrm{e}^2\cos^2\theta)^{3/2}}\mathrm{d}\theta\right]\\
&= R_\Gamma^{-2}\left[\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1+\mathrm{e}^2\cos^2\theta}{(1-\mathrm{e}^2\cos^2\theta)^{3/2}}\mathrm{d}\theta\right].
\end{aligned}
$$

That is, formula (20) is proved. The proof of Lemma 3.3 is completed. □

LEMMA 3.4. (Mean 4-gravity norm formula) *Under the hypotheses in Theorem 2.1, we have*

$$
\overline{\|\mathbf{F}_4\|} = R_\Gamma^{-4}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{1-\mathrm{e}^2\cos^2\theta}\,\mathrm{d}\theta\right)\left[\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta\right]. \tag{27}
$$

Proof. By (21) and (22), we have

$$
\begin{aligned}
\overline{\|\mathbf{F}_4\|} &= \frac{1}{|\Gamma|}\oint_\Gamma\frac{1}{\|A-P\|^4}\mathrm{d}s\\
&= \frac{1}{|\Gamma|}\int_0^{2\pi}\frac{1}{a^4(1+\mathrm{e}\cos\theta)^4}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{|\Gamma|}\int_0^{2\pi} \frac{a\sqrt{1-\mathrm{e}^2\cos^2\theta}}{a^4(1+\mathrm{e}\cos\theta)^4}\mathrm{d}\theta \\
&= \frac{1}{a^3|\Gamma|}\int_0^{2\pi} \frac{\sqrt{1-\mathrm{e}\cos\theta}}{(1+\mathrm{e}\cos\theta)^{7/2}}\mathrm{d}\theta,
\end{aligned}
$$

that is,

$$
\overline{\|\mathbf{F}_4\|} = \frac{1}{a^3|\Gamma|}\int_0^{2\pi} \frac{\sqrt{1-\mathrm{e}\cos\theta}}{(1+\mathrm{e}\cos\theta)^{7/2}}\mathrm{d}\theta. \tag{28}
$$

In (28), set $\theta = \vartheta - \pi$. Since the cosine function $\cos x$ is a periodic function with the period 2π, by Lemma 3.1, we have

$$
\begin{aligned}
\overline{\|\mathbf{F}_4\|} &= \frac{1}{a|\Gamma|}\int_0^{2\pi} \frac{\sqrt{1-\mathrm{e}\cos\theta}}{(1+\mathrm{e}\cos\theta)^{3/2}}\mathrm{d}\theta \\
&= \frac{1}{a^3|\Gamma|}\int_\pi^{2\pi+\pi} \frac{\sqrt{1-\mathrm{e}\cos(\vartheta-\pi)}}{[1+\mathrm{e}\cos(\vartheta-\pi)]^{7/2}}\mathrm{d}(\vartheta-\pi) \\
&= \frac{1}{a^3|\Gamma|}\int_\pi^{2\pi+\pi} \frac{\sqrt{1+\mathrm{e}\cos\vartheta}}{(1-\mathrm{e}\cos\vartheta)^{7/2}}\mathrm{d}\vartheta \\
&= \frac{1}{a^3|\Gamma|}\int_0^{2\pi} \frac{\sqrt{1+\mathrm{e}\cos\theta}}{(1-\mathrm{e}\cos\theta)^{7/2}}\mathrm{d}\theta,
\end{aligned}
$$

that is,

$$
\overline{\|\mathbf{F}_4\|} = \frac{1}{a^3|\Gamma|}\int_0^{2\pi} \frac{\sqrt{1+\mathrm{e}\cos\theta}}{(1-\mathrm{e}\cos\theta)^{7/2}}\mathrm{d}\theta. \tag{29}
$$

By (28) and (29), we get

$$
\begin{aligned}
\overline{\|\mathbf{F}_4\|} &= \frac{1}{a^3|\Gamma|}\int_0^{2\pi} \frac{1}{2}\left[\frac{\sqrt{1-\mathrm{e}\cos\theta}}{(1+\mathrm{e}\cos\theta)^{7/2}} + \frac{\sqrt{1+\mathrm{e}\cos\theta}}{(1-\mathrm{e}\cos\theta)^{7/2}}\right]\mathrm{d}\theta \\
&= \frac{1}{a^3|\Gamma|}\int_0^{2\pi} \frac{1}{2}\frac{(1-\mathrm{e}\cos\theta)^4+(1+\mathrm{e}\cos\theta)^4}{(1+\mathrm{e}\cos\theta)^{7/2}(1-\mathrm{e}\cos\theta)^{7/2}}\mathrm{d}\theta \\
&= \frac{1}{a^3|\Gamma|}\int_0^{2\pi} \frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta,
\end{aligned}
$$

that is,

$$
\overline{\|\mathbf{F}_4\|} = \frac{1}{a^3|\Gamma|}\int_0^{2\pi} \frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta. \tag{30}
$$

According to (26), (30) and Lemma 3.2, we get

$$
\begin{aligned}
\overline{\|\mathbf{F}_4\|} &= \frac{1}{a^3|\Gamma|}\int_0^{2\pi}\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta \\
&= \frac{1}{|\Gamma|^4}\left(\int_0^{2\pi}\sqrt{1-\mathrm{e}^2\cos^2\theta}\mathrm{d}\theta\right)^3\int_0^{2\pi}\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta \\
&= \left(\frac{2\pi}{|\Gamma|}\right)^4\left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-\mathrm{e}^2\cos^2\theta}\mathrm{d}\theta\right)^3\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta\right] \\
&= \left(\frac{2\pi}{|\Gamma|}\right)^4\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{1-\mathrm{e}^2\cos^2\theta}\mathrm{d}\theta\right)^3\left[\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta\right] \\
&= R_\Gamma^{-4}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{1-\mathrm{e}^2\cos^2\theta}\mathrm{d}\theta\right)\left[\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta\right].
\end{aligned}
$$

That is, formula (27) is proved. This ends the proof of Lemma 3.4. □

According to the theory of mathematical analysis, we have the following Lemma 3.5.

LEMMA 3.5. (See Lemma 5 in [16]) *For any positive integer n, we have*

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\mathrm{d}\theta = \frac{(2n-1)!!}{(2n)!!}. \tag{31}$$

where

$$(2n)!! = 2\times4\times6\times\cdots\times(2n) \text{ and } (2n-1)!! = 1\times3\times5\times\cdots\times(2n-1). \tag{32}$$

LEMMA 3.6. *For any* $\mathrm{e}\in[0,1)$, *we have*

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{1-\mathrm{e}^2\cos^2\theta}\mathrm{d}\theta = 1-\mathrm{e}^2\sum_{n=1}^{\infty}\frac{u_n}{4n^2-1}\mathrm{e}^{2(n-1)}, \tag{33}$$

where the sequence $\{u_n\}_{n=0}^{\infty}$ *is defined as follows:*

$$\{u_n\}_{n=0}^{\infty}: u_0 \triangleq 1, u_n \triangleq \prod_{k=1}^{n}\left(1-\frac{1}{4k^2}\right) = \frac{(2n+1)!!(2n-1)!!}{[(2n)!!]^2},\ \forall n\geqslant 1. \tag{34}$$

Proof. By $0\leqslant \mathrm{e}<1$, $0\leqslant \mathrm{e}^2\cos^2\theta<1$ and the Newton formula

$$(1+x)^{\mu} = 1+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}(\mu+1-k)x^n,\ \forall x\in(-1,1), \tag{35}$$

we get

$$\begin{aligned}\sqrt{1-\mathrm{e}^2\cos^2\theta} &= \left(1-\mathrm{e}^2\cos^2\theta\right)^{1/2}\\ &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\left(\frac{1}{2}+1-k\right)\left(-\mathrm{e}^2\cos^2\theta\right)^n\\ &= 1-\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(2n-3)!!}{(2n)!!}\mathrm{e}^{2(n-1)}\cos^{2n}\theta\\ &= 1-\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n-1)(2n)!!}\mathrm{e}^{2(n-1)}\cos^{2n}\theta,\end{aligned}$$

that is

$$\sqrt{1-\mathrm{e}^2\cos^2\theta} = 1-\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n-1)(2n)!!}\mathrm{e}^{2(n-1)}\cos^{2n}\theta. \tag{36}$$

By (36) and Lemma 3.5, we get

$$\begin{aligned}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{1-\mathrm{e}^2\cos^2\theta}\mathrm{d}\theta &= 1-\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n-1)(2n)!!}\mathrm{e}^{2(n-1)}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\mathrm{d}\theta\right)\\ &= 1-\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n-1)(2n)!!}\mathrm{e}^{2(n-1)}\frac{(2n-1)!!}{(2n)!!}\\ &= 1-\mathrm{e}^2\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}\frac{(2n+1)!!(2n-1)!!}{[(2n)!!]^2}\mathrm{e}^{2(n-1)}\\ &= 1-\mathrm{e}^2\sum_{n=1}^{\infty}\frac{u_n}{4n^2-1}\mathrm{e}^{2(n-1)}.\end{aligned}$$

That is, (33) is proved. The proof of Lemma 3.6 is completed. □

LEMMA 3.7. *For any* $\mathrm{e}\in[0,1)$, *we have*

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1+\mathrm{e}^2\cos^2\theta}{\left(1-\mathrm{e}^2\cos^2\theta\right)^{3/2}}\mathrm{d}\theta = 1+\sum_{n=1}^{\infty}\frac{4n+1}{2n+1}u_n\mathrm{e}^{2n}, \tag{37}$$

where the sequence $\{u_n\}_{n=0}^{\infty}$ *is defined by* (34).

Proof. By the Newton formula (35), we get

$$\begin{aligned}\frac{1+\mathrm{e}^2\cos^2\theta}{\left(1-\mathrm{e}^2\cos^2\theta\right)^{3/2}} &= 2\left(1-\mathrm{e}^2\cos^2\theta\right)^{-3/2}-\left(1-\mathrm{e}^2\cos^2\theta\right)^{-1/2}\\ &= 1+2\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\left(-\frac{3}{2}+1-k\right)\left(-\mathrm{e}^2\cos^2\theta\right)^n\\ &\quad -\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\left(-\frac{1}{2}+1-k\right)\left(-\mathrm{e}^2\cos^2\theta\right)^n\end{aligned}$$

$$
\begin{aligned}
&= 1+2\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(2n+1)!!}{(2n)!!}\mathrm{e}^{2(n-1)}\cos^{2n}\theta \\
&\quad -\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\mathrm{e}^{2(n-1)}\cos^{2n}\theta \\
&= 1+\mathrm{e}^2\sum_{n=1}^{\infty}\frac{2(2n+1)!!-(2n-1)!!}{(2n)!!}\mathrm{e}^{2(n-1)}\cos^{2n}\theta \\
&= 1+\mathrm{e}^2\sum_{n=1}^{\infty}\frac{2(2n+1)(2n-1)!!-(2n-1)!!}{(2n)!!}\mathrm{e}^{2(n-1)}\cos^{2n}\theta \\
&= 1+\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(4n+1)(2n-1)!!}{(2n)!!}\mathrm{e}^{2(n-1)}\cos^{2n}\theta,
\end{aligned}
$$

that is

$$
\frac{1+\mathrm{e}^2\cos^2\theta}{(1-\mathrm{e}^2\cos^2\theta)^{3/2}} = 1+\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(4n+1)(2n-1)!!}{(2n)!!}\mathrm{e}^{2(n-1)}\cos^{2n}\theta. \tag{38}
$$

By (38) and Lemma 3.5, we get

$$
\begin{aligned}
\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1+\mathrm{e}^2\cos^2\theta}{(1-\mathrm{e}^2\cos^2\theta)^{3/2}}\mathrm{d}\theta &= 1+\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(4n+1)(2n-1)!!}{(2n)!!}\mathrm{e}^{2(n-1)}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\mathrm{d}\theta\right) \\
&= 1+\mathrm{e}^2\sum_{n=1}^{\infty}\frac{(4n+1)(2n-1)!!}{(2n)!!}\mathrm{e}^{2(n-1)}\frac{(2n-1)!!}{(2n)!!} \\
&= 1+\mathrm{e}^2\sum_{n=1}^{\infty}\frac{4n+1}{2n+1}(2n+1)\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\mathrm{e}^{2(n-1)} \\
&= 1+\mathrm{e}^2\sum_{n=1}^{\infty}\frac{4n+1}{2n+1}\frac{(2n+1)!!(2n-1)!!}{[(2n)!!]^2}\mathrm{e}^{2(n-1)} \\
&= 1+\mathrm{e}^2\sum_{n=1}^{\infty}\frac{4n+1}{2n+1}u_n\mathrm{e}^{2(n-1)}.
\end{aligned}
$$

That is, the identity (37) holds. This ends the proof of Lemma 3.7. □

LEMMA 3.8. *For any* $\mathrm{e}\in[0,1)$, *we have*

$$
\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta = 1+\sum_{n=1}^{\infty}\left[1+\frac{16}{15}n(2n+3)\right]u_n\mathrm{e}^{2n}, \tag{39}
$$

where the sequence $\{u_n\}_{n=0}^{\infty}$ *is defined by* (34).

Proof. By the Newton formula (35), we get

$$
\begin{aligned}
&\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{\left(1-\mathrm{e}^2\cos^2\theta\right)^{7/2}}\\
=&\ \frac{\left(1-\mathrm{e}^2\cos^2\theta\right)^2+8\left(\mathrm{e}^2\cos^2\theta-1\right)+8}{\left(1-\mathrm{e}^2\cos^2\theta\right)^{7/2}}\\
=&\ \left(1-\mathrm{e}^2\cos^2\theta\right)^{-3/2}-8\left(1-\mathrm{e}^2\cos^2\theta\right)^{-5/2}+8\left(1-\mathrm{e}^2\cos^2\theta\right)^{-7/2}\\
=&\ 1+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\left(-\frac{3}{2}+1-k\right)\left(-\mathrm{e}^2\cos^2\theta\right)^n\\
&-8\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\left(-\frac{5}{2}+1-k\right)\left(-\mathrm{e}^2\cos^2\theta\right)^n\right]\\
&+8\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\left(-\frac{7}{2}+1-k\right)\left(-\mathrm{e}^2\cos^2\theta\right)^n\right]\\
=&\ 1+\sum_{n=1}^{\infty}\frac{1}{2^n n!}\prod_{k=1}^{n}(2k+1)\left(\mathrm{e}^2\cos^2\theta\right)^n-8\sum_{n=1}^{\infty}\frac{1}{2^n n!}\prod_{k=1}^{n}(2k+3)\left(\mathrm{e}^2\cos^2\theta\right)^n\\
&+8\sum_{n=1}^{\infty}\frac{1}{2^n n!}\prod_{k=1}^{n}(2k+5)\left(\mathrm{e}^2\cos^2\theta\right)^n\\
=&\ 1+\sum_{n=1}^{\infty}\frac{(2n+1)!!}{(2n)!!}\left(\mathrm{e}^2\cos^2\theta\right)^n-8\sum_{n=1}^{\infty}\frac{2n+3}{3!!}\frac{(2n+1)!!}{(2n)!!}\left(\mathrm{e}^2\cos^2\theta\right)^n\\
&+8\sum_{n=1}^{\infty}\frac{(2n+3)(2n+5)}{5!!}\frac{(2n+1)!!}{(2n)!!}\left(\mathrm{e}^2\cos^2\theta\right)^n\\
=&\ 1+\sum_{n=1}^{\infty}\left[1-\frac{8(2n+3)}{3!!}+\frac{8(2n+3)(2n+5)}{5!!}\right]\frac{(2n+1)!!}{(2n)!!}\left(\mathrm{e}^2\cos^2\theta\right)^n\\
=&\ 1+\mathrm{e}^2\sum_{n=1}^{\infty}\left[1+\frac{16}{15}n(2n+3)\right]\frac{(2n+1)!!}{(2n)!!}\cos^{2n}\theta\,\mathrm{e}^{2(n-1)},
\end{aligned}
$$

that is

$$
\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{\left(1-\mathrm{e}^2\cos^2\theta\right)^{7/2}}=1+\mathrm{e}^2\sum_{n=1}^{\infty}\left[1+\frac{16}{15}n(2n+3)\right]\frac{(2n+1)!!}{(2n)!!}\cos^{2n}\theta\,\mathrm{e}^{2(n-1)}. \tag{40}
$$

By (40) and Lemma 3.5, we get

$$
\begin{aligned}
&\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{\left(1-\mathrm{e}^2\cos^2\theta\right)^{7/2}}\mathrm{d}\theta\\
&=1+\mathrm{e}^2\sum_{n=1}^{\infty}\left[1+\frac{16}{15}n(2n+3)\right]\frac{(2n+1)!!}{(2n)!!}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\,\mathrm{d}\theta\right)\mathrm{e}^{2(n-1)}
\end{aligned}
$$

$$= 1 + \mathrm{e}^2 \sum_{n=1}^{\infty} \left[1 + \frac{16}{15} n(2n+3)\right] \frac{(2n+1)!!(2n-1)!!}{[(2n)!!]^2} \mathrm{e}^{2(n-1)}$$
$$= 1 + \mathrm{e}^2 \sum_{n=1}^{\infty} \left[1 + \frac{16}{15} n(2n+3)\right] u_n \mathrm{e}^{2(n-1)}.$$

That is, the identity (39) is proved. The proof of Lemma 3.8 is completed. □

LEMMA 3.9. *For any* $\mathrm{e} \in [0,1)$, *we have*

$$\mathrm{Var}\|\mathbf{F}\| = R_\Gamma^{-4} \sum_{n=1}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} u_i u_j, \tag{41}$$

where the sequence $\{u_n\}_{n=0}^{\infty}$ *is defined by* (34), *and*

$$w_{i,j} \triangleq -\frac{1}{4i^2-1}\left[1 + \frac{16}{15} j(2j+3)\right] - \frac{4i+1}{2i+1} \times \frac{4j+1}{2j+1}. \tag{42}$$

Proof. Set

$$a_n = -\frac{u_n}{4n^2-1},\ b_n = \left[1 + \frac{16}{15} n(2n+3)\right] u_n,\ c_n = \frac{4n+1}{2n+1} u_n,\ \forall n \geqslant 0. \tag{43}$$

Then $a_0 = b_0 = c_0 = 1$. By Lemmas 3.3, 3.4, 3.6, 3.7 and 3.8, we have

$$\mathrm{Var}\|\mathbf{F}\| = R_\Gamma^{-4} \varpi(\mathrm{e}), \tag{44}$$

where

$$\begin{aligned}
\varpi(\mathrm{e}) &= \left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1-\mathrm{e}^2\cos^2\theta}\,\mathrm{d}\theta\right)\left[\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1+6\mathrm{e}^2\cos^2\theta+\mathrm{e}^4\cos^4\theta}{(1-\mathrm{e}^2\cos^2\theta)^{7/2}}\mathrm{d}\theta\right] \\
&\quad - \left[\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1+\mathrm{e}^2\cos^2\theta}{(1-\mathrm{e}^2\cos^2\theta)^{3/2}}\mathrm{d}\theta\right]^2 \\
&= \left(\sum_{n=0}^{\infty} a_n \mathrm{e}^{2n}\right)\left(\sum_{n=0}^{\infty} b_n \mathrm{e}^{2n}\right) - \left(\sum_{n=0}^{\infty} c_n \mathrm{e}^{2n}\right)^2 \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} a_i b_j - \sum_{n=0}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} c_i c_j \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} (a_i b_j - c_i c_j) \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} \left\{-\frac{1}{4i^2-1}\left[1+\frac{16}{15}j(2j+3)\right] - \frac{4i+1}{2i+1}\times\frac{4j+1}{2j+1}\right\} u_i u_j \\
&= \sum_{n=1}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} \left\{-\frac{1}{4i^2-1}\left[1+\frac{16}{15}j(2j+3)\right] - \frac{4i+1}{2i+1}\times\frac{4j+1}{2j+1}\right\} u_i u_j
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} u_i u_j,$$

that is,

$$\varpi(\mathrm{e}) = \sum_{n=1}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} u_i u_j. \tag{45}$$

By (44) and (45), we see that the identity (41) holds. This ends the proof of Lemma 3.9. □

LEMMA 3.10. *For any positive integer n, we have*

$$\frac{2}{\pi} < u_n < \frac{4}{\pi} \times \frac{2n+1}{4n+1}, \tag{46}$$

where the sequence $\{u_n\}_{n=0}^{\infty}$ is defined by (34).

Proof. Recall the following famous Euler formula:

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right), \ \forall x \in \mathbb{R}. \tag{47}$$

In (47), set $x = \pi/2$, we get

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 - \frac{1}{4k^2}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) = \frac{2}{\pi}. \tag{48}$$

Since the sequence $\{u_n\}_{n=1}^{\infty}$ is strictly decreasing, by (48), we get

$$u_n > \lim_{n\to\infty} u_n = \frac{2}{\pi}.$$

This proves the first inequality in (46).

We define a new sequence $\{u_n^*\}_{n=1}^{\infty}$ as follows:

$$\{u_n^*\}_{n=1}^{\infty} : u_n^* \triangleq \frac{4n+1}{2n+1} u_n. \tag{49}$$

By (48), we have

$$\lim_{n\to\infty} u_n^* = \frac{4}{\pi}. \tag{50}$$

By means of the command Expand[ ] of the Mathematica software, we have

$$\frac{u_{n+1}^*}{u_n^*} = \frac{4n+5}{2n+3} \times \frac{2n+1}{4n+1} \left[1 - \frac{1}{4(n+1)^2}\right] = \frac{16n^3+36n^2+24n+5}{16n^3+36n^2+24n+4} > 1.$$

Hence the sequence $\{u_n^*\}_{n=1}^{\infty}$ is strictly increasing. Therefore, by (50), we have

$$\frac{5}{4} = u_1^* \leqslant u_n^* = \frac{4n+1}{2n+1} u_n < \lim_{n\to\infty} u_n^* = \frac{4}{\pi} \Rightarrow u_n < \frac{4}{\pi} \times \frac{2n+1}{4n+1}. \tag{51}$$

That is, the second inequalities in (46) is also proved. This ends the proof of Lemma 3.10. □

LEMMA 3.11. *For any positive integer n, we have*

$$\frac{2}{\pi}u_{n-1} < u_i u_j \leqslant u_n, \ \forall i, j : i + j = n \text{ and } i, j \geqslant 0, \tag{52}$$

where the sequence $\{u_n\}_{n=0}^{\infty}$ is defined by (34).

Proof. By Lemma 3.10, inequalities (52) is true when $n = 1$. In what follows we assume that $n \geqslant 2$.

When $i = 0$ or $i = n$, we have

$$u_n = u_i u_j = \left(1 - \frac{1}{4n^2}\right) u_{n-1} \geqslant \left(1 - \frac{1}{4 \times 2^2}\right) u_{n-1} = \frac{15}{16}u_{n-1} > \frac{2}{\pi}u_{n-1}.$$

Hence inequalities (52) holds. Now we assume that $1 \leqslant i \leqslant n-1$.

Since the sequence $\{u_i\}_{i=0}^{\infty}$ is strictly decreasing, by Lemma 3.10, we have

$$u_i u_j = u_{n-i} u_i > \frac{2}{\pi}u_i \geqslant \frac{2}{\pi}u_{n-1}.$$

Hence the first inequalities in (52) holds.

Now we prove the second inequalities in (52) also holds as follows.

Since $u_i u_j = u_j u_i$, without loss of generality, we can assume that

$$1 \leqslant i \leqslant j \Leftrightarrow 1 \leqslant i \leqslant \frac{n}{2} \Leftrightarrow 1 \leqslant i \leqslant \left[\frac{n}{2}\right],$$

where the $[\circ]$ is the Gauss function [21].

We define a new sequence $\{v_i\}_{i=0}^{[n/2]}$ as follows:

$$\{v_i\}_{i=0}^{[n/2]} : v_0 \triangleq u_n, \ \ v_i \triangleq u_i u_j = u_i u_{n-i}, \ \forall i \geqslant 1.$$

Then

$$1 \leqslant i \leqslant \left[\frac{n}{2}\right] \Rightarrow 1 \leqslant i \leqslant \frac{n}{2} \Rightarrow 0 < 2i < 2(n-i+1),$$

and

$$\frac{v_i}{v_{i-1}} = \frac{u_i u_{n-i}}{u_{i-1} u_{n-i+1}} = \frac{1 - 1/(2i)^2}{1 - 1/[2(n-i+1)]^2} < 1, \ \forall i : 1 \leqslant i \leqslant \left[\frac{n}{2}\right].$$

Hence the sequence $\{v_i\}_{i=1}^{[n/2]}$ is strictly decreasing. So we get

$$u_i u_j = v_i \leqslant v_1 = u_1 u_{n-1} = \frac{3}{4}\left(1 - \frac{1}{4n^2}\right)^{-1} u_n < \frac{3}{4}\left(1 - \frac{1}{4}\right)^{-1} u_n = u_n, \ \forall i : 1 \leqslant i \leqslant \left[\frac{n}{2}\right].$$

That is, the second inequalities in (52) also holds. This ends the proof of Lemma 3.11. □

LEMMA 3.12. *Let n be a positive integer. Then we have*

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{2i-1}-\frac{1}{2}\log n\right)=\eta, \tag{53}$$

and

$$\eta+\frac{1}{2}\log n<\sum_{i=1}^{n}\frac{1}{2i-1}\leqslant 1+\frac{1}{2}\log n,\ \forall n\geqslant 1, \tag{54}$$

where

$$\eta\triangleq\frac{1}{2}\gamma+\log 2=0.9817550130107103\cdots, \tag{55}$$

and

$$\gamma\triangleq\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i}-\log n\right)=0.5772156649015301\cdots$$

is the Euler's constant.

Proof. We define an auxiliary sequence as follows:

$$\{x_n\}_{n=1}^{\infty}:x_n\triangleq\sum_{i=1}^{n}\frac{1}{2i-1}-\frac{1}{2}\log n. \tag{56}$$

Then

$$x_{n+1}-x_n=\frac{1}{2n+1}-\frac{1}{2}\left[\log(n+1)-\log n\right]\triangleq\varphi(n),$$

where

$$\varphi:[1,\infty)\to\mathbb{R},\ \varphi(t)\triangleq\frac{1}{2t+1}-\frac{1}{2}\left[\log(t+1)-\log t\right].$$

Since

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}t}&=-\frac{2}{(2t+1)^2}-\frac{1}{2}\left(\frac{1}{t+1}-\frac{1}{t}\right)\\&=-\frac{2}{(2t+1)^2}+\frac{1}{2t(t+1)}\\&=\frac{1}{2t(t+1)(2t+1)^2}>0,\ \forall t\geqslant 1,\end{aligned}$$

the function $\varphi:[1,\infty)\to\mathbb{R}$ is strictly increasing. Hence

$$x_{n+1}-x_n=\varphi(n)<\lim_{n\to\infty}\varphi(n)=0,\ \forall n\geqslant 1.$$

That is, the sequence $\{x_n\}_{n=1}^{\infty}$ is strictly descending.

Since the sequence

$$\{y_n\}_{n=1}^{\infty}:y_n\triangleq\sum_{i=1}^{n}\frac{1}{i}-\log n$$

is convergent, and

$$\gamma \triangleq \lim_{n\to\infty} y_n = 0.5772156649015301\cdots,$$

where the γ is the Euler's constant, we see that there exists a constant c such that

$$y_n \geqslant c, \ \forall n \geqslant 1.$$

Hence

$$x_n = \sum_{i=1}^{n} \frac{1}{2i-1} - \frac{1}{2}\log n > \sum_{i=1}^{n} \frac{1}{2i} - \frac{1}{2}\log n = \frac{1}{2} y_n \geqslant \frac{1}{2} c, \ \forall n \geqslant 1.$$

That is, the sequence $\{x_n\}_{n=1}^{\infty}$ has a lower bound $c/2$.

Based on the above analysis, we see that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent. Since

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{2i-1} - \frac{1}{2}\log n \right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{2n} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{2i} - \frac{1}{2}\log n \right) \\
&= \lim_{n\to\infty} \left[y_{2n} + \log 2n - \frac{1}{2}(y_n + \log n) - \frac{1}{2}\log n \right] \\
&= \lim_{n\to\infty} \left(y_{2n} - \frac{1}{2} y_n + \log 2 \right) \\
&= \gamma - \frac{1}{2}\gamma + \log 2 \\
&= \frac{1}{2}\gamma + \log 2,
\end{aligned}$$

we get

$$\lim_{n\to\infty} x_n = \frac{1}{2}\gamma + \log 2 = \eta,$$

and

$$\eta = \lim_{n\to\infty} x_n < x_n = \sum_{i=1}^{n} \frac{1}{2i-1} - \frac{1}{2}\log n \leqslant x_1 = 1, \ \forall n \geqslant 1.$$

That is, (53) and (54) hold. This ends the proof of Lemma 3.12. □

The following calculations are based on the Mathematica software since which are very complex.

LEMMA 3.13. *For any positive integer n, we have*

$$\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} > -\frac{1}{2} - \frac{20n}{3} - \frac{16n^2}{15} + \frac{1}{1+n} + \frac{1}{2(-1+2n)} + \left(\frac{28}{5} + \frac{32n}{15} - \frac{1}{n+1} \right) \left(\eta + \frac{1}{2}\log n \right), \tag{57}$$

and

$$\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} \leqslant -\frac{1}{2}-\frac{20n}{3}-\frac{16n^2}{15}+\frac{1}{1+n}+\frac{1}{2(-1+2n)} + \left(\frac{28}{5}+\frac{32n}{15}-\frac{1}{n+1}\right)\left(1+\frac{1}{2}\log n\right), \tag{58}$$

where the $w_{i,j}$ *is defined by* (42) *and the* η *is defined by* (55).

Proof. From $j=n-i$, and

$$\begin{aligned} w_{i,j} &= -\frac{1}{4i^2-1}\left[1+\frac{16}{15}j(2j+3)\right]-\frac{4i+1}{2i+1}\times\frac{4j+1}{2j+1} \\ &= -\frac{1}{(2i+1)(2i-1)}\left[1+\frac{16}{15}(n-i)(2n+3-2i)\right]-\frac{4i+1}{2i+1}\times\frac{4n+1-4i}{2n+1-2i}, \end{aligned}$$

we see that there exist functions $A(n)$, $B(n)$, $C(n)$, $D(n)$ such that

$$w_{i,j} \equiv A(n)+\frac{B(n)}{2i-1}+\frac{C(n)}{2i+1}+\frac{D(n)}{2n+1-2i},\ \forall i\in\mathbb{R}. \tag{59}$$

By means of the command Limit[ ] of the Mathematica software, we get

$$A(n)=\lim_{i\to\infty} w_{i,j} = -\frac{68}{15}, \tag{60}$$

$$B(n)=\lim_{i\to\frac{1}{2}}(2i-1)w_{i,j} = \frac{1}{30}(1-16n-32n^2), \tag{61}$$

$$C(n)=\lim_{i\to-\frac{1}{2}}(2i+1)w_{i,j} = \frac{107}{30}+\frac{8n}{3}+\frac{16n^2}{15}-\frac{1}{2(n+1)}, \tag{62}$$

$$D(n)=\lim_{i\to n+\frac{1}{2}}(2n+1-2i)w_{i,j} = 2-\frac{1}{2(n+1)}. \tag{63}$$

Hence

$$C(n)+D(n)=\frac{167}{30}+\frac{8n}{3}+\frac{16n^2}{15}-\frac{1}{n+1}, \tag{64}$$

and

$$B(n)+C(n)+D(n)=\frac{28}{5}+\frac{32n}{15}-\frac{1}{n+1}. \tag{65}$$

By (59)–(65) and the command Expand[ ] of the Mathematica software, we get

$$\begin{aligned} &\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} \\ &= \sum_{i=1}^{n-1}\left[A(n)+\frac{B(n)}{2i-1}+\frac{C(n)}{2i+1}+\frac{D(n)}{2n+1-2i}\right] \\ &= (n-1)A(n)+B(n)\sum_{i=1}^{n-1}\frac{1}{2i-1}+C(n)\sum_{i=1}^{n-1}\frac{1}{2i+1}+D(n)\sum_{i=1}^{n-1}\frac{1}{2n+1-2i} \end{aligned}$$

$$
\begin{aligned}
&= (n-1)A(n) + B(n)\sum_{i=1}^{n-1}\frac{1}{2i-1} + C(n)\sum_{i=1}^{n-1}\frac{1}{2i+1} + D(n)\sum_{i=1}^{n-1}\frac{1}{2i+1} \\
&= (n-1)A(n) + B(n)\sum_{i=1}^{n-1}\frac{1}{2i-1} + [C(n)+D(n)]\sum_{i=1}^{n-1}\frac{1}{2i+1} \\
&= (n-1)A(n) + B(n)\left(\sum_{i=1}^{n}\frac{1}{2i-1} - \frac{1}{2n-1}\right) + [C(n)+D(n)]\left(\sum_{i=1}^{n}\frac{1}{2i-1} - 1\right) \\
&= (n-1)A(n) - \frac{B(n)}{2n-1} - [C(n)+D(n)] + [B(n)+C(n)+D(n)]\sum_{i=1}^{n}\frac{1}{2i-1} \\
&= -\frac{68(n-1)}{15} - \frac{1-16n-32n^2}{30(2n-1)} - \left(\frac{167}{30} + \frac{8n}{3} + \frac{16n^2}{15} - \frac{1}{n+1}\right) \\
&\quad + \left(\frac{28}{5} + \frac{32n}{15} - \frac{1}{n+1}\right)\sum_{i=1}^{n}\frac{1}{2i-1} \\
&= -\frac{1}{2} - \frac{20n}{3} - \frac{16n^2}{15} + \frac{1}{1+n} + \frac{1}{2(-1+2n)} + \left(\frac{28}{5} + \frac{32n}{15} - \frac{1}{n+1}\right)\sum_{i=1}^{n}\frac{1}{2i-1},
\end{aligned}
$$

that is,

$$
\begin{aligned}
\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} = &-\frac{1}{2} - \frac{20n}{3} - \frac{16n^2}{15} + \frac{1}{1+n} + \frac{1}{2(-1+2n)} \\
&+ \left(\frac{28}{5} + \frac{32n}{15} - \frac{1}{n+1}\right)\sum_{i=1}^{n}\frac{1}{2i-1}. \qquad (66)
\end{aligned}
$$

We remark here that, if $n=1$, then $\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} \triangleq 0$, and (66) also holds. Since

$$\frac{28}{5} + \frac{32n}{15} - \frac{1}{n+1} > 0,\ \forall n \geqslant 1,$$

by (66) and Lemma 3.12, we see that inequalities (57) and (58) hold. This ends the proof of Lemma 3.13. □

LEMMA 3.14. *For any positive integer n, we have*

$$\frac{32n(n+1)}{15\pi} < \sum_{i+j=n,\ i,j\geqslant 0} w_{i,j}u_iu_j \leqslant n(n+1), \qquad (67)$$

where the $w_{i,j}$ is defined by (42) and the sequence $\{u_n\}_{n=0}^{\infty}$ is defined by (34).

Proof. From

$$w_{i,j} = -\frac{1}{4i^2-1}\left[1 + \frac{16}{15}j(2j+3)\right] - \frac{4i+1}{2i+1} \times \frac{4j+1}{2j+1},$$

we see that

$$1 \leqslant i,j \leqslant n-1,\ n \geqslant 2 \Rightarrow w_{i,j} < 0. \qquad (68)$$

By Lemma 3.11 and (68), we have

$$u_n \sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} \leqslant \sum_{i+j=n,\ i,j\geqslant 1} w_{i,j}u_iu_j \leqslant \frac{2}{\pi}u_{n-1}\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j},\ \forall n\geqslant 1. \tag{69}$$

We remark here that, if $n=1$, then inequalities (69) can be rewritten as $0\leqslant 0\leqslant 0$, which also hold.

Since

$$\begin{aligned}
w_{0,n}+w_{n,0} &= 1+\frac{16}{15}n(2n+3)-\frac{4n+1}{2n+1}-\frac{1}{4n^2-1}-\frac{4n+1}{2n+1}\\
&= -3+\frac{16n}{5}+\frac{32n^2}{15}-\frac{1}{2(2n-1)}+\frac{5}{2(2n+1)}\\
&> 0,\ \forall n\geqslant 1,
\end{aligned}$$

we have,

$$w_{0,n}+w_{n,0} = -3+\frac{16n}{5}+\frac{32n^2}{15}-\frac{1}{2(2n-1)}+\frac{5}{2(2n+1)} > 0,\ \forall n\geqslant 1. \tag{70}$$

By (69), we have

$$\begin{aligned}
\sum_{i+j=n,\ i,j\geqslant 0} w_{i,j}u_iu_j &= (w_{0,n}+w_{n,0})\,u_0u_n+\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j}u_iu_j\\
&\geqslant (w_{0,n}+w_{n,0})\,u_n+u_n\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j}\\
&= u_n\sum_{i+j=n,\ i,j\geqslant 0} w_{i,j},
\end{aligned}$$

that is

$$\sum_{i+j=n,\ i,j\geqslant 0} w_{i,j}u_iu_j \geqslant u_n\sum_{i+j=n,\ i,j\geqslant 0} w_{i,j},\ \forall n\geqslant 1. \tag{71}$$

By (54), (70) and the command Expand[ ] of the Mathematica software, we have

$$\begin{aligned}
\sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} &= w_{0,n}+w_{n,0}+\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j}\\
&\geqslant -3+\frac{16n}{5}+\frac{32n^2}{15}-\frac{1}{2(2n-1)}+\frac{5}{2(2n+1)}\\
&\quad -\frac{1}{2}-\frac{20n}{3}-\frac{16n^2}{15}+\frac{1}{1+n}+\frac{1}{2(-1+2n)}\\
&\quad +\left(\frac{28}{5}+\frac{32n}{15}-\frac{1}{n+1}\right)\left(\eta+\frac{1}{2}\log n\right)\\
&= -\frac{7}{2}-\frac{52n}{15}+\frac{16n^2}{15}+\frac{1}{1+n}+\frac{5}{2(1+2n)}\\
&\quad +\left(\frac{28}{5}+\frac{32n}{15}-\frac{1}{n+1}\right)\left(\eta+\frac{1}{2}\log n\right),
\end{aligned}$$

that is,

$$\sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} \geqslant \varphi_1(n), \forall n \geqslant 1. \tag{72}$$

where

$$\varphi_1(t) \triangleq -\frac{7}{2} - \frac{52t}{15} + \frac{16t^2}{15} + \frac{1}{1+t} + \frac{5}{2(1+2t)} + \left(\frac{28}{5} + \frac{32t}{15} - \frac{1}{t+1}\right)\left(\eta + \frac{1}{2}\log t\right).$$

We define an auxiliary function φ_2 as follows:

$$\varphi_2 : [1,\infty) \to \mathbb{R},\ \varphi_2(t) \triangleq \frac{\varphi_1(t)}{t(t+1)}.$$

By means of the command Plot[ ] of the Mathematica software, we know that the graph of the function $\varphi_2(t),\ t \in [1,100]$, is depicted in Figure 1, and the graph of the function $\varphi_2(t^{-1}),\ t \in (0,1]$, is depicted in Figure 2.

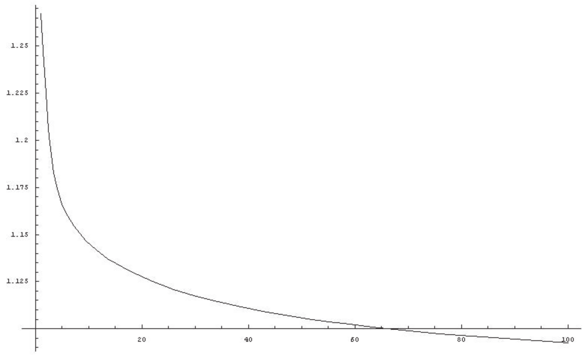

Figure 1: *The graph of the function* $\varphi_2(t),\ t \in [1,100]$.

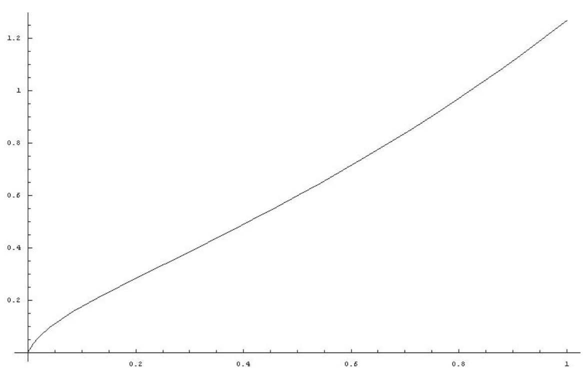

Figure 2: *The graph of the function* $\varphi_2(t^{-1}),\ t \in (0,1]$.

By means of the command Solve[ ] of the Mathematica software, we know that the equation

$$\frac{\mathrm{d}\varphi_2(t)}{\mathrm{d}t} = 0,\ t \in [1,\infty)$$

has no any real roots, and the function $\varphi_2 : [1,\infty)$ is strictly decreasing, so we get

$$\frac{\varphi_1(n)}{n(n+1)} = \varphi_2(n) > \lim_{n\to\infty} \varphi_2(n) = \frac{16}{15},\ \forall n \geqslant 1,$$

that is,

$$\varphi_1(n) > \frac{16}{15} n(n+1),\ \forall n \geqslant 1. \tag{73}$$

According to (71), (72), (73) and Lemma 3.10, we get

$$\begin{aligned}\sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} u_i u_j &\geqslant u_n \sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} \geqslant u_n \varphi_1(n) > u_n \times \frac{16}{15} n(n+1) \\ &> \frac{2}{\pi} \times \frac{16}{15} n(n+1) = \frac{32}{15\pi} n(n+1),\ \forall n \geqslant 1.\end{aligned}$$

This proves the first inequality in (67).

Next, we prove the second inequalities in (67) as follows.

By (69), we have

$$\begin{aligned}\sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} u_i u_j &= (w_{0,n} + w_{n,0}) u_n + \sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} u_i u_j \\ &\leqslant (w_{0,n} + w_{n,0}) u_n + \frac{2}{\pi} u_{n-1} \sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} \\ &= u_n \left[(w_{0,n} + w_{n,0}) + \frac{2}{\pi} \left(1 - \frac{1}{4n^2}\right)^{-1} \sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} \right],\end{aligned}$$

that is

$$\sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} u_i u_j \leqslant u_n \left[(w_{0,n} + w_{n,0}) + \frac{2}{\pi} \left(1 - \frac{1}{4n^2}\right)^{-1} \sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} \right]. \tag{74}$$

By (58) and (70), we have

$$\begin{aligned}&(w_{0,n} + w_{n,0}) + \frac{2}{\pi} \left(1 - \frac{1}{4n^2}\right)^{-1} \sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} \\ &\leqslant -3 + \frac{16n}{5} + \frac{32n^2}{15} - \frac{1}{2(2n-1)} + \frac{5}{2(2n+1)} + \frac{2}{\pi} \left(1 - \frac{1}{4n^2}\right)^{-1} \times \\ &\quad \left[-\frac{1}{2} - \frac{20n}{3} - \frac{16n^2}{15} + \frac{1}{1+n} + \frac{1}{2(-1+2n)} + \left(\frac{28}{5} + \frac{32n}{15} - \frac{1}{n+1}\right)\left(1 + \frac{1}{2}\log n\right) \right],\end{aligned}$$

that is,

$$(w_{0,n}+w_{n,0})+\frac{2}{\pi}\left(1-\frac{1}{4n^2}\right)^{-1}\sum_{i+j=n,\ i,j\geqslant 1} w_{i,j}\leqslant \varphi_3(n), \forall n\geqslant 1, \tag{75}$$

where

$$\varphi_3(t)\triangleq\left[-3+\frac{16t}{5}+\frac{32t^2}{15}-\frac{1}{2(2t-1)}+\frac{5}{2(2t+1)}\right]+\frac{2}{\pi}\left(1-\frac{1}{4t^2}\right)^{-1}\times$$
$$\left[-\frac{1}{2}-\frac{20t}{3}-\frac{16t^2}{15}+\frac{1}{1+t}+\frac{1}{2(-1+2t)}+\left(\frac{28}{5}+\frac{32t}{15}-\frac{1}{t+1}\right)\left(1+\frac{1}{2}\log t\right)\right].$$

We define a auxiliary sequence $\{z_n\}_{n=1}^{\infty}$ and a auxiliary function $\varphi_4(t)$ as follows:

$$\{z_n\}_{n=1}^{\infty}: z_n\triangleq\frac{u_n\varphi_3(n)}{n(n+1)},$$

and

$$\varphi_4:[1,\infty)\to\mathbb{R},\ \varphi_4(t)\triangleq\left(1-\frac{1}{4t^2}\right)\frac{t}{t+2}\frac{\varphi_3(t+1)}{\varphi_3(t)}.$$

By means of the command Plot[ ] of the Mathematica software, we know that the graph of the function $\varphi_4(t)$, $t\in[1,100]$, is depicted in Figure 3, and the graph of the function $\varphi_4(t^{-1})$, $t\in(0,1]$, is depicted in Figure 4.

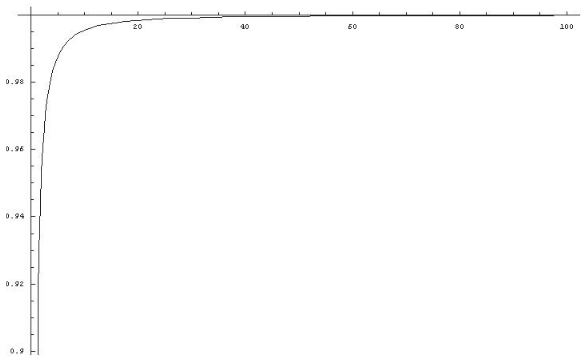

Figure 3: *The graph of the function* $\varphi_4(t)$, $t\in[1,100]$.

Since

$$\varphi_4(n)=\frac{z_{n+1}}{z_n},\ \forall n\in\mathbb{N}\triangleq\{1,2,\ldots,m,\ldots\}, \tag{76}$$

by means of the command Solve[ ] of the Mathematica software, we know that the equation

$$\varphi_4(t)=1,\ t\in[1,\infty)$$

has no any real roots, and

$$\varphi_4(t)<1,\ \forall t\in[1,\infty). \tag{77}$$

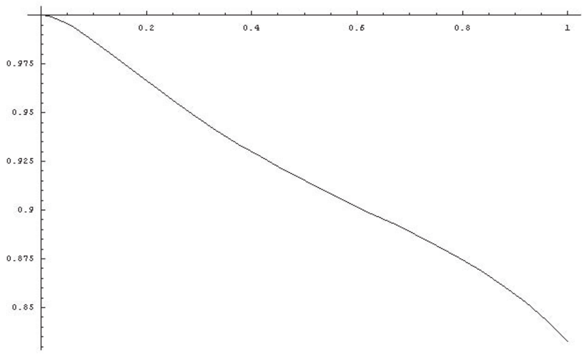

Figure 4: *The graph of the function* $\varphi_4(t^{-1})$, $t \in (0,1]$.

From (76) and (77), we see that

$$z_{n+1} < z_n, \ \forall n \in \mathbb{N}. \tag{78}$$

Hence the sequence $\{z_n\}_{n=1}^{\infty}$ is strictly decreasing. So we get

$$\frac{u_n \varphi_3(n)}{n(n+1)} = z_n \leqslant z_1 = 1, \ \forall n \in \mathbb{N},$$

and

$$\frac{u_n \varphi_3(n)}{n(n+1)} = z_n > \lim_{n\to\infty} z_n = \frac{64(\pi-1)}{15\pi^2}, \ \forall n \in \mathbb{N}.$$

Therefore,

$$0 < \frac{64(\pi-1)}{15\pi^2} n(n+1) < u_n \varphi_3(n) \leqslant n(n+1), \ \forall n \in \mathbb{N}. \tag{79}$$

According to (74), (75) and (79), we get

$$\begin{aligned} \sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} u_i u_j &\leqslant u_n \left[(w_{0,n} + w_{n,0}) + \frac{2}{\pi}\left(1 - \frac{1}{4n^2}\right)^{-1} \sum_{i+j=n,\ i,j\geqslant 1} w_{i,j} \right] \\ &\leqslant u_n \varphi_3(n) \leqslant n(n+1), \ \forall n \in \mathbb{N}. \end{aligned}$$

That is, the second inequalities in (67) also holds. The proof of Lemma 3.14 is completed. □

According to the theory of power series in mathematical analysis, we can easily get the following lemma.

LEMMA 3.15. *For any real number* $\mathrm{e} \in [0,1)$, *we have*

$$\sum_{n=1}^{\infty} n(n+1)\mathrm{e}^{2n} = \frac{2\mathrm{e}^2}{(1-\mathrm{e}^2)^3}, \tag{80}$$

and

$$\sum_{n=1}^{\infty} \mathrm{e}^{2n} = \frac{\mathrm{e}^2}{1-\mathrm{e}^2}. \tag{81}$$

LEMMA 3.16. *Under the hypotheses in Theorem 2.1, we have*

$$\frac{8}{\sqrt{15\pi}} \times \frac{\mathrm{e}}{(1-\mathrm{e}^2)^{3/2}} \times R_\Gamma^{-2} \leqslant \overline{\mathrm{Var}}\|\mathbf{F}\| \leqslant \sqrt{2} \times \frac{\mathrm{e}}{(1-\mathrm{e}^2)^{3/2}} \times R_\Gamma^{-2}. \tag{82}$$

Proof. By Lemmas 3.9, 3.14 and 3.15, we get

$$\begin{aligned}
\mathrm{Var}\|\mathbf{F}\| &= R_\Gamma^{-4} \sum_{n=1}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} u_i u_j \\
&\geqslant R_\Gamma^{-4} \sum_{n=1}^{\infty} \mathrm{e}^{2n} \frac{32n(n+1)}{15\pi} \\
&= R_\Gamma^{-4} \times \frac{32}{15\pi} \times \frac{2\mathrm{e}^2}{(1-\mathrm{e}^2)^3} \\
&= \frac{64}{15\pi} \times \frac{\mathrm{e}^2 R_\Gamma^{-4}}{(1-\mathrm{e}^2)^3},
\end{aligned}$$

and

$$\mathrm{Var}\|\mathbf{F}\| = R_\Gamma^{-4} \sum_{n=1}^{\infty} \mathrm{e}^{2n} \sum_{i+j=n,\ i,j\geqslant 0} w_{i,j} u_i u_j \leqslant R_\Gamma^{-4} \sum_{n=1}^{\infty} \mathrm{e}^{2n} n(n+1) = \frac{2\mathrm{e}^2 R_\Gamma^{-4}}{(1-\mathrm{e}^2)^3},$$

that is, (82) hold. This ends the proof of Lemma 3.16. □

LEMMA 3.17. *Under the hypotheses in Theorem 2.1, we have*

$$\left(1 + \frac{5}{4} \times \frac{\mathrm{e}^2}{1-\mathrm{e}^2}\right) \times R_\Gamma^{-2} \leqslant \overline{\|\mathbf{F}\|} \leqslant \left(1 + \frac{4}{\pi} \times \frac{\mathrm{e}^2}{1-\mathrm{e}^2}\right) \times R_\Gamma^{-2}. \tag{83}$$

Proof. By Lemmas 3.3, 3.7, 3.10, 3.15 and (51), we get

$$\begin{aligned}
\overline{\|\mathbf{F}\|} &= R_\Gamma^{-2} \left[\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1+\mathrm{e}^2\cos^2\theta}{(1-\mathrm{e}^2\cos^2\theta)^{3/2}} \mathrm{d}\theta\right] \\
&= R_\Gamma^{-2} \left(1 + \sum_{n=1}^{\infty} \frac{4n+1}{2n+1} u_n \mathrm{e}^{2n}\right) \\
&\geqslant R_\Gamma^{-2} \left(1 + \sum_{n=1}^{\infty} \frac{5}{4} \mathrm{e}^{2n}\right) \\
&= R_\Gamma^{-2} \left(1 + \frac{5}{4} \times \frac{\mathrm{e}^2}{1-\mathrm{e}^2}\right),
\end{aligned}$$

and

$$\overline{\|\mathbf{F}\|} = R_\Gamma^{-2}\left(1+\sum_{n=1}^{\infty}\frac{4n+1}{2n+1}u_n \mathrm{e}^{2n}\right) \leqslant R_\Gamma^{-2}\left(1+\sum_{n=1}^{\infty}\frac{4}{\pi}\mathrm{e}^{2n}\right) = R_\Gamma^{-2}\left(1+\frac{4}{\pi}\times\frac{\mathrm{e}^2}{1-\mathrm{e}^2}\right),$$

that is, (83) hold. The proof is completed. □

4. Proof of Theorem 2.1

Now we prove Theorem 2.1 as follows.

Proof. According Lemmas 3.16 and 3.17, we have

$$\begin{aligned}
\widetilde{\|\mathbf{F}\|} = \frac{\overline{\mathrm{Var}}\|\mathbf{F}\|}{\overline{\|\mathbf{F}\|}} &\geqslant \frac{\frac{8}{\sqrt{15\pi}}\frac{\mathrm{e}R_\Gamma^{-2}}{(1-\mathrm{e}^2)^{3/2}}}{R_\Gamma^{-2}\left(1+\frac{4}{\pi}\times\frac{\mathrm{e}^2}{1-\mathrm{e}^2}\right)} \\
&= \frac{8}{\sqrt{15\pi}}\times\frac{\mathrm{e}}{\left[1+\left(\frac{4}{\pi}-1\right)\mathrm{e}^2\right]\sqrt{1-\mathrm{e}^2}} \\
&\geqslant \frac{8}{\sqrt{15\pi}}\times\frac{\mathrm{e}}{\left[1+\left(\frac{4}{\pi}-1\right)\right]\sqrt{1-\mathrm{e}^2}} \\
&= \sqrt{\frac{4\pi}{15}}\times\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}},
\end{aligned}$$

and

$$\widetilde{\|\mathbf{F}\|} = \frac{\overline{\mathrm{Var}}\|\mathbf{F}\|}{\overline{\|\mathbf{F}\|}} \leqslant \frac{\sqrt{2}\times\frac{\mathrm{e}R_\Gamma^{-2}}{(1-\mathrm{e}^2)^{3/2}}}{R_\Gamma^{-2}\left(1+\frac{5}{4}\times\frac{\mathrm{e}^2}{1-\mathrm{e}^2}\right)} = \sqrt{2}\times\frac{\mathrm{e}}{\left(1+\frac{1}{4}\mathrm{e}^2\right)\sqrt{1-\mathrm{e}^2}} \leqslant \sqrt{2}\times\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}},$$

that is, inequalities (15) is proved.

By (20), (41) and (70), we have

$$\begin{aligned}
\lim_{\mathrm{e}\to 0}\widetilde{\|\mathbf{F}\|}\left(\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}\right)^{-1} &= \lim_{\mathrm{e}\to 0}\frac{\overline{\mathrm{Var}}\|\mathbf{F}\|}{\overline{\|\mathbf{F}\|}}\left(\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}\right)^{-1} = \lim_{\mathrm{e}\to 0}\frac{\overline{\mathrm{Var}}\|\mathbf{F}\|}{\overline{\|\mathbf{F}\|}}\mathrm{e}^{-1} \\
&= \frac{\lim_{\mathrm{e}\to 0}\mathrm{e}^{-1}\overline{\mathrm{Var}}\|\mathbf{F}\|}{\lim_{\mathrm{e}\to 0}\overline{\|\mathbf{F}\|}} = \frac{\lim_{\mathrm{e}\to 0}\mathrm{e}^{-1}\overline{\mathrm{Var}}\|\mathbf{F}\|}{R_\Gamma^{-2}} \\
&= \frac{\lim_{\mathrm{e}\to 0}\mathrm{e}^{-1}\sqrt{R_\Gamma^{-4}\sum_{n=1}^{\infty}\mathrm{e}^{2n}\sum_{i+j=n,\ i,j\geqslant 0}w_{i,j}u_iu_j}}{R_\Gamma^{-2}} \\
&= \lim_{\mathrm{e}\to 0}\sqrt{\sum_{n=1}^{\infty}\mathrm{e}^{2n-2}\sum_{i+j=n,\ i,j\geqslant 0}w_{i,j}u_iu_j} \\
&= \sqrt{\sum_{i+j=1,\ i,j\geqslant 0}w_{i,j}u_iu_j} = \sqrt{(w_{1,0}+w_{0,1})u_1}
\end{aligned}$$

$$= \sqrt{\left[-3+\frac{16n}{5}+\frac{32n^2}{15}-\frac{1}{2(2n-1)}+\frac{5}{2(2n+1)}\right]_{n=1}\times\frac{3}{4}}$$
$$= \sqrt{2}.$$

Therefore, the coefficient $\sqrt{2}$ of $\mathrm{e}/\sqrt{1-\mathrm{e}^2}$ in (15) is the best constant. This completes the proof of Theorem 2.1. □

5. Applications

Let $S^{(2)}\{P,\Gamma\}$ be a centered surround system, where the Γ is an ellipse and the P is one of the foci of the ellipse. Then we may also think that the $A\in\Gamma$ as a planet (such as the Mercury, Venus, Earth, etc.) and P as the Sun, and the ellipse Γ as the motion trajectory of the planet. Assume that the radiation energy of the Sun P to the planet A is 1, then, according to the optical laws, the radiant energy received by the planet A is $C/\|A-P\|^2 = C\|\mathbf{F}\|$, and the reflect radiation energy of the planet received by the Sun P is $C/\|A-P\|^4 = C\|\mathbf{F}_4\|$, where the $C>0$ is a constant of the radiation energy. This is the another significance of the 4-gravity $\mathbf{F}_4$ in space science.

Suppose that the planet A is regarded as a particle, and the temperature on the planet A at a certain moment is $T=T(A)$, and the mean temperature on the planet is $\overline{T}$. Then, based on the above analysis, there exists constant $C^*>0$ such that

$$T:\Gamma\to(0,\infty),\ T=C^*\|\mathbf{F}\|. \tag{84}$$

Without loss of generality, here we assume that $C^*=1$.

In the centered surround system $S^{(2)}\{P,\Gamma\}$, we may also think that $T=\|\mathbf{F}\|$ as a random variable which follows a uniform distribution, that is , its probability density function is $p:\Gamma\to(0,\infty),\ p=1/|\Gamma|$. Then, by (14) and (84), we have

$$\mathrm{E}T=\overline{\|\mathbf{F}\|},\ \overline{\mathrm{Var}}T=\sqrt{\mathrm{Var}\,\|\mathbf{F}\|}\text{ and }\widetilde{T}=\frac{\overline{\mathrm{Var}}T}{\mathrm{E}T}=\widetilde{\|\mathbf{F}\|}. \tag{85}$$

According to Theorem 2.1 and (85), we have

$$\sqrt{\frac{4\pi}{15}}\times\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}\leqslant\widetilde{T}\leqslant\sqrt{2}\times\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}. \tag{86}$$

By (86), we know that there exists real function $\chi(\mathrm{e})$ such that

$$\widetilde{T}=\chi(\mathrm{e})\times\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}, \tag{87}$$

where

$$0.9152912328637689\cdots=\sqrt{\frac{4\pi}{15}}\leqslant\chi(\mathrm{e})\leqslant\sqrt{2}=1.4142135623730951\cdots. \tag{88}$$

Since the error

$$\sqrt{2}-\sqrt{\frac{4\pi}{15}}=0.4989223295093262\,5\cdots$$

is not very large, by (88), we see that

$$\chi(\mathrm{e}) \approx \frac{1}{2}\left(\sqrt{\frac{4\pi}{15}}+\sqrt{2}\right) = 1.164752397618432\cdots, \ \forall \mathrm{e} \in (0,1). \tag{89}$$

According to (87) and (89), we obtain the following *approximate temperature coefficient of variation formula*:

$$\widetilde{T} \approx 1.164752397618432\cdots \times \frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}, \tag{90}$$

where $\widetilde{T}$ is called *temperature coefficient of variation*.

Similarly, according to Lemma 3.16, we have

$$\frac{8}{\sqrt{15\pi}} \times \frac{\mathrm{e}}{(1-\mathrm{e}^2)^{3/2}} \times R_\Gamma^{-2} \leqslant \overline{\mathrm{Var}} T \leqslant \sqrt{2} \times \frac{\mathrm{e}}{(1-\mathrm{e}^2)^{3/2}} \times R_\Gamma^{-2}. \tag{91}$$

Since

$$\begin{aligned}\sqrt{2} - \frac{8}{\sqrt{15\pi}} &= 1.4142135623730951\cdots - 1.1653849926315512\cdots \\ &= 0.248828569741544\cdots\end{aligned}$$

is very small, and

$$\frac{1}{2}\left(\sqrt{2}+\frac{8}{\sqrt{15\pi}}\right) = 1.2897992775023233\cdots,$$

we have the following *approximate temperature mean variance formula*:

$$\overline{\mathrm{Var}} T \approx 1.2897992775023233\cdots \times \frac{\mathrm{e}}{(1-\mathrm{e}^2)^{3/2}} \times R_\Gamma^{-2}, \tag{92}$$

where $R_\Gamma \triangleq \sqrt{a|\Gamma|/(2\pi)} \in [b,a]$ is a quasi-radius of the ellipse Γ, and $\overline{\mathrm{Var}} T$ is called *temperature mean variance*.

Based on (90) and (92), we know that Theorem 2.1 and Lemma 3.16 can be used to study the temperature change on a planet and the climate change on the Earth.

Competing interests. The authors declare that they have no conflicts of interest in this joint work.

Authors' contributions. All authors contributed equally and significantly in this paper. All authors read and approved the final manuscript.

Acknowledgements. The authors would like to acknowledge the support from the National Natural Science Foundation of China (No. 11161024).

REFERENCES

[1] A. ENCISO, D. POYATO AND J. SOLER, *Stability Results, Almost Global Generalized Beltrami Fields and Applications to Vortex Structures in the Euler Equations*, Comm. Math. Phys. **360** (1) (2018), 197–269.

[2] P. ZHAO, *Quantum variance of Maass-Hecke cusp forms*, Comm. Math. Phys. **297** (2) (2010), 475–514.

[3] J. J. WEN, T. Y. HAN AND S. S. CHENG, *Inequalities involving Dresher variance mean*, J. Inequal. Appl., **2013**: 366, 29 pages.

[4] J. J. WEN, Y. HUANG AND S. S. CHENG, *Theory of ϕ-Jensen variance and its applications in higher education*, J. Inequal. Appl., **2015**: 270 (2015), 40 pages.

[5] W. MENDENHALL AND T. SINCICH, *Statistics for Engineers and the Sciences* (Fifth Edition), Upper Saddle River, New Jersey: Pearson Prentice Hall, 2006.

[6] C. F. PAGE, *Allocation proportional to coefficients of variation when estimating the product of parameters*, J. Amer. Statist. Assoc., **85** (412) (1990), 1134–1139.

[7] B. FLEURY, O. GUÉDON AND G. PAOURIS, *A stability result for mean width of Lp-centroid bodies*, Adv. Math. **214** (2) (2007), 865–877.

[8] VAN DE MOORTEL, MAXIME, *Stability and Instability of the Subextremal Reissner-Nordström Black Hole Interior for the Einstein-Maxwell-Klein-Gordon Equations in Spherical Symmetry*, Comm. Math. Phys. **360** (1) (2018), 103–168.

[9] B. DUCOMET AND E. FEIREISL, *The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars*, Comm. Math. Phys. **266** (3) (2006), 595–629.

[10] C. D. SIJOY AND S. CHATURVEDI, *Combining node-centered parallel radiation transport and higher-order multi-material cell-centered hydrodynamics methods in three-temperature radiation hydrodynamics code TRHD*, Comput. Phys. Commun. **203** (2016), 94–109.

[11] R. BRUNETTI, K. FREDENHAGEN AND K. REJZNER, *Quantum gravity from the point of view of locally covariant quantum field theory*, Comm. Math. Phys. **345** (3) (2016), 741–779.

[12] J. J. WEN AND C. B. GAO, *Geometric inequalities involving the central distance of the centered 2-surround system*, Acta. Math. Sinica. **51** (4) (2008), 815–832. (in Chinese)

[13] C. B. GAO AND J. J. WEN, *Theory of surround system and associated inequalities*, Comput. Math. Appl. **63** (12) (2012), 1621–1640.

[14] C. B. GAO AND J. J. WEN, *Theories and inequalities on the satellite system*, ISRN Math. Anal., **2011**, Article ID 909261, 22 pages.

[15] J. J. WEN, J. YAN AND S. H. WU, *Isoperimetric inequalities in surround system and space science*, J. Inequal. Appl., **2016**: 74 (2016), 28 pages.

[16] J. J. WEN, J. YUAN, S. H. WU AND T. Y. HAN, *Gravity inequalities and the mean temperature on a planet*, J. Inequal. Appl., **2016**: 264. 18 pages.

[17] J. J. WEN, S. H. WU, J. YUAN AND T. Y. HAN, *Mean central distance–central distance inequalities*, J. Math. Inequal., **11** (4) (2017), 1131–1149.

[18] J. J. WEN, T. Y. HAN AND C. B. GAO, *Convergence tests on constant Dirichlet series*, Comput. Math. Appl., **62** (9) (2011), 3472–3489.

[19] D. KING-HELE, *Theory of satellite orbits in an atmosphere*, Butterworths, London, 1964.

[20] C. B. GAO AND J. J. WEN, *A dimensionality reduction principle on the optimization of function*, J. Math. Inequal., **7** (3) (2013), 357–375.

[21] J. J. WEN, S. H. WU AND C. B. GAO, *Sharp lower bounds involving circuit layout system*, J. Inequal. Appl., **2013**: 592 (2013), 22 pages.

[22] T. Y. HAN, S. H. WU AND J. J. WEN, *Sharp upper bound involving circuit layout system*, J. Nonlinear Sci. Appl. **9** (4) (2016), 1922–1935.

[23] F. R. K. CHUNG AND PRASAD TETALI, *Isoperimetric inequalities for cartesian products of graphs*, Combinatorics, Probability and Computing, **7** (2) (1998), 141–148.

[24] F. DELLA PIETRA AND N. GAVITONE, *The Relative isoperimetric inequality: the anisotropic case*, J. Convex Anal., **20** (1) (2013), 157–180.

[25] J. J. WEN AND W. L. WANG, *The inequalities involving generalized interpolation polynomial*, Comput. Math. Appl., **56** (4) (2008), 1045–1058.

[26] J. J. WEN AND S. S. CHENG, *Closed balls for interpolating quasi-polynomials*, Comput. Appl. Math., **30** (3) (2011), 545–570.

[27] Z. H. YANG, Y. M. CHU AND W. ZHANG, *Accurate approximations for the complete elliptic integral of the second kind*, J. Math. Anal. Appl. **438** (2) (2016), 875–888.

[28] J. J. WEN, J. YUAN AND S. F. YUAN, *An optimal version of an inequality involving the third symmetric means*, Proc. Indian Acad. Sci. Math. Sci., **118** (4) (2008), 505–516.

[29] J. J. WEN AND W. L. WANG, *The optimization for the inequalities of power means*, J. Inequal. Appl., volume **2006**, Article ID 46782, 25 pages.

[30] J. J. WEN AND W. L. WANG, *Chebyshev type inequalities involving permanents and their applications*, Linear Algebra Appl., **422** (1) (2007), 295–303.

[31] J. J. WEN, J. E. PEČARIĆ AND Y. H. TIAN, *Weak monotonicity and Chebyshev type inequality*, Math. Inequal. Appl., **18** (1) (2015), 217–231.

[32] J. J. WEN, C. B. GAO AND W. L. WANG, *Inequalities of J-P-S-F type*, J. Math. Inequal., **7** (2) (2013), 213–225.

[33] P. S. BULLEN, P. S. MITRINOVIĆ AND P. M. VASIĆ, *Means and their inequalities*, Reidel: Dordrecht/Boston/Lancaster/Toyyo, 1988.

(Received June 9, 2018)

Jiajin Wen
College of Mathematics and Computer Science
Chengdu University
Chengdu, 610106, P. R. China
e-mail: wenjiajin623@163.com

Tianyong Han
College of Mathematics and Computer Science
Chengdu University
Chengdu, 610106, P. R. China
e-mail: hantian123_123@163.com

Jun Yuan
School of Information Engineering
Nanjing Xiaozhuang University
Nanjing, 211171, P. R. China
e-mail: yuanjun_math@126.com