Journal of Mathematical Inequalities
Volume 14, Number 4 (2020), 1039–1054 doi:10.7153/jmi-2020-14-67

EQUIVALENT STATEMENTS OF A HILBERT–TYPE INTEGRAL INEQUALITY WITH THE EXTENDED HURWITZ ZETA FUNCTION IN THE WHOLE PLANE

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(*Communicated by Y.-H. Kim*)

Abstract. By using the way of real analysis and the weight functions, a few equivalent statements of a Hilbert-type integral inequality with the nonhomogeneous kernel in the whole plane are obtained. The constant factor related the extended Hurwitz zeta function is proved to be the best possible. As applications, a few equivalent statements of a Hilbert-type integral inequality with the homogeneous kernel in the whole plane are deduced. We also consider the operator expressions and some corollaries.

1. Introduction

Suppose that $f(x),\ g(y) \geqslant 0,\ 0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(y)dy < \infty$. We have the following Hilbert's integral inequality (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \tag{1}$$

with the best possible constant factor π. By means of the weight functions, some extensions of (1) were given by [2], [3]. A few Hilbert-type inequalities with the homogenous and nonhomogenous kernels were provided by [4]–[9]. In 2017, Hong [10] also gave two equivalent statements between a Hilbert-type inequalities with the homogenous kernel and parameters. Some other kinds of Hilbert-type inequalities were obtained by [11]–[15].

In 2007, Yang [16] gave a Hilbert-type integral inequality in the whole plane as follows:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda} dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \left(\int_{-\infty}^\infty e^{-\lambda x} f^2(x)dx \int_{-\infty}^\infty e^{-\lambda y} g^2(y)dy \right)^{\frac{1}{2}}, \tag{2}$$

Mathematics subject classification (2010): 26D15, 65B10.
Keywords and phrases: Hilbert-type integral inequality, weight function, equivalent statement, operator, Hurwitz zeta function.

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with the best possible constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ $(\lambda>0,\ B(u,v)$ is the beta function) (see [17]). He et al. [18]–[28] proved a few Hilbert-type integral inequalities in the whole plane with the best possible constant factors.

In this paper, by means of the way of real analysis and the weight functions, a few equivalent statements of a Hilbert-type integral inequality with the nonhomogeneous kernel in the whole plane similar to (2) are obtained. The constant factor related to the extended Hurwitz zeta function is proved to be the best possible. As applications, a few equivalent statements of a Hilbert-type integral inequality with the homogeneous kernel in the whole plane are deduced. We also consider the operator expressions and some corollaries.

2. An example and two lemmas

EXAMPLE 1. For $\gamma>0$, we set $H(u):=\frac{|\ln u|^{\gamma}(\min\{u,1\})^{\alpha+\beta}}{|u^{\lambda+\alpha}-1|(\max\{u,1\})^{\beta}}$ $(u>0)$, and for $a,b\neq 0$,

$$H(e^{ax+by})=\frac{|ax+by|^{\gamma}(\min\{e^{ax+by},1\})^{\alpha+\beta}}{|e^{(\lambda+\alpha)(ax+by)}-1|(\max\{e^{ax+by},1\})^{\beta}},\quad (x,y\in\mathbf{R}). \tag{3}$$

For $\sigma,\ \mu>-\alpha-\beta,\ \sigma+\mu=\lambda>-\alpha$, it follows that $H(v^{-1})v^{-1-\sigma}=H(v)v^{\mu-1}$ $(0<v<1)$, and

$$\begin{aligned}
k_{\lambda}(\sigma)&:=\int_0^{\infty}H(u)u^{\sigma-1}du=\int_0^1 H(u)(u^{\sigma-1}+u^{\mu-1})du\\
&=\int_0^1\frac{(-\ln u)^{\gamma}(\min\{u,1\})^{\alpha+\beta}}{(1-u^{\lambda+\alpha})(\max\{u,1\}^{\beta}}(u^{\sigma-1}+u^{\mu-1})du\\
&=\int_0^1\frac{(-\ln u)^{\gamma}}{1-u^{\lambda+\alpha}}(u^{\alpha+\beta+\sigma-1}+u^{\alpha+\beta+\mu-1})du\\
&=\int_0^1(-\ln u)^{\gamma}\sum_{k=0}^{\infty}u^{k(\lambda+\alpha)}(u^{\alpha+\beta+\sigma-1}+u^{\alpha+\beta+\mu-1})du.
\end{aligned}$$

By Lebesgue term by term integration theorem (cf. [29]), setting $v=-\ln u$, we find

$$\begin{aligned}
k_{\lambda}(\sigma)&=\sum_{k=0}^{\infty}\int_0^1(-\ln u)^{\gamma}[u^{k(\lambda+\alpha)+\alpha+\beta+\sigma-1}+u^{k(\lambda+\alpha)+\alpha+\beta+\mu-1}]du\\
&=\sum_{k=0}^{\infty}\int_0^{\infty}v^{\gamma}\{e^{-[k(\lambda+\alpha)+\alpha+\beta+\sigma]v}+e^{-[k(\lambda+\alpha)+\alpha+\beta+\mu]v}\}dv\\
&=\int_0^{\infty}\frac{t^{\gamma}}{e^t}dt\sum_{k=0}^{\infty}\left\{\frac{1}{[k(\lambda+\alpha)+\alpha+\beta+\sigma]^{\gamma+1}}+\frac{1}{[k(\lambda+\alpha)+\alpha+\beta+\mu]^{\gamma+1}}\right\}\\
&=\frac{\Gamma(\gamma+1)}{(\lambda+\alpha)^{\gamma+1}}\left(\zeta\left(\gamma+1,\frac{\alpha+\beta+\sigma}{\lambda+\alpha}\right)+\zeta\left(\gamma+1,\frac{\alpha+\beta+\mu}{\lambda+\alpha}\right)\right)\\
&\in\mathbf{R}_{+},
\end{aligned}\tag{4}$$

where, $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt$ $(\mathrm{Re}\, s > 0)$ is the gamma function, and

$$\zeta(s,c) := \sum_{k=0}^{\infty} \frac{1}{(k+c)^s} \ (\mathrm{Re}\, s > 1;\ c > 0)$$

is the extended Hurwitz zeta function (Note. for $0 < c \leqslant 1$, $\zeta(s,c)$ is called the Hurwitz zeta function (cf. [17])).

In particular, (i) for $\alpha = 0$, we have $\sigma,\ \mu > -\beta,\ \sigma + \mu = \lambda > 0,\ H_1(u) = \frac{|\ln u|^\gamma (\min\{u,1\})^\beta}{|u^\lambda - 1|(\max\{u,1\})^\beta}$ $(u > 0)$, and

$$k_\lambda^{(1)}(\sigma) = \frac{\Gamma(\gamma+1)}{\lambda^{\gamma+1}} \left(\zeta\left(\gamma+1, \frac{\beta+\sigma}{\lambda}\right) + \zeta\left(\gamma+1, \frac{\beta+\mu}{\lambda}\right)\right);$$

(ii) for $\beta = 0$, we have $\sigma,\ \mu > -\alpha,\ \sigma + \mu = \lambda > -\alpha,\ H_2(u) = \frac{|\ln u|^\gamma (\min\{u,1\})^\alpha}{|u^{\lambda+\alpha} - 1|}$ $(u > 0)$, and

$$k_\lambda^{(2)}(\sigma) = \frac{\Gamma(\gamma+1)}{(\lambda+\alpha)^{\gamma+1}} \left(\zeta\left(\gamma+1, \frac{\alpha+\sigma}{\lambda+\alpha}\right) + \zeta\left(\gamma+1, \frac{\alpha+\mu}{\lambda+\alpha}\right)\right);$$

(iii) for $\beta = -\alpha$, we have $\sigma,\ \mu > 0,\ \sigma + \mu = \lambda > -\alpha,\ H_3(u) = \frac{|\ln u|^\gamma (\max\{u,1\})^\alpha}{|u^{\lambda+\alpha} - 1|}$ $(u > 0)$, and

$$k_\lambda^{(3)}(\sigma) = \frac{\Gamma(\gamma+1)}{(\lambda+\alpha)^{\gamma+1}} \left(\zeta\left(\gamma+1, \frac{\sigma}{\lambda+\alpha}\right) + \zeta\left(\gamma+1, \frac{\mu}{\lambda+\alpha}\right)\right).$$

In (iii), for $\alpha = 0$, we have $\sigma,\ \mu > 0,\ \sigma + \mu = \lambda > 0, H_4(u) = \frac{|\ln u|^\gamma}{|u^\lambda - 1|}$ $(u > 0)$, and

$$k_\lambda^{(4)}(\sigma) = \frac{\Gamma(\gamma+1)}{\lambda^{\gamma+1}} \left(\zeta\left(\gamma+1, \frac{\sigma}{\lambda}\right) + \zeta\left(\gamma+1, \frac{\mu}{\lambda}\right)\right).$$

In the following, we assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b \neq 0$, σ_1, $\sigma \in \mathbf{R} = (-\infty, \infty)$, $\gamma > 0$, $\sigma,\ \mu > -\alpha - \beta,\ \sigma + \mu = \lambda > -\alpha$, and

$$\begin{aligned} K_\lambda(\sigma) &:= \frac{1}{|a|^{1/q}|b|^{1/p}} k_\lambda(\sigma) = \frac{\Gamma(\gamma+1)}{|a|^{1/q}|b|^{1/p}(\lambda+\alpha)^{\gamma+1}} \\ &\times \left(\zeta\left(\gamma+1, \frac{\alpha+\beta+\sigma}{\lambda+\alpha}\right) + \zeta\left(\gamma+1, \frac{\alpha+\beta+\mu}{\lambda+\alpha}\right)\right). \end{aligned} \tag{5}$$

For $n \in \mathbf{N} = \{1, 2, \ldots\}$, we define $E_c := \{t \in \mathbf{R}; ct \geqslant 0\}$, $F_c := \{t \in \mathbf{R}; ct < 0\}$ $(c = a, b)$, and the following two expressions:

$$I_1 := \int_{F_b} e^{(\sigma_1 + \frac{1}{qn})by} \left[\int_{E_a} H(e^{ax+by}) e^{(\sigma - \frac{1}{pn})ax} dx\right] dy, \tag{6}$$

$$I_2 := \int_{E_b} e^{(\sigma_1 - \frac{1}{qn})by} \left[\int_{F_a} H(e^{ax+by}) e^{(\sigma + \frac{1}{pn})ax} dx \right] dy. \tag{7}$$

Setting $u = e^{ax+by}$ in (6), by Fubini theorem (cf. [29]), it follows that

$$\begin{aligned} I_1 &= \frac{1}{|a|} \int_{F_b} e^{(\sigma_1 - \sigma + \frac{1}{n})by} \left(\int_{e^{by}}^{\infty} H(u) u^{\sigma - \frac{1}{pn} - 1} du \right) dy \\ &= \frac{1}{|ab|} \int_0^1 v^{\sigma_1 - \sigma + \frac{1}{n} - 1} \left(\int_v^{\infty} H(u) u^{\sigma - \frac{1}{pn} - 1} du \right) dv \ \ (v = e^{by}). \end{aligned} \tag{8}$$

In the same way, we find that

$$\begin{aligned} I_2 &= \frac{1}{|a|} \int_{E_b} e^{(\sigma_1 - \sigma - \frac{1}{n})by} \left(\int_0^{e^{by}} H(u) u^{\sigma + \frac{1}{pn} - 1} du \right) dy \\ &= \frac{1}{|ab|} \int_1^{\infty} v^{\sigma_1 - \sigma - \frac{1}{n} - 1} \left(\int_0^v H(u) u^{\sigma + \frac{1}{pn} - 1} du \right) dv. \end{aligned} \tag{9}$$

LEMMA 1. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, the following inequality*

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(e^{ax+by}) f(x) g(y) dx dy \\ &\leqslant M \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma_1 by}} \right)^q dy \right]^{\frac{1}{q}} \end{aligned} \tag{10}$$

holds true, then we have $\sigma_1 = \sigma$.

Proof. If $\sigma_1 < \sigma$, then for $n > \frac{1}{\sigma - \sigma_1}$ $(n \in \mathbf{N})$, we set two functions

$$f_n(x) := \begin{cases} e^{(\sigma - \frac{1}{pn})ax}, & x \in E_a \\ 0, & x \in F_a \end{cases}, \quad g_n(y) := \begin{cases} 0, & y \in E_b \\ e^{(\sigma_1 + \frac{1}{qn})by}, & y \in F_b \end{cases},$$

and find

$$\begin{aligned} J_2 &:= \left(\int_{-\infty}^{\infty} e^{-p\sigma ax} f_n^p(x) dx \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} e^{-q\sigma_1 by} g_n^q(y) dy \right)^{\frac{1}{q}} \\ &= \left(\int_{E_a} e^{-\frac{ax}{n}} dx \right)^{\frac{1}{p}} \left(\int_{F_b} e^{\frac{by}{n}} dy \right)^{\frac{1}{q}} = \frac{n}{|a|^{1/p} |b|^{1/q}}. \end{aligned}$$

By (6), (8) and (10), we have

$$\begin{aligned} &\frac{1}{|ab|} \int_0^1 v^{\sigma_1 - \sigma + \frac{1}{n} - 1} dv \int_1^{\infty} H(u) u^{\sigma - \frac{1}{pn} - 1} du \\ &\leqslant I_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(e^{ax+by}) f_n(x) g_n(y) dx dy \leqslant M J_2 = \frac{Mn}{|a|^{1/p} |b|^{1/q}}. \end{aligned} \tag{11}$$

Since for any $n > \frac{1}{\sigma-\sigma_1}$ $(n \in \mathbf{N})$, $\sigma_1 - \sigma + \frac{1}{n} < 0$, it follows that $\int_0^1 v^{\sigma_1-\sigma+\frac{1}{n}-1} dv = \infty$. In view of $\int_1^\infty H(u) u^{\sigma-\frac{1}{pn}-1} du > 0$, by (11), we find that $\infty \leqslant \frac{Mn}{|a|^{1/p}|b|^{1/q}} < \infty$, which is a contradiction.

If $\sigma_1 > \sigma$, then for $n > \frac{1}{\sigma_1-\sigma}$ $(n \in \mathbf{N})$, we set functions

$$\widetilde{f}_n(x) := \begin{cases} 0, & x \in E_a \\ e^{(\sigma+\frac{1}{pn})ax}, & x \in F_a \end{cases}, \quad \widetilde{g}_n(y) := \begin{cases} e^{(\sigma_1-\frac{1}{qn})by}, & y \in E_b \\ 0, & y \in F_b \end{cases},$$

and find

$$\begin{aligned} \widetilde{J}_2 &:= \left(\int_{-\infty}^{\infty} e^{-p\sigma ax} \widetilde{f}_n^p(x) dx \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} e^{-q\sigma_1 by} \widetilde{g}_n^q(y) dy \right)^{\frac{1}{q}} \\ &= \left(\int_{F_a} e^{\frac{ax}{n}} dx \right)^{\frac{1}{p}} \left(\int_{E_b} e^{-\frac{by}{n}} dy \right)^{\frac{1}{q}} = \frac{n}{|a|^{1/p}|b|^{1/q}}. \end{aligned}$$

By (9) and (10), we have

$$\begin{aligned} &\frac{1}{|ab|} \int_1^\infty v^{\sigma_1-\sigma-\frac{1}{n}-1} dv \int_0^1 H(u) u^{\sigma+\frac{1}{pn}-1} du \\ &\leqslant I_2 = \int_0^\infty \int_0^\infty H(e^{ax+by}) \widetilde{f}_n(x) \widetilde{g}_n(y) dx dy \leqslant M \widetilde{J}_2 = \frac{Mn}{|a|^{1/p}|b|^{1/q}}. \end{aligned} \tag{12}$$

Since for $n > \frac{1}{\sigma_1-\sigma}$ $(n \in \mathbf{N})$, $\sigma_1 - \sigma - \frac{1}{n} > 0$, it follows that $\int_1^\infty v^{\sigma_1-\sigma-\frac{1}{n}-1} dv = \infty$. By (12), in view of $\int_0^1 H(u) u^{\sigma+\frac{1}{pn}-1} du > 0$, we have $\infty \leqslant \frac{Mn}{|a|^{1/p}|b|^{1/q}} < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

The lemma is proved. □

For $\sigma_1 = \sigma$, we have

LEMMA 2. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, the following inequality*

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(e^{ax+by}) f(x) g(y) dx dy \\ &\leqslant M \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma by}} \right)^q dy \right]^{\frac{1}{q}} \end{aligned} \tag{13}$$

holds true, then we have $M \geqslant K_\lambda(\sigma) (> 0)$.

Proof. By (8), for $\sigma_1 = \sigma$, we obtain

$$\begin{aligned} I_1 &= \frac{1}{|ab|}\int_0^1 v^{\frac{1}{n}-1}\left(\int_v^1 H(u)u^{\sigma-\frac{1}{pn}-1}du\right)dv \\ &+\frac{1}{|ab|}\int_0^1 v^{\frac{1}{n}-1}\left(\int_1^\infty H(u)u^{\sigma-\frac{1}{pn}-1}du\right)dv \\ &= \frac{1}{|ab|}\int_0^1\left(\int_0^u v^{\frac{1}{n}-1}dv\right)H(u)u^{\sigma-\frac{1}{pn}-1}du \\ &+\frac{n}{|ab|}\int_1^\infty H(u)u^{\sigma-\frac{1}{pn}-1}du \\ &= \frac{n}{|ab|}\left(\int_0^1 H(u)u^{\sigma+\frac{1}{qn}-1}du+\int_1^\infty H(u)u^{\sigma-\frac{1}{pn}-1}du\right). \end{aligned}$$

We use inequality $I_1 \leqslant M\widetilde{J_2}$ (when $\sigma_1 = \sigma$) as follows

$$\begin{aligned} \frac{1}{n}I_1 &= \frac{1}{|ab|}\left(\int_0^1 H(u)u^{\sigma+\frac{1}{qn}-1}du+\int_1^\infty H(u)u^{\sigma-\frac{1}{pn}-1}du\right) \\ &\leqslant \frac{M}{|a|^{1/p}|b|^{1/q}}. \end{aligned} \tag{14}$$

By Fatou lemma (cf. [29]) and (14), we find

$$\begin{aligned} K_\lambda(\sigma) &= \frac{1}{|a|^{1/q}|b|^{1/p}}\left(\int_0^1 \lim_{n\to\infty} H(u)u^{\sigma+\frac{1}{qn}-1}du+\int_1^\infty \lim_{n\to\infty} H(u)u^{\sigma-\frac{1}{pn}-1}du\right) \\ &\leqslant \underline{\lim}_{n\to\infty}\frac{|a|^{1/p}|b|^{1/q}}{n}I_1 \leqslant M. \end{aligned}$$

The lemma is proved. □

3. Main results and some corollaries

THEOREM 1. *If M is a constant, then the following statements (i), (ii) and (iii) are equivalent:*

(i) For any nonnegative measurable function $f(x)$ in **R**, *we have the following inequality:*

$$\begin{aligned} J &:= \left[\int_{-\infty}^\infty e^{p\sigma_1 by}\left(\int_{-\infty}^\infty H(e^{ax+by})f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ &\leqslant M\left[\int_{-\infty}^\infty \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}. \end{aligned} \tag{15}$$

(ii) For any nonnegative measurable functions $f(x)$ and $g(y)$ in **R**, *we have the*

following inequality:

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(e^{ax+by}) f(x)g(y)dxdy$$
$$\leqslant M\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma_1 by}}\right)^q dy\right]^{\frac{1}{q}}. \tag{16}$$

(iii) $\sigma_1 = \sigma$, *and* $M \geqslant K_\lambda(\sigma)(>0)$.

Proof. $(i) \implies (ii)$. By Hölder's inequality (see [30]), we have

$$I = \int_{-\infty}^{\infty}\left(e^{\sigma_1 by}\int_{-\infty}^{\infty} H(e^{ax+by}) f(x)dx\right)\left(e^{-\sigma_1 by} g(y)\right)dy$$
$$\leqslant J\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma_1 by}}\right)^q dy\right]^{\frac{1}{q}}. \tag{17}$$

Then by (15), we have (16).

$(ii) \implies (iii)$. By Lemma 1, we have $\sigma_1 = \sigma$. Then by Lemma 2, we have $M \geqslant K_\lambda(\sigma)(>0)$.

$(iii) \implies (i)$. Setting $u = e^{ax+by}$, we obtain the following weight functions: For $y, x \in \mathbf{R}$,

$$\omega(\sigma, y) := e^{\sigma by}\int_{-\infty}^{\infty} H(e^{ax+by}) e^{\sigma ax} dx$$
$$= \frac{1}{|a|}\int_0^{\infty} H(u)u^{\sigma-1}du = \frac{1}{|a|}k_\lambda(\sigma), \tag{18}$$
$$\varpi(\sigma, x) := e^{\sigma ax}\int_{-\infty}^{\infty} H(e^{ax+by}) e^{\sigma by} dy = \frac{1}{|b|}k_\lambda(\sigma). \tag{19}$$

By Hölder's inequality with weight and (18), we have

$$\left(\int_{-\infty}^{\infty} H(e^{ax+by}) f(x)dx\right)^p$$
$$= \left[\int_{-\infty}^{\infty} H(e^{ax+by})\left(\frac{e^{\sigma by/p}}{e^{\sigma ax/q}} f(x)\right)\left(\frac{e^{\sigma ax/q}}{e^{\sigma by/p}}\right)dx\right]^p$$
$$\leqslant \int_{-\infty}^{\infty} H(e^{ax+by})\frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x)dx\left(\int_{-\infty}^{\infty} H(e^{ax+by})\frac{e^{\sigma ax}}{e^{\sigma byq/p}}dx\right)^{p/q}$$
$$= \left[\omega(\sigma, y)e^{-q\sigma by}\right]^{p-1}\int_{-\infty}^{\infty} H(e^{ax+by})\frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x)dx$$
$$= \left(\frac{1}{|a|}k_\lambda(\sigma)\right)^{p-1} e^{-p\sigma by}\int_{-\infty}^{\infty} H(e^{ax+by})\frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x)dx. \tag{20}$$

For $\sigma_1 = \sigma$, by Fubini theorem (see [29]) and (19), we have

$$\begin{aligned} J &\leqslant \left(\frac{1}{|a|}k_\lambda(\sigma)\right)^{\frac{1}{q}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(e^{ax+by})\frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x)dxdy\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{|a|}k_\lambda(\sigma)\right)^{\frac{1}{q}} \left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} H(e^{ax+by})\frac{e^{\sigma by}}{e^{\sigma axp/q}}dy\right) f^p(x)dx\right]^{\frac{1}{p}} \\ &= \left(\frac{1}{|a|}k_\lambda(\sigma)\right)^{\frac{1}{q}} \left(\int_{-\infty}^{\infty} \omega(\sigma,x)e^{-p\sigma ax} f^p(x)dx\right)^{\frac{1}{p}} \\ &= K_\lambda(\sigma)\left(\int_{-\infty}^{\infty} e^{-p\sigma ax} f^p(x)dx\right)^{\frac{1}{p}}. \end{aligned}$$

For $K_\lambda(\sigma) \leqslant M$, we have (15).

Therefore, the statements (i), (ii) and (iii) are equivalent.

The theorem is proved. □

THEOREM 2. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geqslant 0$, *satisfying* $0 < \int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx < \infty$, *we have the following inequality:*

$$\begin{aligned} J_1 &= \left\{\int_{-\infty}^{\infty} e^{p\sigma by}\left[\int_{-\infty}^{\infty} \frac{|ax+by|^\gamma(\min\{e^{ax+by},1\})^{\alpha+\beta} f(x)}{|e^{(\lambda+\alpha)(ax+by)}-1|(\max\{e^{ax+by},1\})^\beta}dx\right]^p dy\right\}^{\frac{1}{p}} \\ &< K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}. \end{aligned} \tag{21}$$

(ii) For any $f(x) \geqslant 0$, *satisfying* $0 < \int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx < \infty$ *and* $g(y) \geqslant 0$, *satisfying* $0 < \int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy < \infty$, *we have the following inequality:*

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|ax+by|^\gamma(\min\{e^{ax+by},1\})^{\alpha+\beta}}{|e^{(\lambda+\alpha)(ax+by)}-1|(\max\{e^{ax+by},1\})^\beta} f(x)g(y)dxdy \\ &< K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy\right]^{\frac{1}{q}}. \end{aligned} \tag{22}$$

Moreover, the constant factor $K_\lambda(\sigma)$ *in* (21) *and* (22) *is the best possible.*

In particular, for $\alpha = \beta = 0$, σ, $\mu > 0$, $\sigma + \mu = \lambda$,

$$\widetilde{K}_\lambda(\sigma) := \frac{\Gamma(\gamma+1)}{|a|^{1/q}|b|^{1/p}\lambda^{\gamma+1}}\left(\zeta\left(\gamma+1,\frac{\sigma}{\lambda}\right)+\zeta\left(\gamma+1,\frac{\mu}{\lambda}\right)\right), \tag{23}$$

we have the following equivalent inequalities with the best possible constant factor

$\widetilde{K}_\lambda(\sigma)$:

$$\left\{\int_{-\infty}^{\infty} e^{p\sigma by}\left[\int_{-\infty}^{\infty}\frac{|ax+by|^\gamma f(x)}{|e^{\lambda(ax+by)}-1|}dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$< \widetilde{K}_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}, \tag{24}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|ax+by|^\gamma}{|e^{\lambda(ax+by)}-1|}f(x)g(y)dxdy$$
$$< \widetilde{K}_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy\right]^{\frac{1}{q}}. \tag{25}$$

Proof. We first prove that (21) is valid. If (20) takes the form of equality for a $y \in \mathbf{R}$, then (see [30]), there exists constants A and B, such that they are not all zero, and
$$A\frac{e^{\sigma by}}{e^{\sigma axp/q}}f^p(x) = B\frac{e^{\sigma ax}}{e^{\sigma byq/p}}\ a.e. \text{ in } \mathbf{R}.$$
We suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that
$$\left(\frac{f(x)}{e^{\sigma ax}}\right)^p = e^{-q\sigma by}\frac{B}{A}\ a.e. \text{ in } \mathbf{R},$$
which contradicts the fact that $0 < \int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx < \infty$. Hence, (20) takes the form of strict inequality. For $\sigma_1 = \sigma$ by the proof of Theorem 1, we obtain (21).

$(i) \implies (ii)$. By (17) (for $\sigma_1 = \sigma$) and (21), we have (22).

$(ii) \implies (i)$. We set the following function:
$$g(y) := e^{p\sigma by}\left(\int_{-\infty}^{\infty} H(e^{ax+by})f(x)dx\right)^{p-1}\ (y \in \mathbf{R}).$$
If $J_1 = \infty$, then it is impossible since (21) is valid; if $J_1 = 0$, then (21) is trivially valid. In the following, we suppose that $0 < J_1 < \infty$. By (22), we have
$$0 < \int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy = J_1^p = I$$
$$< K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy\right]^{\frac{1}{q}} < \infty,$$
$$J_1 = \left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy\right]^{\frac{1}{p}} < K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}},$$
namely, (21) follows, which is equivalent to (22).

Hence, statements (i) and (ii) are valid and equivalent.

If there exists a constant $M \leqslant K_{\lambda}(\sigma)$, such that (22) is valid when replacing $K_{\lambda}(\sigma)$ by M, then by Lemma 2, we have $K_{\lambda}(\sigma) \leqslant M$. Hence, the constant factor $M = K_{\lambda}(\sigma)$ in (22) is the best possible.

The constant factor $K_{\lambda}(\sigma)$ in (21) is still the best possible. Otherwise, by (17) (for $\sigma_1 = \sigma$), we would reach a contradiction that the constant factor $K_{\lambda}(\sigma)$ in (22) is not the best possible.

The theorem is proved. $\square$

For $g(y) = e^{-\lambda by}G(y)$, and $\mu_1 = \lambda - \sigma_1$ in Theorem 1 and Theorem 2, then replacing b (rep. $G(y)$) by $-b$ (rep. $g(y)$), setting

$$K_{\lambda}(e^{ax}, e^{by}) := \frac{|ax - by|^{\gamma}(\min\{e^{ax}, e^{by}\})^{\alpha+\beta}}{|e^{(\lambda+\alpha)ax} - e^{(\lambda+\alpha)by}|(\max\{e^{ax}, e^{by}\})^{\beta}} \quad (x, y \in \mathbf{R}), \tag{26}$$

we have the following corollaries:

COROLLARY 1. *If M is a constant, then the following statements (i), (ii) and (iii) are equivalent:*

(i) For any nonnegative measurable function $f(x)$ in $\mathbf{R}$, we have the following inequality:

$$\left[\int_{-\infty}^{\infty} e^{p\mu_1 by}\left(\int_{-\infty}^{\infty} K_{\lambda}(e^{ax}, e^{by}) f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$\leqslant M\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}. \tag{27}$$

(ii) For any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\lambda}(e^{ax}, e^{by}) f(x)g(y)dxdy$$
$$\leqslant M\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\mu_1 by}}\right)^q dy\right]^{\frac{1}{q}}. \tag{28}$$

(iii) $\mu_1 = \mu$, and $M \geqslant K_{\lambda}(\sigma)(> 0)$.

Proof. Replacing b by $-b$, we obtain

$$\begin{aligned}
H(e^{ax-by}) &= \frac{|ax - by|^{\gamma}(\min\{e^{ax-by}, 1\})^{\alpha+\beta}}{|e^{(\lambda+\alpha)(ax-by)} - 1|(\max\{e^{ax-by}, 1\})^{\beta}} \\
&= \frac{|ax - by|^{\gamma}(\min\{e^{ax}, e^{by}\})^{\alpha+\beta} e^{-b(\alpha+\beta)y}}{e^{(\lambda+\alpha)(-by)}|e^{(\lambda+\alpha)ax} - e^{(\lambda+\alpha)by}|(\max\{e^{ax}, e^{by}\})^{\beta} e^{-b\beta y}} \\
&= \frac{|ax - by|^{\gamma}(\min\{e^{ax}, e^{by}\})^{\alpha+\beta} e^{-b(\alpha+\beta)y}}{e^{(\lambda+\alpha)(-by)}|e^{(\lambda+\alpha)ax} - e^{(\lambda+\alpha)by}|(\max\{e^{ax}, e^{by}\})^{\beta} e^{-b\beta y}} \\
&= e^{\lambda by} K_{\lambda}(e^{ax}, e^{by}).
\end{aligned}$$

For $\mu_1 = \lambda - \sigma_1$, we have

$$\left[\int_{-\infty}^{\infty} e^{-p\sigma_1 by}\left(\int_{-\infty}^{\infty} e^{\lambda by} K_\lambda(e^{ax},e^{by})f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$=\left[\int_{-\infty}^{\infty} e^{-p\sigma_1 by+p\lambda by}\left(\int_{-\infty}^{\infty} K_\lambda(e^{ax},e^{by})f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$=\left[\int_{-\infty}^{\infty} e^{p\mu_1 by}\left(\int_{-\infty}^{\infty} K_\lambda(e^{ax},e^{by})f(x)dx\right)^p dy\right]^{\frac{1}{p}}.$$

Then (15) (replacing b by $-b$) reduces to (27). Since for $\mu_1 = \lambda - \sigma_1$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(e^{ax-by})f(x)g(y)dxdy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\lambda by}K_\lambda(e^{ax},e^{by})f(x)g(y)dxdy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\lambda(e^{ax},e^{by})f(x)G(y)dxdy,$$

$$\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{-\sigma_1 by}}\right)^q dy=\int_{-\infty}^{\infty}\left(\frac{e^{-\lambda by}G(y)}{e^{-\sigma_1 by}}\right)^q dy=\int_{-\infty}^{\infty}\left(\frac{G(y)}{e^{\mu_1 by}}\right)^q dy,$$

then replace $G(y)$ by $g(y)$, (16) (replacing b by $-b$) reduces to (28).

Hence, by Theorem 1, we have Corollary 1.

The corollary is proved. □

COROLLARY 2. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geqslant 0$, *satisfying* $0 < \int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx < \infty$, *we have the following inequality:*

$$\left\{\int_{-\infty}^{\infty} e^{pb\mu y}\left[\int_{-\infty}^{\infty}\frac{|ax-by|^\gamma(\min\{e^{ax},e^{by}\})^{\alpha+\beta}f(x)}{|e^{(\lambda+\alpha)ax}-e^{(\lambda+\alpha)by}|(\max\{e^{ax},e^{by}\})^\beta}dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$< K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}. \tag{29}$$

(ii) For any $f(x) \geqslant 0$, *satisfying* $0 < \int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx < \infty$, *and* $g(y) \geqslant 0$, *satisfying* $0 < \int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\mu by}}\right)^q dy < \infty$, *we have the following inequality:*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|ax-by|^\gamma(\min\{e^{ax},e^{by}\})^{\alpha+\beta}f(x)g(y)}{|e^{(\lambda+\alpha)ax}-e^{(\lambda+\alpha)by}|(\max\{e^{ax},e^{by}\})^\beta}dxdy$$
$$< K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\mu by}}\right)^q dy\right]^{\frac{1}{q}}. \tag{30}$$

Moreover, the constant factor $K_\lambda(\sigma)$ in (29) and (30) is the best possible.

In particular, for $\alpha = \beta = 0$, σ, $\mu > 0$, we have the following equivalent inequalities with the best possible constant factor $\widetilde{K}_\lambda(\sigma)$:

$$\left[\int_{-\infty}^{\infty} e^{p\mu by}\left(\int_{-\infty}^{\infty} \frac{|ax-by|^\gamma}{|e^{\lambda ax}-e^{\lambda by}|} f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< \widetilde{K}_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}, \tag{31}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|ax-by|^\gamma}{|e^{\lambda ax}-e^{\lambda by}|} f(x)g(y)dxdy$$
$$< \widetilde{K}_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\mu by}}\right)^q dy\right]^{\frac{1}{q}}. \tag{32}$$

In (24) and (25), setting $F(x) = e^{\frac{\lambda a}{2}x} f(x)$, $G(y) = e^{\frac{\lambda b}{2}y} g(y)$, then replacing back $F(x)$ $(G(y))$ by $f(x)$ $(g(y))$, and introducing the hyperbolic sine function as $\sinh(s) = \frac{e^s - e^{-s}}{2}$, we have

COROLLARY 3. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geqslant 0$, satisfying $0 < \int_{-\infty}^{\infty}\left[e^{(\frac{\lambda}{2}-\sigma)ax} f(x)\right]^p dx < \infty$, we have the following inequality:

$$\left\{\int_{-\infty}^{\infty} e^{p(\sigma-\frac{\lambda}{2})by}\left[\int_{-\infty}^{\infty} \frac{|ax+by|^\gamma}{|\sinh(\frac{\lambda(ax+by)}{2})|} f(x)dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$< 2\widetilde{K}_\lambda(\sigma)\left\{\int_{-\infty}^{\infty}\left[e^{(\frac{\lambda}{2}-\sigma)ax} f(x)\right]^p dx\right\}^{\frac{1}{p}}. \tag{33}$$

(ii) For any $f(x) \geqslant 0$, satisfying $0 < \int_{-\infty}^{\infty}\left[e^{(\frac{\lambda}{2}-\sigma)ax} f(x)\right]^p dx < \infty$ and $g(y) \geqslant 0$, satisfying $0 < \int_{-\infty}^{\infty}\left[e^{(\frac{\lambda}{2}-\sigma)by} g(y)\right]^q dy < \infty$, we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|ax+by|^\gamma}{|\sinh(\frac{\lambda(ax+by)}{2})|} f(x)g(y)dxdy$$
$$< 2\widetilde{K}_\lambda(\sigma)\left\{\int_{-\infty}^{\infty}\left[e^{(\frac{\lambda}{2}-\sigma)ax} f(x)\right]^p dx\right\}^{\frac{1}{p}}\left\{\int_{-\infty}^{\infty}\left[e^{(\frac{\lambda}{2}-\sigma)by} g(y)\right]^q dy\right\}^{\frac{1}{q}}. \tag{34}$$

Moreover, the constant factor $2\widetilde{K}_\lambda(\sigma)$ in (33) and (34) is the best possible.

4. Operator expressions

We set the following functions: $\varphi(x) := e^{-p\sigma ax}$, $\psi(y) := e^{-q\sigma by}$, $\phi(y) := e^{-q\mu by}$, wherefrom, $\psi^{1-p}(y) = e^{p\sigma by}$, $\phi^{1-p}(y) = e^{p\mu by}$ $(x, y \in \mathbf{R})$, and define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}) := \left\{ f : ||f||_{p,\varphi} := \left(\int_{-\infty}^{\infty} \varphi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

wherefrom,

$$L_{q,\psi}(\mathbf{R}) = \left\{ g : ||g||_{q,\psi} = \left(\int_{-\infty}^{\infty} \psi(y)|g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{q,\phi}(\mathbf{R}) = \left\{ g : ||g||_{q,\phi} = \left(\int_{-\infty}^{\infty} \phi(y)|g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{p,\psi^{1-p}}(\mathbf{R}) = \left\{ h : ||h||_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\phi^{1-p}}(\mathbf{R}) = \left\{ h : ||h||_{p,\phi^{1-p}} = \left(\int_{-\infty}^{\infty} \phi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}.$$

(a) In view of Theorem 2, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$H_1(y) := \int_{-\infty}^{\infty} H(e^{ax+by}) f(x) dx \, (y \in \mathbf{R}),$$

by (21), we have

$$||H_1||_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y) H_1^p(y) dy \right)^{\frac{1}{p}} \leqslant K_\lambda(\sigma) ||f||_{p,\varphi} < \infty. \tag{35}$$

DEFINITION 1. Define a Hilbert-type integral operator with the nonhomogeneous kernel $T^{(1)} : L_{p,\varphi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T^{(1)} f = H_1 \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying for any $y \in \mathbf{R}$, $T^{(1)} f(y) = H_1(y)$.

In view of (35), it follows that

$$||T^{(1)} f||_{p,\psi^{1-p}} = ||H_1||_{p,\psi^{1-p}} \leqslant K_\lambda(\sigma) ||f||_{p,\varphi},$$

and then the operator $T^{(1)}$ is bounded satisfying

$$||T^{(1)}|| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R})} \frac{||T^{(1)} f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leqslant K_\lambda(\sigma).$$

If we define the formal inner product of $T^{(1)}f$ and g as follows:

$$(T^{(1)}f,g) := \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} H(e^{ax+by})f(x)dx\right)g(y)dy,$$

then we can rewrite Theorem 2 as follows:

THEOREM 3. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geqslant 0,\ f \in L_{p,\varphi}(\mathbf{R}),$ satisfying $||f||_{p,\varphi} > 0,$ we have the following inequality:

$$||T^{(1)}f||_{p,\psi^{1-p}} < K_{\lambda}(\sigma)||f||_{p,\varphi}. \tag{36}$$

(ii) For any $f(x), g(y) \geqslant 0,\ f \in L_{p,\varphi}(\mathbf{R}),\ g \in L_{q,\psi}(\mathbf{R}),$ satisfying $||f||_{p,\varphi} > 0,$ and $||g||_{q,\psi} > 0,$ we have the following inequality:

$$(T^{(1)}f,g) < K_{\lambda}(\sigma)||f||_{p,\varphi}||g||_{q,\psi}. \tag{37}$$

Moreover, the constant factor $K_{\lambda}(\sigma)$ in (36) and (37) is the best possible, namely,

$$||T^{(1)}|| = K_{\lambda}(\sigma).$$

(b) In view of Corollary 2, for $f \in L_{p,\varphi}(\mathbf{R}),$ setting

$$H_2(y) := \int_{-\infty}^{\infty} K_{\lambda}(e^{ax}, e^{by})f(x)dx\ (y \in \mathbf{R}),$$

by (29), we have

$$||H_2||_{p,\phi^{1-p}} = \left[\int_{-\infty}^{\infty} \phi^{1-p}(y)H_2^p(y)dy\right]^{\frac{1}{p}} \leqslant K_{\lambda}(\sigma)||f||_{p,\varphi} < \infty. \tag{38}$$

DEFINITION 2. Define a Hilbert-type integral operator with the homogeneous kernel $T^{(2)} : L_{p,\varphi}(\mathbf{R}) \to L_{p,\phi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R}),$ there exists a unique representation $T^{(2)}f = H_2 \in L_{p,\phi^{1-p}}(\mathbf{R}),$ satisfying for any $y \in \mathbf{R},\ T^{(2)}f(y) = H_2(y).$

In view of (38), it follows that

$$||T^{(2)}f||_{p,\phi^{1-p}} = ||H_2||_{p,\phi^{1-p}} \leqslant K_{\lambda}(\sigma)||f||_{p,\varphi},$$

and then the operator $T^{(2)}$ is bounded satisfying

$$||T^{(2)}|| = \sup_{(f \neq \theta) \in L_{p,\varphi}(\mathbf{R})} \frac{||T^{(2)}f||_{p,\phi^{1-p}}}{||f||_{p,\varphi}} \leqslant K_{\lambda}(\sigma).$$

If we define the formal inner product of $T^{(2)}f$ and g as follows:

$$(T^{(2)}f,g) := \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} K_{\lambda}(e^{ax}, e^{by})f(x)dx\right)g(y)dy,$$

then we can rewrite Corollary 2 as follows:

COROLLARY 4. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geqslant 0,\ f \in L_{p,\varphi}(\mathbf{R}),$ *satisfying* $||f||_{p,\varphi} > 0,$ *we have the following inequality:*

$$||T^{(2)}f||_{p,\phi^{1-p}} < K_\lambda(\sigma)||f||_{p,\varphi}. \tag{39}$$

(ii) For any $f(x),\ g(y) \geqslant 0,\ f \in L_{p,\varphi}(\mathbf{R}),\ g \in L_{q,\phi}(\mathbf{R}),$ *satisfying* $||f||_{p,\varphi} > 0,$ *and* $||g||_{q,\phi} > 0,$ *we have the following inequality:*

$$(T^{(2)}f, g) < K_\lambda(\sigma)||f||_{p,\varphi}||g||_{q,\phi}. \tag{40}$$

Moreover, the constant factor $K_\lambda(\sigma)$ *in* (39) *and* (40) *is the best possible, namely,*

$$||T^{(2)}|| = K_\lambda(\sigma).$$

Acknowledgements. This work is supported by the National Natural Science Foundation (No. 61772140), and Science and Technology Planning Project Item of Guangzhou City (No. 201707010229). We are grateful for this help.

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(Received September 5, 2018)

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