Journal of Mathematical Inequalities
Volume 14, Number 4 (2020), 1083–1094 doi:10.7153/jmi-2020-14-70

SOME NECESSARY AND SUFFICIENT CONDITIONS FOR A VMO FUNCTION

GUANGQING WANG AND JINHUI LI*

(*Communicated by A. Meskhi*)

Abstract. Let $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ $(W\widetilde{L}_{q,\lambda}(\mathbb{R}^n))$ be the (weak) modified Morrey spaces. In this paper, for some appropriate indices p,λ and q, we firstly prove that the commutator $[b,I_\alpha]$, generated by the symbol b and the fractional intergal operator I_α, is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if b belongs to $\mathrm{VMO}(\mathbb{R}^n)$. Besides, for the fractional maximal commutator $M_{\alpha,b}$, the result still holds. Moreover, commutators of fractional maximal functions with symbol b are investigated. More precisely, it is shown that commutators $[b,M_\alpha]$ is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if b belongs to $\mathrm{VMO}(\mathbb{R}^n)$ with the negative part of b equals to zero almost everywhere.

1. Introduction

In 1961, John and Nirenberg [1] defined the Bounded Mean Oscillation spaces $\mathrm{BMO}(\mathbb{R}^n)$. Let $Q=Q(x,r)$ be a cube in $\mathbb{R}^n$ centered at x with sides parallel to the axes having sidelength r. A locally integrable function f is said to be in BMO spaces if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}=\sup_{r>0,x\in\mathbb{R}^n}\frac{1}{|Q(x,r)|}\int_{Q(x,r)}|f(y)-f_Q|dy<\infty,$$

where f_Q means the average of f on Q. And a function f in $\mathrm{BMO}(\mathbb{R}^n)$ is said to have vanishing mean oscillation, or to belong to the Sarason class $\mathrm{VMO}(\mathbb{R}^n)$ (see [2]), if

$$\lim_{\rho\to 0}\eta(\rho)=0,$$

where $\eta(\rho)$ $(\rho>0)$, the VMO modulus of f, is defined by

$$\eta(\rho)=\sup_{r\leqslant\rho,x\in\mathbb{R}^n}\frac{1}{|Q(x,r)|}\int_{Q(x,r)}|f(y)-f_Q|dy.$$

It is obvious that VMO space contains all uniformly continuous functions in $\mathrm{BMO}(\mathbb{R}^n)$ and is a closed subspace of BMO space (see [2]).

Mathematics subject classification (2010): 42B35, 47B47.
Keywords and phrases: Characterization, commutator, VMO.
* Corresponding author.

Let I_α $(0<\alpha<n)$ be the fractional integral operator of order α, defined by

$$I_\alpha f(x):=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

For a locally integrable function b, Chanillo [3] proved that if $b\in \mathrm{BMO}(\mathbb{R}^n)$, the commutator

$$[b,I_\alpha]f(x):=b(x)I_\alpha f(x)-I_\alpha(bf)(x),$$

is bounded from Lebesgue spaces $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$ and that the reverse is also valid, if $n-\alpha$ is even. A complete characterization of $\mathrm{BMO}(\mathbb{R}^n)$ via the commutator $[b,I_\alpha]$ was shown by Ding [4]. Similarly, the boundedness of $[b,I_\alpha]$ from classical (or generalized) Morrey spaces to itself [5, 6, 7, 4] or from Morrey spaces to its predual [8] can be used to characterized the BMO spaces as well. And there are a number of classical results that demonstrate BMO spaces are the right collections to do harmonic analysis on the boundedness of commutators, see [9, 10, 11, 12].

Then, a natural question occurs to us: does there exist a kind of boundedness of $[b,I_\alpha]$ to characterize the $\mathrm{VMO}(\mathbb{R}^n)$ spaces? In this paper, we will give an affirmative answer. We show that $b\in \mathrm{VMO}(\mathbb{R}^n)$ if and only if the commutator $[b,I_\alpha]$ is bounded from the modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to the weak modified Morrey spaces $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Another subject of this paper is to show characterization of the VMO spaces via fractional maximal commutator and commutator of factional maximal functions that are defined by

$$M_{\alpha,b}f(x):=M_\alpha\big((b(x)-b)f\big)(x)$$

and

$$[b,M_\alpha]f(x):=M_\alpha(bf)(x)-b(x)M_\alpha f(x),$$

respectively, where M_α denotes fractional maximal function given by

$$M_\alpha f(x)=\sup_{t>0}|B(x,t)|^{-1+\frac{\alpha}{n}}\int_{B(x,t)}|f(y)|dy.$$

What should be emphasized here is that the operators $M_{\alpha,b}$ and $[b,M_\alpha]$ are different from each other. For example, $M_{\alpha,b}$ is positive and sublinear, but $[b,M_\alpha]$ is neither positive nor sublinear. Both of them play an important role in the study of commutators of singular operators with BMO symbols and have been investigated, intensively (see [14, 15, 16, 17, 18, 19, 20, 26, 27]). One of the applications is the characterization of BMO functions via the strong-type boundedness of them acting on Morrey spaces or Lebesgue spaces. Readers interested can refer to [15, 20, 27, 26] for instance. In this paper, we apply weak-type boundedness for character VMO functions; More precisely, it is shown that the commutator $M_{\alpha,b}$ is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if b is in $\mathrm{VMO}(\mathbb{R}^n)$, and that the commutator $[b,M_\alpha]$ is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if b is in $\mathrm{VMO}(\mathbb{R}^n)$ with the negative part of b equals to zero almost everywhere. Therefore, our results imply that both $M_{\alpha,b}$ and $[b,M_\alpha]$ have properties that the strong-type and weak-type boundedness are equal on the modified Morrey spaces.

2. Notations and preliminaries

Throughout the paper, the letter C denote a positive constant that may vary from line to line but remains independent of the main variables. We write $A \lesssim B$ to indicate that A is majorized by B times a constant independent of A and B, and that $Q = Q(x,r)$ denotes the cube centered at x with side length r. Given a Lebesgue measurable set E, χ_E will denote the characteristic function of E and $|E|$ is the Lebesgue measure of E.

DEFINITION 2.1. (see [21]) Let $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant n$ and $[r]_1 = \min(1,r)$. We denote by $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified Morrey spaces, and by $W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the weak modified Morrey spaces, as the set of locally integrable function f, with the finite norms

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \left([r]_1^{-\lambda} \int_{Q(x,r)} |f(y)|^p dy\right)^{\frac{1}{p}},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)} = \sup_{t>0} t \Big(\sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\lambda} |\{y \in Q(x,r) : \ |f(y)| > t\}| \Big)^{\frac{1}{p}},$$

respectively.

For convenience, we denote $\widetilde{L}_{p,\lambda} := \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ and $W\widetilde{L}_{p,\lambda} := W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$. From [21], we have

$$\widetilde{L}_{p,\lambda} \subset\!\succ L_{p,\lambda} \cap L_p \quad \text{and} \quad \max(\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}) \leqslant \|f\|_{\widetilde{L}_{p,\lambda}}, \tag{2.1}$$

where $L_{p,\lambda}$ denotes the classical Morrey spaces whose definitions can be found in many papers such as [5, 6, 7, 8, 12]. And, the opposite of (2.1) is true as well [13, Lemma 3.1]. So the following lemma is valid.

LEMMA 2.1. *Let* $1 \leqslant p < \infty$ *and* $0 \leqslant \lambda \leqslant n$. *Then*

$$\widetilde{L}_{p,\lambda} = L_{p,\lambda} \cap L_p \quad \text{and} \quad \|f\|_{\widetilde{L}_{p,\lambda}} = \max(\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}).$$

Given function $f \in L^1_{loc}(\mathbb{R}^n)$. Denote by $M^\sharp$ the Sharp Maximal Function:

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

And Strömberg related commutators with Sharp Maximal function in [22], firstly. Using $M^\sharp$ to act operator $[b, I_\alpha]$, Shirai [5] get the following lemma.

LEMMA 2.2. (see [5]) *Let* $0 < \alpha < n$, $1 < r < \infty$ *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$. *Then there exists a constant* $C \geqslant 0$, *independent of* b *and* f, *such that*

$$M^\sharp([b, I_\alpha]f)(x) \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \big\{ I_\alpha(|f|)(x) + I_{\alpha,r}(|f|)(x) \big\},$$

for almost all x *and every* $f \in C_c^\infty(\mathbb{R}^n)$, *where* $I_{\alpha,r}(|f|)(x) = (I_{r\alpha}(|f|)^r(x))^{\frac{1}{r}}$.

The bounedeness of $M_{\alpha,b}$ can be easily obtained by the inequality $M_{\alpha,b}f(x) \leqslant I_{\alpha,b}f(x)$, where the operator $I_{\alpha,b}$, having similar properties with the commutator $[b,I_\alpha]$, is defined as follows

$$I_{\alpha,b}f(x) = \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|}{|x-y|^{n-\alpha}} f(y)dy.$$

LEMMA 2.3. *Let $0<\alpha<n$, $1<r<\infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then there exists a constant $C \geqslant 0$, independent of b and f, such that*

$$M^\sharp(I_{\alpha,b}f)(x) \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\{I_\alpha(|f|)(x)+I_{\alpha,r}(|f|)(x)\}$$

for almost all x and every $f \in C_c^\infty(\mathbb{R}^n)$, where $I_{\alpha,r}(|f|)(x) = (I_{r\alpha}(|f|)^r(x))^{\frac{1}{r}}$.

The proof of the lemma is similar to the proof of Lemma 4.2 in [5] and be omitted here.

LEMMA 2.4. *Let $0<\lambda<n$ and $1<p<\infty$. Then there exists a constant $C \geqslant 0$, independent of f, such that*

$$\|Mf\|_{\widetilde{L}_{p,\lambda}} \leqslant C\|M^\sharp f\|_{\widetilde{L}_{p,\lambda}},$$

for almost all x and every $f \in \widetilde{L}_{p,\lambda}$.

Proof. From Lemma 2.1, the inequalities $\|Mf\|_{L_{p,\lambda}} \leqslant C\|M^\sharp f\|_{L_{p,\lambda}}$ (see [6]) and $\|Mf\|_{L_p} \leqslant C\|M^\sharp f\|_{L_p}$ (see [23]), it is obvious that

$$\|Mf\|_{\widetilde{L}_{p,\lambda}} \leqslant \max(C_1\|M^\sharp f\|_{\widetilde{L}_{p,\lambda}}, C_2\|M^\sharp f\|_{\widetilde{L}_{p,\lambda}}) \leqslant C\|M^\sharp f\|_{\widetilde{L}_{p,\lambda}},$$

which implies the proof is completed. □

To get $b \in \mathrm{VMO}(\mathbb{R}^n)$ by weak-type boundedness of $[b,I_\alpha]$, the following lemma will be our main tool.

LEMMA 2.5. *Let $0<\lambda<n$ and $1 \leqslant p' < p$. If $f \in W\widetilde{L}_{p,\lambda}$, then, for any cube $Q = Q(x,r)$, we have*

$$\int_Q |f(y)|^{p'}dy \lesssim r^{n-\frac{np'}{p}}[r]_1^{\frac{\lambda p'}{p}}\|f\|_{W\widetilde{L}_{p,\lambda}}^{p'}.$$

Proof. Let Q be a fixed cube and $f \in W\widetilde{L}_{p,\lambda}$. Choose $N = |Q|^{-\frac{1}{p}}[r]_1^{\frac{\lambda}{p}}\|f\|_{W\widetilde{L}_{p,\lambda}}$, then

$$\begin{aligned}\int_Q |f(y)|^{p'}dy &= \int_0^\infty t^{p'-1}|\{y\in Q:\ |f(y)|>t\}|dt \\ &= \Big(\int_0^N + \int_N^\infty\Big)t^{p'-1}|\{y\in Q:\ |f(y)|>t\}|dt\end{aligned}$$

$$\leqslant |Q|N^{p'} + [r]_1^{\lambda} N^{p'-p} \|f\|^p_{W\widetilde{L}_{p,\lambda}}$$

$$= r^{n-\frac{np'}{p}} [r]_1^{\frac{\lambda p'}{p}} \|f\|^{p'}_{W\widetilde{L}_{p,\lambda}},$$

which implies that the proof of the lemma is completed. □

REMARK 2.1. In Lemma 2.4, if $0<\lambda'<n$ satisfying $\frac{n-\lambda'}{p'}<\frac{n-\lambda}{p}$, then $W\widetilde{L}_{p,\lambda}\subset \widetilde{L}_{p',\lambda'}$. Moveover, $\|f\|_{\widetilde{L}_{p',\lambda'}}\leqslant \|f\|_{W\widetilde{L}_{p,\lambda}}$. In fact, from lemma 2.4, it follows that

$$\Big(\frac{1}{[r]_1^{\lambda'}}\int_Q |f(y)|^{p'}dy\Big)^{\frac{1}{p'}} \leqslant r^{\frac{n}{p'}-\frac{n}{p}}[r]_1^{\frac{\lambda}{p}-\frac{\lambda'}{p'}}\|f\|_{W\widetilde{L}_{p,\lambda}} \leqslant \|f\|_{W\widetilde{L}_{p,\lambda}},$$

which implies the conclusion.

3. Main results

Next, we show our main results as well as their proofs. The first results about the modified Morrey estimates for the Riesz potential I_α can be deduced by the result of Guliyev et al. [21, Theorem 2]. And we give a new proof here.

THEOREM 3.1. *Let $0<\alpha<n$, $0\leqslant\lambda<n-\alpha$, $1<p<(n-\lambda)/\alpha$ and $1/q=1/p-\alpha/(n-\lambda)$. Then I_α is bounded from $\widetilde{L}_{p,\lambda}$ to $\widetilde{L}_{q,\lambda}$.*

Proof. From Lemma 2.1, the $(L_{p,\lambda},L_{q,\lambda})$ and (L_p,L_q) boundedness of I_α, the conclusion follows immediately. □

COROLLARY 3.1. *Let $0<\alpha<n$, $0\leqslant\lambda<n-\alpha$, $1<p<(n-\lambda)/\alpha$ and $1/q=1/p-\alpha/(n-\lambda)$. Then M_α is bounded from $\widetilde{L}_{p,\lambda}$ to $\widetilde{L}_{q,\lambda}$.*

Now we state our results about commutator $[b,I_\alpha]$ with a symbol b, that is, the characterization of VMO spaces via the commutators $[b,I_\alpha]$, $M_{\alpha,b}$ and $[b,M_\alpha]$.

THEOREM 3.2. *Let $0<\alpha<n$, $0<\lambda<n-\alpha$, $1<p<(n-\lambda)/\alpha$ and $1/q=1/p-\alpha/(n-\lambda)$. Then the following statements are equivalent:*

1. $b\in \mathrm{VMO}(\mathbb{R}^n)$.
2. *$[b,I_\alpha]$ is bounded from $\widetilde{L}_{p,\lambda}$ to $\widetilde{L}_{q,\lambda}$.*
3. *$[b,I_\alpha]$ is bounded from $\widetilde{L}_{p,\lambda}$ to $W\widetilde{L}_{q,\lambda}$.*

Proof. Since that $(2)\Rightarrow(3)$ is clear, it suffices to prove the following assertions.

$(1)\Rightarrow(2)$: Let $b\in\mathrm{VMO}(\mathbb{R}^n)$, then $b\in\mathrm{BMO}(\mathbb{R}^n)$. From Lemma 2.1, the $(L_{p,\lambda},L_{q,\lambda})$ and (L_p,L_q) boundedness of $[b,I_\alpha]$, the conclusion follows immediately

$(3) \Rightarrow (1)$: Follow the method in [22]. Choose $z_0 \in \mathbb{R}^n \backslash \{0\}$ and $\delta > 0$ such that $|x|^{n-\alpha}$ has an absolutely convergent Fourier series in the cube $Q(z_0, \delta)$,

$$|x|^{n-\alpha} = \sum_m a_m e^{i\langle v_m, x\rangle},$$

with $\sum_m |a_m| < \infty$, where the exact form of the vectors v_m is unrelated, since $|x|^{n-\alpha} \in C^\infty(Q(z_0,\delta))$. Set $z_1 = \delta^{-1} z_0$. If $|z - z_1| < \sqrt{n}$, we have the expansion

$$|z|^{n-\alpha} = \delta^{\alpha-n}|\delta z|^{n-\alpha} = \delta^{\alpha-n}\sum_m a_m e^{i\langle v_m, \delta z\rangle}.$$

Let $Q_0 = Q(x_0, r)$ be a fixed cube. we consider $f = \chi_{Q_0}$. It is clear that

$$\|\chi_{Q_0}\|_{\widetilde{L}^{p,\lambda}} = r^{\frac{n}{p}}[r]_1^{-\frac{\lambda}{p}}. \tag{3.1}$$

Denote $Q' = Q(x_0 - rz_1, r)$. Then, for any $x, y \in Q$ and $z \in Q'$,

$$\left|\frac{x-y}{r} - z_1\right| \leqslant \left|\frac{x-x_0}{r}\right| + \left|\frac{y-(x_0-rz_1)}{r}\right| < \sqrt{n}.$$

Let $s(x) = \overline{\operatorname{sgn}(\int_{Q'}(b(x)-b(y))dy)}$. Then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |b(x) - b_{Q'}|dx \\
&= \delta^{\alpha-n}\frac{1}{|Q|^2}\int_Q\int_{Q'} s(x)(b(x)-b(y))r^{n-\alpha}|x-y|^{\alpha-n}\left|\frac{(x-y)\delta}{r}\right|^{n-\alpha}dydx \\
&= Cr^{-\alpha-n}\sum_m a_m \int_Q\int_{\mathbb{R}^n}(b(x)-b(y))|x-y|^{\alpha-n}e^{i\langle v_m, \frac{\delta}{r}x\rangle}s(x)\chi_Q(x)e^{-i\langle v_m, \frac{\delta}{r}y\rangle}\chi_{Q'}(y)dydx.
\end{aligned}$$

Set $g_m(y) = e^{-i\langle v_m, \frac{\delta}{r}y\rangle}\chi_{Q'}(y)$ and $h_m(x) = e^{i\langle v_m, \frac{\delta}{r}x\rangle}s(x)\chi_Q(x)$, then

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |b(x) - b_{Q'}|dx &= \frac{C}{|Q|^{1+\alpha/n}}\sum_m a_m\int_Q\int_{\mathbb{R}^n}(b(x)-b(y))|x-y|^{\alpha-n}g_m(y)h_m(x)dydx \\
&= \frac{C}{|Q|^{1+\alpha/n}}\sum_m a_m\int_Q [b, I_\alpha](g_m)(x)h_m(x)dx.
\end{aligned}$$

Choose $1 < q' < q$. By the Lemma 2.5 and (3.1), it follows that

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |b(x) - b_Q|dx &\leqslant \frac{2}{|Q|}\int_Q |b(x) - b_{Q'}|dx \\
&\lesssim r^{-\alpha-\frac{n}{q'}}\sum_m |a_m|\Big(\int_Q |[b, I_\alpha](g_m)(x)|^{q'}dx\Big)^{\frac{1}{q'}} \\
&\leqslant r^{-\alpha-\frac{n}{q'}}\sum_m |a_m| r^{\frac{n}{q'}-\frac{n}{q}}[r]_1^{\frac{\lambda}{q}}\|[b, I_\alpha](g_m)\|_{W\widetilde{L}_{q,\lambda}}
\end{aligned}$$

$$\lesssim r^{-\alpha-\frac{n}{q}}[r]_1^{\frac{\lambda}{q}}\|\chi_Q\|_{\widetilde{L}_{p,\lambda}}\sum_m |a_m|\|[b,I_\alpha]\|_{\widetilde{L}_{p,\lambda}\to W\widetilde{L}_{q,\lambda}}$$
$$\lesssim \min(r^{(n-\lambda)(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n-\lambda})}, r^{n(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n})}).$$

Let $r<1$. Observing that

$$\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n}>\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n-\lambda}=0,$$

we have

$$\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx \leqslant \min\left(1, r^{n(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n})}\right)=r^{n\left(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n}\right)}\to 0 \quad \text{as} \quad r\to 0,$$

which implies $b\in \mathrm{VMO}(\mathbb{R}^n)$. This completes the proof. $\square$

THEOREM 3.3. *Let $0<\alpha<n$, $0<\lambda<n-\alpha$, $1<p<(n-\lambda)/\alpha$ and $1/q=1/p-\alpha/(n-\lambda)$. Then the following statements are equivalent:*

1. $b\in \mathrm{VMO}(\mathbb{R}^n)$.
2. $M_{\alpha,b}$ *is bounded from* $\widetilde{L}_{p,\lambda}$ *to* $\widetilde{L}_{q,\lambda}$.
3. $M_{\alpha,b}$ *is bounded from* $\widetilde{L}_{p,\lambda}$ *to* $W\widetilde{L}_{q,\lambda}$.

Proof. It suffices to show the following assertions.

$(1)\Rightarrow(2)$: Let $1<r<p$. By Lemma 2.4 and 2.3, it is concluded that

$$\begin{aligned}\|M_{\alpha,b}f\|_{\widetilde{L}_{q,\lambda}} &\leqslant \|M(I_{\alpha,b}f)\|_{\widetilde{L}_{q,\lambda}} \leqslant \|M^\sharp(I_{\alpha,b}f)\|_{\widetilde{L}_{q,\lambda}}\\ &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\left\{\|I_\alpha(|f|)\|_{\widetilde{L}_{q,\lambda}}+\|I_{\alpha r}(|f|^r)\|_{\widetilde{L}_{\frac{q}{r},\lambda}}^{\frac{1}{r}}\right\}\\ &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{L}_{p,\lambda}}.\end{aligned}$$

$(3)\Rightarrow(1)$: For any $x\in Q$, we have

$$\begin{aligned}|Q|^{\frac{\alpha}{n}}|b(x)-b_Q| &\leqslant \frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q |b(x)-b(y)|dy\\ &\leqslant \sup_{x\in Q'}\frac{1}{|Q'|^{1-\frac{\alpha}{n}}}\int_{Q'\cap Q}|b(x)-b(y)|dy\\ &=M_{\alpha,b}\chi_Q(x).\end{aligned}$$

Let $N>0$ and $0<\delta\leqslant 1$. Noticing $q>p>1$, we have

$$\begin{aligned}\int_Q |b(x)-b_Q|dx &= \delta\left\{\int_0^N+\int_N^\infty\right\}|\{x\in Q:\ |b(x)-b_Q|>t\}|dt\\ &\leqslant |Q|N+[r]_1^{\lambda}r^{-\alpha q}\|M_{\alpha,b}\chi_Q\|_{W\widetilde{L}_{q,\lambda}}^q N^{1-q}\end{aligned}$$

$$\lesssim |Q|N + [r]_1^{\lambda - \frac{\lambda q}{p}} r^{-\alpha q + \frac{nq}{p}} N^{1-q}$$
$$\leqslant \min(r^n N + r^{\lambda - \frac{\lambda q}{p} - \alpha q + \frac{nq}{p}} N^{1-q}, r^n N + r^{-\alpha q + \frac{nq}{p}} N^{1-q}),$$

Observing that $r^n N + r^{\lambda - \frac{\lambda q}{p} - \alpha q + \frac{nq}{p}} N^{1-q}$ and $r^n N + r^{-\alpha q + \frac{nq}{p}} N^{1-q}$ gain the minimal value with respect to N, at

$$N = r^{\frac{n-\lambda}{p} - \frac{n-\lambda}{q} - \alpha} \quad \text{and} \quad N = r^{\frac{n}{p} - \frac{n}{q} - \alpha},$$

respectively. Then we have

$$\frac{1}{|Q|}\int_Q |b(x) - b_Q| dx \leqslant \min(r^{(n-\lambda)(\frac{1}{p} - \frac{1}{q} - \frac{\alpha}{n-\lambda})}, r^{n(\frac{1}{p} - \frac{1}{q} - \frac{\alpha}{n})}).$$

Hence we obtain that $b \in \mathrm{VMO}(\mathbb{R}^n)$. The proof of the theorem is completed. □

In order to investigate the boundedness of $[b, M_\alpha]$ on Morrey spaces, we need the following relationships between $[b, M_\alpha]$ and $M_{\alpha,b}$.

LEMMA 3.1. *Let b be any non-negative locally integral function in $\mathbb{R}^n$. Then*

$$|[b, M_\alpha] f(x)| \leqslant M_{\alpha,b} f(x)$$

for all $f \in L^1_{loc}(\mathbb{R}^n)$.

Proof. Since the pointwise estimate $|M_\alpha f(x) - M_\alpha g(x)| \leqslant M_\alpha (f-g)(x)$ holds for any $f, g \in L^1_{loc}(\mathbb{R}^n)$, we can write

$$\begin{aligned} |[b, M_\alpha] f(x)| &= |M_\alpha(bf)(x) - b(x) M_\alpha f(x)| = |M_\alpha(bf)(x) - M_\alpha(b(x)f)(x)| \\ &\leqslant |M_\alpha\big(bf - b(x)f\big)(x)| = M_{\alpha,b} f(x), \end{aligned}$$

for any non-negative function b. □

Combining to Theorem 3.3, this lemma easily yields the following corollaries.

COROLLARY 3.2. *Let $0 < \alpha < n$, $0 < \lambda < n - \alpha$, $1 < p < (n-\lambda)/\alpha$ and $1/q = 1/p - \alpha/(n-\lambda)$. If non-negative function b is in $\mathrm{BMO}(\mathbb{R}^n)$, then $[b, M_\alpha]$ is bounded from $\widetilde{L}_{p,\lambda}$ to $\widetilde{L}_{q,\lambda}$.*

Also, we need introduce the following tool

$$M_{\alpha,Q} f(x) = \sup_{Q_0 \ni x, Q_0 \subset Q} \frac{1}{|Q_0|^{1-\frac{\alpha}{n}}} \int_{Q_0} |f(y)| dy.$$

When $\alpha = 0$, we denote $M_Q = M_{0,Q}$ that is used by many people (see [24, 20, 25, 26, 28] for instance). Denote by $b^+ = \max\{b, 0\}$ and $b^- = -\min\{b, 0\}$. Then $b = b^+ - b^-$ and $|b| = b^+ + b^-$.

THEOREM 3.4. *Let $0<\alpha<n$, $0<\lambda<n-\alpha$, $1<p<(n-\lambda)/\alpha$ and $1/q=1/p-\alpha/(n-\lambda)$. Then the following statements are equivalent:*

1. $b\in \mathrm{VMO}(\mathbb{R}^n)$ *and* $b^-(x)=0 \quad a.e.\ x\in\mathbb{R}^n$.
2. $[b,M_\alpha]$ *is bounded from* $\widetilde{L}_{p,\lambda}$ *to* $\widetilde{L}_{q,\lambda}$.
3. $[b,M_\alpha]$ *is bounded from* $\widetilde{L}_{p,\lambda}$ *to* $W\widetilde{L}_{q,\lambda}$.
4. *For any* $0<\delta\leqslant 1$, *we have*

$$\lim_{|Q|\to 0}\frac{1}{|Q|}\int_Q |b(x)-|Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}b(x)|^\delta dx=0.$$

Proof. $(1)\Rightarrow(2)$: Since $(2)\Rightarrow(3)$ is clear, suffices to prove the following assertions. For $b\in VMO(\mathbb{R}^n)$ and $b^-(x)=0$ $a.e.\ x\in\mathbb{R}^n$, we have

$$|[b,M_\alpha]f(x)|\leqslant|[M_\alpha,|b|]f(x)|\leqslant M_{\alpha,|b|}f(x)\leqslant M_{\alpha,b}f(x).$$

So, (2) follows from Theorem 3.3.

$(3)\Rightarrow(4)$: Let Q be a fixed cube. We have (see (2.4) in [27])

$$M_\alpha\chi_Q(x)=M_{\alpha,Q}\chi_Q(x)=|Q|^{\frac{\alpha}{n}},\quad M_\alpha(b\chi_Q)(x)=M_{\alpha,Q}b(x),\quad \forall x\in Q.$$

Choose $N=r^{n/p-n/q}$. Similar to the proof of $(3)\Rightarrow(1)$ in Theorem 3.3, we have

$$\begin{aligned}
&|Q|^{-1}\int_Q|b(x)-|Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}b(x)|^\delta dx\\
&=|Q|^{-1-\frac{\delta\alpha}{n}}\int_Q|b(x)M_\alpha\chi_Q(x)-M_{\alpha,Q}(b\chi_Q)(x)|^\delta dx\\
&=\delta|Q|^{-1-\frac{\delta\alpha}{n}}\left\{\int_0^N+\int_N^\infty\right\}t^{\delta-1}|\{x\in Q:\ |[b,M_\alpha]\chi_Q(x)|>t\}|dt\\
&\leqslant|Q|^{-\frac{\delta\alpha}{n}}N^\delta+r^{-n-\delta\alpha}[r]_1^\lambda\|[b,M_\alpha]f\|^q_{W\widetilde{L}_{q,\lambda}}N^{\delta-q}\\
&\leqslant r^{-\delta\alpha}N^\delta+r^{-n-\delta\alpha}[r]_1^\lambda\|\chi_Q\|^q_{\widetilde{L}_{p,\lambda}}N^{\delta-q}\\
&\leqslant\min(r^{-\delta\alpha}N^\delta+r^{\lambda-n-\delta\alpha+\frac{nq}{p}-\frac{\lambda q}{p}}N^{\delta-q},r^{-\delta\alpha}N^\delta+r^{-n-\delta\alpha+\frac{nq}{p}}N^{\delta-q}).
\end{aligned}$$

Moreover, it can be seen that

$$r^{-\delta\alpha}N^\delta+r^{\lambda-n-\delta\alpha+\frac{nq}{p}-\frac{\lambda q}{p}}N^{\delta-q}\ \text{ and }\ r^{-\delta\alpha}N^\delta+r^{-n-\delta\alpha+\frac{nq}{p}}N^{\delta-q}$$

gain the minimal value with respect to N, at

$$N=r^{\frac{n-\lambda}{p}-\frac{n-\lambda}{q}}\quad\text{and}\quad N=r^{\frac{n}{p}-\frac{n}{q}},$$

respectively. Then we get

$$|Q|^{-1}\int_Q |b(x)-|Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}b(x)|^{\delta}dx \leqslant \min(r^{\delta(n-\lambda)(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n-\lambda})}, r^{\delta n(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n})}).$$

Similarly, let $r<1$. The fact

$$\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n} > \frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n-\lambda}=0,$$

yields

$$|Q|^{-1}\int_Q |b(x)-|Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}b(x)|^{\delta}dx \leqslant r^{n\left(\frac{n}{p}-\frac{n}{q}-\frac{\alpha}{n}\right)} \to 0 \quad \text{as} \quad r\to 0.$$

(4) $\Rightarrow$ (1): Take $\delta=1$. Let $E=\{x\in Q:\ b(x)\leqslant b_Q\}$ and $F=\{x\in Q:\ b(x)>b_Q\}$. Then $\int_E |b(x)-b_Q|dx=\int_F |b(x)-b_Q|dx$.

Observing $b(x)\leqslant b_Q\leqslant |b_Q|\leqslant |Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}b(x)$ for any $x\in E$, we have that

$$\begin{aligned}\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx &= \frac{2}{|Q|}\int_E |b(x)-b_Q|dx\\ &= \frac{2}{|Q|}\int_E |b(x)-|Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}b(x)|dx\\ &= \frac{2}{|Q|}\int_Q |b(x)-|Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}b(x)|dx\end{aligned}$$

and that $b\in \mathrm{VMO}(\mathbb{R}^n)$ follows immediately.

Now we show that $b^-(x)=0$ $a.e.\ x\in\mathbb{R}^n$. Let $Q=Q(x_0,r)$ be a fixed cube. Note that $|b|\leqslant M_Q b$ in Q. Therefore, in Q, we have

$$0\leqslant b^- = |b|-b^+ \leqslant M_Q b - b^+ + b^- = M_Q b - b,$$

which follows that

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |b^-(x)|dx\\ &\leqslant \frac{1}{|Q|}\int_Q |M_Q b(x)-b(x)|dx\\ &\lesssim \frac{1}{|Q|}\int_Q |b(x)-|Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}b(x)|dx+\frac{1}{|Q|}\int_Q ||Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}b(x)-M_Q b(x)|dx\\ &=: M_1+M_2.\end{aligned}$$

It is clear that $M_1\to 0$ as $|Q|\to 0$ (follows from the statement (4)). So, in the following we only show $M_2\to 0$ as $|Q|\to 0$.

Also, the fact holds in Q that $M_Q b(x)=M(b\chi_Q)(x)$, $M\chi_Q(x)=M_Q\chi_Q(x)=\chi_Q(x)=1$, $M_{\alpha,Q}b(x)=M_\alpha(b\chi_Q)(x)$ and $|Q|^{\frac{\alpha}{n}}=M_\alpha\chi_Q(x)$. Therefore, we have

$$\begin{aligned}M_2 &= \frac{1}{|Q|^{1+\frac{\alpha}{n}}}\int_Q |M_\alpha(|b|\chi_Q)(x)-|b(x)|M_\alpha\chi_Q(x)\\ &\quad +|b(x)|M_\alpha\chi_Q(x)M\chi_Q(x)-M_\alpha\chi_Q(x)M(|b|\chi_Q)(x)|dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |[M_\alpha, |b|]\chi_Q(x) - M_\alpha \chi_Q(x)[M, |b|]\chi_Q(x)|dx \\
&\leqslant \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |[M_\alpha, |b|]\chi_Q(x)|dx + \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |M_\alpha \chi_Q(x)[|b|, M]\chi_Q(x)|dx \\
&=: M_{2,1} + M_{2,2}.
\end{aligned}
$$

Firstly, from Hölder's inequality and Corollary 3.2, it follows that

$$
\begin{aligned}
M_{2,1} &\leqslant r^{-\alpha-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}} \|[|b|, M_\alpha]\chi_Q\|_{\widetilde{L}_{q,\lambda}} \lesssim r^{-\alpha-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}} \|\chi_Q\|_{\widetilde{L}_{p,\lambda}} \\
&\leqslant \min(r^{(n-\lambda)(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n-\lambda})}, r^{n(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n})}).
\end{aligned}
$$

So, we get $M_{2,1} \to 0$ as $r \to 0$.

Then by Hölder's inequality with exponent q and q', we have

$$
\begin{aligned}
M_{2,2} &\leqslant \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \Big(\int_Q |M_\alpha \chi_Q(x)|^q dx\Big)^{\frac{1}{q}} \Big(\int_Q |[|b|, M]\chi_Q(x)|^{q'} dx\Big)^{\frac{1}{q'}} \\
&\leqslant r^{-n-\alpha} [r]_1^{\frac{\lambda}{q}+\frac{\lambda}{q'}} \|M_\alpha \chi_Q\|_{\widetilde{L}_{q,\lambda}} \|[|b|, M]\chi_Q\|_{\widetilde{L}_{q',\lambda}}.
\end{aligned}
$$

Besides,

$$
M_{2,2} \lesssim r^{-n-\alpha} [r]_1^{\lambda} \|\chi_Q\|_{\widetilde{L}_{p,\lambda}} \|\chi_Q\|_{\widetilde{L}_{q',\lambda}} \leqslant \min(r^{(n-\lambda)(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n-\lambda})}, r^{n(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n})}) \to 0 \quad \text{as} \quad r \to 0,
$$

by corollaries 3.1 and 3.2. Hence, $M_2 \to 0$ as $|Q| \to 0$. Therefore, for any cube Q,

$$
\tfrac{1}{|Q|} \int_Q |b^-(x)|dx \to 0 \quad \text{as} \quad |Q| \to 0,
$$

that is, $b^-(x) = 0$ $a.e.$ $x \in \mathbb{R}^n$ follows from Lebesgue's differentiation theorem. □

REFERENCES

[1] F. JOHN, L. NIRENBERG, *On functions of bounded mean oscillation*, Communications on pure and applied Mathematics, 1961, 14 (3): 415–426.

[2] D. SARASON, *Functions of vanishing mean oscillation*, Transactions of the American Mathematical Society, 1975, 207: 391–405.

[3] S. CHANILLO, *A note on commutators*, Indiana University Mathematics Journal, 1982, 31 (1): 7–16.

[4] Y. DING, *A characterization of BMO via commutators for some operators*, Northeast. Math. J, 1997, 13 (4): 422–432.

[5] S. SHIRAI, *Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces*, Hokkaido mathematical journal, 2006, 35 (3): 683–696.

[6] G. DI FAZIO, M. A. RAGUSA, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital, 1991, 5 (7): 323–332.

[7] Y. KOMORI, T. MIZUHARA, *Notes on commutators and Morrey spaces*, Hokkaido mathematical journal, 2003, 32 (2): 345–353.

[8] D. R. ADAMS, *Commutators of Morrey Potentials Morrey Spaces*, Springer International Publishing, 2015: 77–83.

[9] R. R. COIFMAN, R. ROCHBERG, G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Annals of Mathematics, 1976, 103: 611–635.

[10] S. JANSON, M. TAIBLESON, G. WEISS, *Elementary characterizations of the Morrey-Campanato spaces*, Lecture Notes in Math, 1983, 992: 101–114.

[11] M. PALUSZYŃSKI, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana University Mathematics Journal, 1995, 44: 1–17.

[12] S. SHI, S. LU, *Some characterizations of Campanato spaces via commutators on Morrey spaces*, Pacific Journal of Mathematics, 2013, 264 (1): 221–234.

[13] V.-S. GULIYEV AND K.-R. RAHIMOVA, *Parabolic fractional maximal operator in modified parabolic Morrey spaces*, J. Funct. Space. Appl, **2012** (2012).

[14] A. M. ALPHONSE, *An end point estimate for maximal commutators*, Journal of Fourier Analysis and Applications, 2000, 6 (4): 449–456.

[15] J. GARCIA-CUERVA, E. HARBOURE, C. SEGOVIA, J. L. TORREA, *Weighted norm inequalities for commutators of strong singular integral*, Indiana University Mathematics Journal, 1991, 40 (4): 1397–1420.

[16] D. LI, G. HU, X. SHI, *Weighted norm inequalities for the maximal commutators of singular integral operators*, Journal of mathematical analysis and applications, 2006, 319 (2): 509–521.

[17] C. SEGOVIA, J. L. TORREA, *Weighted inequalities for commutators of fractional and singular integrals*, Publicacions Matemàtiques, 1991, 35 (1): 209–235.

[18] C. SEGOVIA, J. L. TORREA, *Higher order commutators for vector-valued Calderón-Zygmund operators*, Transactions of the American Mathematical Society, 1993, 336 (2): 537–556.

[19] M. MILMAN, T. SCHONBEK, *Second order estimates in interpolation theory and applications*, Proceedings of the American Mathematical Society, 1990, 110 (4): 961–969.

[20] J. BASTERO, M. MILMAN, F. RUIZ, *Commutators for the maximal and sharp functions*, Proceedings of the American Mathematical Society, 2000, 128 (11): 3329–3334.

[21] V. S. GULIYEV, J. J. HASANOV, Y. ZEREN, *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, Journal of the Mathematical Inequalities, 2011, 5 (4): 491–506.

[22] S. JANSON, *Mean oscillation and commutators of singular integral operators*, Arkiv för Matematik, 1978, 16 (1): 263–270.

[23] R. R. COIFMAN, R. ROCHBERG, *Another characterization of BMO*, Proceedings of the American Mathematical Society, 1980: 249–254.

[24] M. AGCAYAZI, A. GOGATISHVILI, K. KOCA, et al. *A note on maximal commutators and commutators of maximal functions*, Journal of the Mathematical Society of Japan, 2015, 67 (2): 581–593.

[25] C. XIE, *Some estimates of commutators*, Real Analysis Exchange, 2010, 36 (2): 405–416.

[26] P. ZHANG, J. WU, *Commutators of the fractional maximal function on variable exponent Lebesgue spaces*, Czechoslovak Mathematical Journal, 2014, 64 (1): 183–197.

[27] P. ZHANG, J. L. WU, *Commutators of the fractional maximal functions*, Acta Math. Sin., Chin. Ser, 2009, 52: 1235–1238.

[28] C. AYKOL, H. ARMUTCU, M. N. OMAROVA, *Maximal commutator and commutator of maximal function on modified Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics, 2016, 36 (1): 29–35.

(Received October 29, 2018)

Guangqing Wang
School of Mathematics and Statistics
Wuhan University
Wuhan, 430072, Republic of China
e-mail: 2017102010002@whu.edu.cn

Jinhui Li
School of Mathematics and Statistics
Central China Normal University
Wuhan, 430072, Republic of China
e-mail: lijinhui2050@mails.ccnu.edu.cn