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REAL INTERPOLATION WITH A FUNCTION PARAMETER FOR MARTINGALE HARDY–LORENTZ AND BMO SPACES

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Abstract. This paper is devoted to the study of real interpolation between martingale Hardy-Lorentz and BMO spaces in the framework of interpolation with a function parameter. We first establish some inequalities for the sharp functions of martingales. With the aid of these inequalities, some new interpolation theorems which generalize some fundamental interpolation theorems in classical martingale H_p theory are proved. In particular, we show that

$$(H^s_{p_0,q_0}, BMO_2)_{\rho,q} = \Lambda^s_q(t^{\frac{1}{p_0}}/\rho(t^{\frac{1}{p_0}})),$$

where $0 < p_0 < \infty$, $0 < q_0, q \leqslant \infty$ and $\rho \in Q(0,1)$.

1. Introduction and preliminaries

As is well-known, interpolation theory has been applied as a powerful tool in many branches of mathematics, such as partial differential equations, numerical analysis and approximation theory. In classical interpolation theory, one of important results is real interpolation between the classical Hardy and BMO spaces. It was proved by Hanks [1] and Bennett, Sharpley [2] that

$$(H_{p_0,q_0}, BMO)_{\theta,q} = H_{p,q}, \quad \frac{1}{p} = \frac{1-\theta}{p_0},$$

$$0 < \theta < 1, \quad 0 < p_0 < \infty, \quad 0 < q_0, q \leqslant \infty.$$

In classical martingale H_p theory, a fundamental interpolation theorem corresponding to the above result was due to Weisz [3]. He proved that real interpolation spaces between martingale Hardy-Lorentz and BMO spaces are martingale Hardy-Lorentz spaces:

$$(H^s_{p_0,q_0}, BMO_2)_{\theta,q} = H^s_{p,q}, \quad \frac{1}{p} = \frac{1-\theta}{p_0},$$

$$0 < \theta < 1, \quad 0 < p_0 < \infty, \quad 0 < q_0, q \leqslant \infty.$$

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The purpose of this paper is to make a study of real interpolation between martingale Hardy-Lorentz and BMO spaces in the framework of interpolation with a function parameter. Interpolation with a function parameter is one of interesting and attractive fields in the study of interpolation theory. The theory of interpolation space $(X_0,X_1)_{\varphi,q}$ with a function parameter $\varphi(t)$ is an extension of the theory of interpolation space $(X_0,X_1)_{\theta,q}$ originated from Lions and Peetre [4]. It was derived from the work of Kalugina [5] and was systematically developed by Gustavsson [6], Janson [7], Merucci [8], Persson [9] and so on. In the theory of interpolation with a function parameter, there are four important function classes, namely, B_K, B_ψ, $\mathfrak{P}^{+-}$ and $Q(a_0,a_1)$. It was proved by Gustavsson [6] that $B_\psi \subset B_K$ and that if $f \in B_K$, then there exists a function $g \in B_\psi$ such that f is equivalent to g. Persson [9] proved that $B_\psi \subset Q(0,1) \subset \mathfrak{P}^{+-}$ and that if $\varphi \in \mathfrak{P}^{+-}$, then there exists a function $\psi \in B_\psi$ such that φ is equivalent to ψ. From these relationships, one can find that real interpolation with a function parameter belonging to B_K, B_ψ or $\mathfrak{P}^{+-}$ can be transferred to real interpolation with a function parameter belonging to $Q(0,1)$. For this reason, in this paper, we will only consider real interpolation with a function parameter belonging to $Q(0,1)$ for martingale Hardy-Lorentz and BMO spaces.

Interpolation of martingale spaces is one of the main parts in martingale H_p theory, and its theory has been successfully applied to Fourier analysis. More and more attentions have been paid to this topic in recent years, for example see [19, 20, 21, 22]. We have studied real interpolation with a function parameter belonging to $Q(0,1)$ for Lorentz martingale spaces in [23], and for martingale Hardy and BMO spaces in [24]. However, there is still an unsolved problem in [24]. That is, does Theorem 4.1 in [24] still hold for $0<p\leqslant 1$? In this section, we will give an affirmative answer. Moreover, real interpolation with a function parameter belonging to $Q(0,1)$ for martingale Hardy-Lorentz and BMO spaces is identified. These interpolation theorems generalize some fundamental interpolation theorems in classical martingale H_p theory (see [3]). In this paper, we first establish some inequalities for the sharp functions of martingales. With the aid of these inequalities, some new interpolation theorems which generalize some fundamental interpolation theorems in classical martingale H_p theory are proved. In particular, we show that

$$(H^s_{p_0,q_0},BMO_2)_{\rho,q} = \Lambda^s_q(t^{\frac{1}{p_0}}/\rho(t^{\frac{1}{p_0}})),$$

where $0<p_0<\infty$, $0<q_0,q\leqslant\infty$ and $\rho\in Q(0,1)$.

This paper is organized as follows. Some definitions and notations will be given in the remainder of this section. In Section 2, we establish some inequalities for the sharp functions of martingales. As the main results of this paper, some interpolation theorems are proved in Section 3.

Now let us recall some definitions and notations.

Let (X,μ) be a σ-finite measure space, $\mathscr{M}(X)$ the space of all measurable functions on X. For $f\in\mathscr{M}(X)$, denote its distribution function by

$$\lambda_f(t) = \mu(x:|f(x)|>t),\ \ t\geqslant 0,$$

and its decreasing rearrangement function f^* is defined as

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leqslant t\}, \ \ t \geqslant 0.$$

For $0 < q \leqslant \infty$, let φ be a non-negative and locally integrable function on $[0,\infty)$ ($\varphi \not\equiv 0$), the classical Lorentz spaces are defined as

$$\Lambda_q(\varphi) = \{f \in \mathscr{M}(X) : \|f\|_{\Lambda_q(\varphi)} < \infty\},$$

where

$$\|f\|_{\Lambda_q(\varphi)} = \begin{cases} (\int_0^\infty (f^*(t)\varphi(t))^q \frac{dt}{t})^{\frac{1}{q}}, & q < \infty, \\ \sup_{t>0} f^*(t)\varphi(t), & q = \infty. \end{cases}$$

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, and $\{\mathscr{F}_n\}_{n\geqslant 0}$ a nondecreasing sequence of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F} = \sigma(\bigcup_n \mathscr{F}_n)$. The conditional expectation operators relative to $\mathscr{F}_n$ are denoted by $\mathbb{E}_n$. For a martingale $f = (f_n)_{n\geqslant 0}$ relative to $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_n)_{n\geqslant 0})$, denote its martingale differences by $df_i = f_i - f_{i-1}$ ($i \geqslant 0$, with convention $df_0 = 0$) and its conditional quadratic variation by

$$s_n(f) = (\sum_{i=1}^n \mathbb{E}_{i-1}|df_i|^2)^{\frac{1}{2}}, \qquad s(f) = (\sum_{i=1}^\infty \mathbb{E}_{i-1}|df_i|^2)^{\frac{1}{2}}.$$

The sharp function f_r^s of a martingale $f = (f_n)_{n\geqslant 0}$ is defined as

$$f_r^s = \sup_{n\geqslant 0}(\mathbb{E}_n[s^2(f) - s_n^2(f)]^{\frac{r}{2}})^{\frac{1}{r}}, \quad 0 < r < \infty.$$

Let $0 < p < \infty$, $0 < q \leqslant \infty$, define martingale Lorentz and BMO spaces as follows:

$$\Lambda_q^s(\varphi) = \{f = (f_n)_{n\geqslant 0} : \|f\|_{\Lambda_q^s(\varphi)} = \|s(f)\|_{\Lambda_q(\varphi)} < \infty\};$$

$$BMO_2 = \{f = (f_n)_{n\geqslant 0} : \|f\|_{BMO_2} = \sup_{n\geqslant 0}\|(\mathbb{E}_n \mid f - f_n \mid^2)^{\frac{1}{2}}\|_\infty < \infty\};$$

$$BMO_p^s = \{f = (f_n)_{n\geqslant 0} : \|f\|_{BMO_p^s} = \sup_{n\geqslant 0}\|(\mathbb{E}_n[s^2(f) - s_n^2(f)]^{\frac{p}{2}})^{\frac{1}{p}}\|_\infty < \infty\}.$$

REMARK 1.1. One can easily show that Lorentz martingale spaces $\Lambda_q^s(\varphi)$ are quasi-normed spaces. We recall that $BMO_p^s \sim BMO_2^s = BMO_2$ for $0 < p < \infty$. For real-valued BMO martingale theory, we refer to Weisz [3] and Long [16]. It is clear that if $\varphi(t) = t^{\frac{1}{p}}$, then $\Lambda_q^s(\varphi) = H_{p,q}^s$. In particular, if $\varphi(t) = t^{\frac{1}{q}}$, then $\Lambda_q^s(t^{\frac{1}{q}}) = H_q^s$, see [3, 16].

Now let us recall a function parameter class introduced by Persson [9] as follows.

Let a_0 and a_1 be real numbers such that $a_0 < a_1$. The class $Q[a_0, a_1]$ consists of all non-negative functions $\varphi(t)$ on $(0,\infty)$ such that $\varphi(t)t^{-a_0}$ is nondecreasing and $\varphi(t)t^{-a_1}$ is non-increasing. A function is said to belong to the $Q(a_0, a_1)$, if $\varphi(t) \in$

$Q[a_0+\varepsilon, a_1+\varepsilon]$ for some $\varepsilon > 0$. The notation $\varphi(t) \in Q(a_0,-)$ ($\varphi(t) \in Q(-,a_1)$) means that $\varphi(t) \in Q(a_0,b)$ ($\varphi(t) \in Q(b,a_1)$) for some real number b.

Let us recall some notations in interpolation theory. For more details we refer to [2] and [13]. Suppose that A_0 and A_1 are two quasi-normed spaces embedded continuously into a topological space A. The K-functional is defined as

$$K(t,f;A_0,A_1) = \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1, f_i \in A_i, i = 0,1\},$$

where the infimum takes over all possible decompositions with $f = f_0 + f_1$, $f_i \in A_i$, $i = 0,1$. For φ a function parameter, $0 < q \leqslant \infty$, the interpolation spaces $(A_0,A_1)_{\varphi,q}$ between A_0 and A_1 are defined as the space of all functions $f \in A_0 + A_1$ such that $\|f\|_{(A_0,A_1)_{\varphi,q}} < \infty$, where

$$\|f\|_{(A_0,A_1)_{\varphi,q}} = \begin{cases} \left(\int_0^\infty \left(\frac{K(t,f;A_0,A_1)}{\varphi(t)} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{t>0} \frac{K(t,f;A_0,A_1)}{\varphi(t)}, & q = \infty. \end{cases}$$

Throughout this paper, we use C to denote some constant and may be different at each occurrence. The equivalence $a \approx b$ means that $C_1 a \leqslant b \leqslant C_2 a$ for some positive constants C_1 and C_2. Two quasi-normed spaces, A and B, are considered as equal and we write $A = B$ whenever their quasi-norms are equivalent. The relation $A \subseteq B$ means that we have a continuous embedding.

2. Some inequalities for the sharp functions of martingales

As is well-known, the sharp functions play an important role in the study of interpolation between Hardy spaces and BMO spaces, see [2, 3]. In this section, we will devote ourself to establishing some inequalities for the sharp functions of martingales.

We first establish an inequality for the sharp functions of martingales with respect to the Lorentz norm. Here we need the following lemmas:

LEMMA 2.1. *Let $0 < r \leqslant 1$. Then for any martingale $f = (f_n)_{n\geqslant 0}$ we have*

$$s(f)^*(t) \leqslant 4^{\frac{1}{r}} f_r^{s*}\left(\frac{t}{2}\right) + s(f)^*(2t), \quad t > 0.$$

Proof. It is enough to prove

$$s(f)^{*2}(t) \leqslant 16^{\frac{1}{r}} f_r^{s*2}\left(\frac{t}{2}\right) + s(f)^{*2}(2t), \quad t > 0. \tag{2.1}$$

Set $\eta_n = (\mathbb{E}_n(s^2(f) - s_n^2(f))^{\frac{r}{2}})^{\frac{1}{r}}$, $0 < r \leqslant 1$. We define stopping times as follows:

$$\tau = \inf\left\{n \in \mathbf{N} : \eta_n > f_r^{s*}\left(\frac{t}{2}\right)\right\},$$

$$\mu = \inf\{n \in \mathbf{N} : s_{n+1}(f) > s(f)^*(2t)\},$$

then we have

$$\left\{\tau<\infty\right\}=\left\{f_r^s>f_r^{s*}\left(\frac{t}{2}\right)\right\},$$
$$\{\mu<\infty\}=\{s(f)>s(f)^*(2t)\},$$
$$\mathbb{P}(\tau<\infty)\leqslant\frac{t}{2},\quad \mathbb{P}(\mu<\infty)\leqslant 2t.$$

Since

$$\left\{(s(f))^2>16^{\frac{1}{r}}f_r^{s*2}\left(\frac{t}{2}\right)+s(f)^{*2}(2t)\right\}$$
$$\subseteq\{\tau<\infty\}\cup\left\{\mu<\tau,(s(f))^2-(s_\mu(f))^2>16^{\frac{1}{r}}f_r^{s*2}\left(\frac{t}{2}\right)\right\}$$

and notice that $0<r\leqslant 1$, we get

$$\begin{aligned}&\mathbb{P}((s(f))^2>16^{\frac{1}{r}}f_r^{s*2}\left(\frac{t}{2}\right)+s(f)^{*2}(2t))\\ &\leqslant\mathbb{P}(\tau<\infty)+\frac{1}{4f_r^{s*r}(\frac{t}{2})}\int_{\{\mu<\tau\}}((s(f))^2-(s_\mu(f))^2)^{\frac{r}{2}}d\mathbb{P}\\ &=\mathbb{P}(\tau<\infty)+\frac{1}{4f_r^{s*r}(\frac{t}{2})}\int_{\{\mu<\tau\}}\mathbb{E}(((s(f))^2-(s_\mu(f))^2)^{\frac{r}{2}}\mid\mathscr{F}_\mu)d\mathbb{P}\\ &\leqslant\mathbb{P}(\tau<\infty)+\frac{1}{4f_r^{s*r}(\frac{t}{2})}\left(\int_{\{\mu<\tau\}}(\mathbb{E}(((s(f))^2-(s_\mu(f))^2)^{\frac{r}{2}}\mid\mathscr{F}_\mu))^{\frac{1}{r}}d\mathbb{P}\right)^r\left(\int_{\{\mu<\tau\}}d\mathbb{P}\right)^{1-r}\\ &\leqslant\mathbb{P}(\tau<\infty)+\frac{1}{4}\mathbb{P}(\mu<\infty)\leqslant t.\end{aligned}$$

Thus (2.1) holds. The proof is completed. □

LEMMA 2.2. [9] *Let* $\varphi(t)\in Q[a_0,a_1]$. *Then* $\varphi(t^\alpha)\in Q[a_0\alpha,a_1\alpha]$, $\alpha>0$.

LEMMA 2.3. [16] *Let* (F,G) *be a pair of non-negative measurable functions on* $(\Omega,\mathscr{F},\mathbb{P})$. *If* (F,G) *satisfies the rearrangement inequality*

$$F^*(t)\leqslant CG^*\left(\frac{t}{2}\right)+F^*(2t),\quad \forall t>0.$$

Then with the same C, *we have*

$$F^*(t)\leqslant 2CG^*\left(\frac{t}{2}\right)+\frac{C}{\log 2}\int_t^\infty\frac{G^*(s)}{s}ds,\quad \forall t>0.$$

LEMMA 2.4. [9] *Let* $0<q\leqslant\infty$, $0<r<\infty$, $\psi(t)\in Q(-,-)$, *and* $h(t)$ *a positive and non-increasing function on* $(0,\infty)$.

1. If $\varphi(t)\in Q(-,0)$, *then*

$$\left(\int_0^\infty(\varphi(t))^q\left(\int_0^t(h(u)\psi(u))^r\frac{du}{u}\right)^{\frac{q}{r}}\frac{dt}{t}\right)^{\frac{1}{q}}\leqslant C\left(\int_0^\infty(\varphi(t)h(t)\psi(t))^q\frac{dt}{t}\right)^{\frac{1}{q}};$$

2. *If* $\varphi(t) \in Q(0,-)$, *then*

$$\Big(\int_0^\infty (\varphi(t))^q \Big(\int_t^\infty (h(u)\psi(u))^r \frac{du}{u}\Big)^{\frac{q}{r}} \frac{dt}{t}\Big)^{\frac{1}{q}} \leqslant C\Big(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \frac{dt}{t}\Big)^{\frac{1}{q}}.$$

Now we can formulate the following inequality:

THEOREM 2.1. *Let* $0 < p, r < \infty$, $0 < q \leqslant \infty$ *and* $\rho \in Q(0,1)$. *Then*

$$\| s(f) \|_{\Lambda_q(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}}))} \leqslant C \| f_r^s \|_{\Lambda_q(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}}))}.$$

Proof. The case for $1 \leqslant r < \infty$ was proved in [24]. We only need to prove this theorem for $0 < r < 1$.

Since $\rho(t^{\frac{1}{p}}) \in Q(0, \frac{1}{p})$ by Lemma 2.2, then $\rho(t^{\frac{1}{p}})t^{-\varepsilon}$ is nondecreasing for some $\varepsilon > 0$. So we have $\rho(t^{\frac{1}{p}}) \geqslant C\rho((\frac{t}{2})^{\frac{1}{p}})$ for $t > 0$. It follows from Lemma 2.1, 2.2, 2.3 and 2.4 that

$$\begin{aligned}
\| s(f) \|_{\Lambda_q(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}}))} &= \Big(\int_0^\infty \Big(\frac{t^{\frac{1}{p}} s(f)^*(t)}{\rho(t^{1/p})}\Big)^q \frac{dt}{t}\Big)^{1/q} \\
&\leqslant C\Big(\Big(\int_0^\infty \Big(\frac{t^{\frac{1}{p}} f_r^{s*}(\frac{t}{2})}{\rho(t^{1/p})}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\quad + \Big(\int_0^\infty \Big(\frac{t^{\frac{1}{p}}}{\rho(t^{1/p})}\Big)^q \Big(\int_t^\infty \frac{f_r^{s*}(s)}{s}\, ds\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}\Big) \\
&\leqslant C\Big(\int_0^\infty \Big(\frac{t^{\frac{1}{p}} f_r^{s*}(t)}{\rho(t^{1/p})}\Big)^q \frac{dt}{t}\Big)^{1/q} \\
&= C \| f_r^s \|_{\Lambda_q(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}}))}.
\end{aligned}$$

The proof is completed. □

Now let us turn to the $L_{p,\infty}$-norm inequalities for the sharp functions of martingales.

LEMMA 2.5. [3]

(i) Let $0 < r < p < \infty$, *then for any martingale* $f = (f_n)_{n \geqslant 0}$ *we have*

$$\| f_r^s \|_p \leqslant C_p \| s(f) \|_p;$$

(ii) Let $2 \leqslant p < \infty$, *then for any martingale* $f = (f_n)_{n \geqslant 0}$ *we have*

$$\|s(f)\|_p \leqslant C_p \|f\|_p.$$

DEFINITION 2.1. [17] A measurable function a is called a w-1-atom, if there exists a stopping time ν (ν is called the stopping time associated with a) such that
(i) $a_n = \mathbb{E}_n a = 0$ if $\nu \geqslant n$,
(ii) $\|s(a)\|_\infty < \infty$.

LEMMA 2.6. [17] *Let* $1 \leqslant q \leqslant 2$ *and* $T : L_q \to L_q$ *a bounded sublinear operator. If*

$$\mathbb{P}(|Ta| > 0) \leqslant C_q \mathbb{P}(\nu < \infty)$$

for all w-1-atoms a*, where* ν *is the stopping time associated with* a*, then for* $0 < p < q$ *we have*

$$\|Tf\|_{L_{p,\infty}} \leqslant C_{p,\,q} \|s(f)\|_{L_{p,\infty}}.$$

According to the proof of Lemma 2.8 in [17], it is easy to see that the conclusion still holds for quasi-linear operators.

THEOREM 2.2. *Let* $0 < p, r < 2$*. Then*

$$\| f_r^s \|_{L_{p,\infty}} \leqslant C_p \| s(f) \|_{L_{p,\infty}} .$$

Proof. The sharp function $Tf = f_r^s$ is quasi-linear. By Lemma 2.5, T is L_2-bounded. If a is a w-1-atom and ν is the stopping time associated with a, then $(|Ta| > 0) = (a_r^s > 0) \subseteq (\nu < \infty)$. Hence, $\mathbb{P}(|Ta| > 0) \leqslant \mathbb{P}(\nu < \infty)$. It follows from Lemma 2.6 that

$$\|f_r^s\|_{L_{p,\infty}} = \|Tf\|_{L_{p,\infty}} \leqslant C_p \|s(f)\|_{L_{p,\infty}}.$$

The proof is completed. □

A weak type Doob's maximal inequality was proved by Liu [18]:

$$\| M(f) \|_{L_{p,\infty}} \leqslant C_p \| f \|_{L_{p,\infty}}, 1 < p < \infty,$$

where $M(f) = \sup_{n \geqslant 0} | f_n |$ is the Doob's maximal function of a martingale $f = (f_n)_{n \geqslant 0}$. From this inequality and Theorem 2.2, we obtain

COROLLARY 2.1. *Let* $0 < p < \infty$*. Then*

$$\| f_1^s \|_{L_{p,\infty}} \leqslant C_p \| s(f) \|_{L_{p,\infty}} .$$

It was proved by Weisz in [3] that $\| f_r^s \|_p \approx \| s(f) \|_p$ for $0 < r < p < \infty$. In the following let us consider the reverse inequality in Theorem 2.2.

DEFINITION 2.2. [14] Let (f, g) be a pair of non-negative measurable functions on $(\Omega, \mathscr{F}, \mathbb{P})$. We say that it satisfies the good λ-inequality, if there is $\alpha > 1$, and for all $\beta > 0$ small enough, there exist constants ε_β satisfying $\lim_{\beta \to 0} \varepsilon_\beta = 0$, such that

$$\mathbb{P}(f > \alpha\lambda) \leqslant \varepsilon_\beta \mathbb{P}(f > \lambda) + \delta_\beta \mathbb{P}(g > \beta\lambda), \quad \lambda > 0.$$

The weak L_p-norm of a measurable function f is defined by

$$\| f \|_{wL_p} = \sup_{\lambda>0} \lambda \mathbb{P}(|f| > \lambda)^{\frac{1}{p}}.$$

As is well-known, the weak L_p-norm of f is equivalent to its $L_{p,\infty}$-norm(see for example in [3]). The following Lemma indicates that the good λ-inequality is a sufficient condition for the type of inequality $L_{p,\infty}-L_{p,\infty}$ to hold.

LEMMA 2.7. *Let $0 < p < \infty$. If a pair of nonnegative measurable functions (f,g) satisfies the good λ-inequality, then*

$$\| f \|_{L_{p,\infty}} \leqslant C_p \| g \|_{L_{p,\infty}} .$$

Proof. Let $\theta_p(\lambda) = \lambda \mathbb{P}(f > \lambda)^{\frac{1}{p}}$. Since (f,g) satisfies the good λ-inequality, we have

$$\theta_p(\alpha\lambda) \leqslant C_p\left(\alpha\varepsilon_\beta^{\frac{1}{p}}\theta_p(\lambda) + \alpha\beta^{-1}\delta_\beta^{\frac{1}{p}} \| g \|_{wL_p}\right).$$

Hence,

$$\begin{aligned}\theta_p(\lambda) &\leqslant \alpha\widetilde{\varepsilon_{\beta,p}}\theta_p\left(\frac{\lambda}{\alpha}\right) + C_{\alpha,\beta,p} \| g \|_{wL_p} \\ &\leqslant (\alpha\widetilde{\varepsilon_{\beta,p}})^n\theta_p\left(\frac{\lambda}{\alpha^n}\right) + C_{\alpha,\beta,p}(1+\cdots+(\alpha\widetilde{\varepsilon_{\beta,p}})^{n-1}) \| g \|_{wL_p},\end{aligned}$$

where $\widetilde{\varepsilon_{\beta,p}} = C_p\varepsilon_\beta^{\frac{1}{p}}$ and $C_{\alpha,\beta,p} = C_p\alpha\beta^{-1}\delta_\beta^{\frac{1}{p}}$. Now let β small enough such that $\alpha\widetilde{\varepsilon_{\beta,p}} < 1$ and let $n \to \infty$, we get

$$\| f \|_{\mathrm{w}L_p} \leqslant C_p \| g \|_{\mathrm{w}L_p} .$$

The proof is completed. $\square$

LEMMA 2.8. *Let $0 < r < \infty$. Then the pair $(s(f), f_r^s)$ satisfies the good λ-inequality.*

Proof. For $0 < r \leqslant 1$, let $\alpha > 1$ and define three stopping times as follows:

$$T = \inf\{n \in \mathbf{N} : s_{n+1}(f) > \sqrt{\alpha}\lambda\},$$

$$S = \inf\{n \in \mathbf{N} : s_{n+1}(f) > \lambda\},$$

$$R = \inf\{n \in \mathbf{N} : \eta_n(f) > \beta\lambda\}.$$

Obviously, $S \leqslant T$. Since

$$\begin{aligned}\{T<\infty\} &\subseteq \{T<\infty, S\leqslant T, S<R\}\bigcup\{T<\infty, S\leqslant T, R\leqslant S\} \\ &\subseteq \{T<\infty, S\leqslant T, S<R\}\bigcup\{R<\infty\},\end{aligned}$$

we have

$$
\begin{aligned}
&\mathbb{P}(T<\infty)\\
\leqslant\ & \frac{1}{(\alpha-1)^{\frac{r}{2}}\lambda^r}\int_{\{S\leqslant T\}\bigcap\{S<R\}}\mathbb{E}((s^2(f)-s_S^2(f))^{\frac{r}{2}}\mid \mathscr{F}_S)d\mathbb{P}+\mathbb{P}(R<\infty)\\
\leqslant\ & \frac{1}{(\alpha-1)^{\frac{r}{2}}\lambda^r}\Big(\int_{\{S\leqslant T\}\bigcap\{S<R\}}(\mathbb{E}((s^2(f)-s_S^2(f))^{\frac{r}{2}}\mid \mathscr{F}_S))^{\frac{1}{r}}d\mathbb{P}\Big)^r\\
&\times\Big(\int_{\{S\leqslant T\}\bigcap\{S<R\}}d\mathbb{P}\Big)^{1-r}+\mathbb{P}(R<\infty)\\
\leqslant\ & \frac{\beta^r}{(\alpha-1)^{\frac{r}{2}}}\mathbb{P}(S<\infty)+\mathbb{P}(R<\infty).
\end{aligned}
$$

That is

$$
\mathbb{P}(s(f)>\sqrt{\alpha}\lambda)\leqslant\frac{\beta^r}{(\alpha-1)^{\frac{r}{2}}}\mathbb{P}(s(f)>\lambda)+\mathbb{P}(f_r^s>\beta\lambda).
$$

For $1\leqslant r<\infty$, let $\alpha>1$, we define the following three stopping times at this time:

$$
\begin{aligned}
T&=\inf\{n\in\mathbf{N}:s_{n+1}(f)>\alpha\lambda\},\\
S&=\inf\{n\in\mathbf{N}:s_{n+1}(f)>\lambda\},\\
R&=\inf\{n\in\mathbf{N}:\eta_n(f)>\beta\lambda\}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\mathbb{P}(T<\infty)\\
\leqslant\ & \frac{1}{(\alpha-1)\lambda}\int_{\{S\leqslant T\}\bigcap\{S<R\}}(s(f)-s_S(f))d\mathbb{P}+\mathbb{P}(R<\infty)\\
\leqslant\ & \frac{1}{(\alpha-1)\lambda}\int_{\{S\leqslant T\}\bigcap\{S<R\}}\mathbb{E}((s^2(f)-s_S^2(f))^{\frac{1}{2}}\mid \mathscr{F}_S)d\mathbb{P}+\mathbb{P}(R<\infty)\\
\leqslant\ & \frac{1}{(\alpha-1)\lambda}\int_{\{S\leqslant T\}\bigcap\{S<R\}}(\mathbb{E}((s^2(f)-s_S^2(f))^{\frac{r}{2}}\mid \mathscr{F}_S))^{\frac{1}{r}}d\mathbb{P}+\mathbb{P}(R<\infty)\\
\leqslant\ & \frac{\beta}{\alpha-1}\mathbb{P}(S<\infty)+\mathbb{P}(R<\infty).
\end{aligned}
$$

That is

$$
\mathbb{P}(s(f)>\alpha\lambda)\leqslant\frac{\beta}{\alpha-1}\mathbb{P}(s(f)>\lambda)+\mathbb{P}(f_r^s>\beta\lambda).
$$

The proof is completed. □

By Lemma 2.7 and 2.8, we obtain

THEOREM 2.3. *Let* $0<r<\infty$, $0<p<\infty$. *Then*

$$
\| s(f)\|_{L_{p,\infty}}\leqslant C_p\| f_r^s\|_{L_{p,\infty}}.
$$

It follows from Theorem 2.2, 2.3 and Corollary 2.1 that

COROLLARY 2.2. *(i) Let* $0 < p, r < 2$. *Then*

$$\| s(f) \|_{L_{p,\infty}} \approx \| f_r^s \|_{L_{p,\infty}};$$

(ii) Let $0 < p < \infty$. *Then*

$$\| s(f) \|_{L_{p,\infty}} \approx \| f_1^s \|_{L_{p,\infty}}.$$

3. Some interpolation theorems for martingale Hardy-Lorentz spaces

In this section, some new interpolation theorems will be proved. The results generalize some fundamental interpolation theorems in classical martingale H_p theory are proved. In particular, we obtain an interpolation theorem for martingale Hardy-Lorentz and BMO spaces (Theorem 3.3).

LEMMA 3.1. [23] *Let* $0 < p < \infty$, $0 < q \leqslant \infty$ *and* $\rho \in Q(0,1)$. *Then*

$$(H_p^s, H_\infty^s)_{\rho,q} = \Lambda_q^s(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}})).$$

THEOREM 3.1. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$ *and* $\rho \in Q(0,1)$. *Then*

$$(H_p^s, BMO_2)_{\rho,q} = \Lambda_q^s(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}})).$$

Proof. By the equivalence between BMO_2 and BMO_1^s, we have

$$\begin{aligned} \| f \|_{BMO_2} &\leqslant C \| f \|_{BMO_1^s} \leqslant C \sup_{n \geqslant 0} \| \mathbb{E}_n s(f) \|_\infty \\ &\leqslant C \| s(f) \|_\infty = C \| f \|_{H_\infty^s}. \end{aligned}$$

Thus by Lemma 3.1 we have

$$\| f \|_{(H_p^s, BMO_2)_{\rho,q}} \leqslant C \| f \|_{(H_p^s, H_\infty^s)_{\rho,q}} \leqslant C \| f \|_{\Lambda_q^s(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}}))}, \tag{3.1}$$

from which we get $\Lambda_q^s(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}})) \subseteq (H_p^s, BMO_2)_{\rho,q}$.

For the converse, let $f \in H_p^s + BMO$, $f = g + h$, with $g \in H_p^s$, $h \in BMO$, then

$$f_r^s \leqslant C(g_r^s + h_r^s) \leqslant C(g_r^s + \| h \|_{BMO_2}).$$

Let $r = 1$, by Corollary 2.1, then for any $t > 0$,

$$\begin{aligned} t f_1^{s*}(t^p) &\leqslant C(t g_1^{s*}(t^p) + t \| h \|_{BMO_2}) \\ &\leqslant C(\| g_1^s \|_{L_{p,\infty}} + t \| h \|_{BMO_2}) \\ &\leqslant C(\| s(g) \|_{L_{p,\infty}} + t \| h \|_{BMO_2}) \\ &\leqslant C(\| s(g) \|_p + t \| h \|_{BMO_2}) \end{aligned} \tag{3.2}$$

Taking the infimum over all decompositions $f = g + h \in H_p^s + BMO_2$, we obtain

$$t f_r^{s*}(t^p) \leqslant CK(t, f; H_p^s, BMO_2).$$

Hence, by Theorem 2.1 we get

$$\begin{aligned}\| f \|_{\Lambda_q^s(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}}))} &\leqslant C \| f_r^s \|_{\Lambda_q(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}}))} \\ &\leqslant C \| f \|_{(H_p^s, BMO_2)_{\rho,q}},\end{aligned}$$

from which we get $(H_p^s, BMO_2)_{\rho,q} \subseteq \Lambda_q^s(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}}))$. The proof is completed. □

Since $\| f \|_{(H_{p,\infty}^s, BMO_2)_{\rho,q}} \leqslant C \| f \|_{(H_p^s, BMO_2)_{\rho,q}}$, it follows from (3.1) and (3.2) that

COROLLARY 3.1. *Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $\rho \in Q(0,1)$. Then*

$$(H_{p,\infty}^s, BMO_2)_{\rho,q} = \Lambda_q^s(t^{\frac{1}{p}}/\rho(t^{\frac{1}{p}})).$$

LEMMA 3.2. [9] *Let $\varphi(t) \in Q[a_0, a_1]$, then $t^\alpha(\varphi(t))^\beta \in Q[\alpha + a_1\beta, \alpha + a_0\beta]$, $\alpha \in \mathbf{R}$, $\beta < 0$.*

The following lemma was obtained by Persson [9] for quasi-Banach spaces. By use of the method in [10] and [12], it is easy to verify that the lemma also holds for quasi-normed spaces.

LEMMA 3.3. *Let $\rho(t)$ and $\rho_0(t)$ be in the class $Q(0,1)$, $0 < q_0 < \infty$, $0 < q \leqslant \infty$. If we put $\rho_1(t) = \rho_0(t)\rho(t/\rho_0(t))$. Then*

$$((A_0, A_1)_{\rho_0, q_0}, A_1)_{\rho,q} = (A_0, A_1)_{\rho_1, q}.$$

THEOREM 3.2. *Let $\varphi_0(t)$ and $\rho(t)$ be in the class $Q(0,1)$, $0 < q_0 < \infty$, $0 < q \leqslant \infty$. Then*

$$(\Lambda_{q_0}^s(\varphi_0), BMO_2)_{\rho,q} = \Lambda_q^s(\varphi),$$

where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t))$.

Proof. Put $\rho_0(t) = t/\varphi_0(t^p)$, by Lemma 2.2 and 3.2, we can choose p so small that $\rho_0(t) \in Q(0,1)$. Then by Lemma 3.3 and Theorem 3.1 we obtain

$$\begin{aligned}(\Lambda_{q_0}^s(\varphi_0), BMO_2)_{\rho,q} &= ((H_p^s, BMO_2)_{\rho_0,q_0}, BMO_2)_{\rho,q} \\ &= (H_p^s, BMO_2)_{\rho_0(t)\rho(t/\rho_0(t)),q} = \Lambda_q^s(\varphi),\end{aligned}$$

where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t))$. The proof is completed. □

It follows from Corollary 3.1 and Theorem 3.2 that we obtain

THEOREM 3.3. *Let $0 < p_0 < \infty$, $0 < q_0, q \leqslant \infty$ and $\rho \in Q(0,1)$. Then*

$$(H_{p_0,q_0}^s, BMO_2)_{\rho,q} = \Lambda_q^s(t^{\frac{1}{p_0}}/\rho(t^{\frac{1}{p_0}})).$$

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