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HARNACK INEQUALITY FOR STOCHASTIC HEAT EQUATION DRIVEN BY FRACTIONAL NOISE WITH HURST INDEX $H > \frac{1}{2}$

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Abstract. In this short note, we establish the dimensional-free Harnack inequality for stochastic heat equation with Dirichlet boundary condition:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + b(u(t,x)) + \dot{W}^H(t,x), \; 0 < t \leqslant T, \; 0 < x < 1, \\ u(t,0) = u(t,1) = 0, \; 0 < t \leqslant T, \\ u(0,x) = f(x), \; 0 \leqslant x \leqslant 1, \end{cases}$$

where $T > 0$, $f(x) \in L^2([0,1])$ and $W^H(t,x)$ is the fractional noise with Hurst index $H \in (\frac{1}{2}, 1)$. The strong Feller property is also obtained.

1. Introduction

The classical Harnack inequality was introduced by Harnack [9] for positive harmonic functions, after that, many scholars have made further research on it, especially, Wang [17] established the dimensional-free Harnack inequality. The original work of Wang has become a important tool in probability, see, for example, [22, 12, 13] for strong Feller property and contractivity properties; [1, 2] for short times behaviors of infinite dimensional diffusions; [3, 8] for heat kernel estimates, entropy-cost inequalities, and transportation cost inequalities.

The dimensional-free Harnack inequality can also be applied to establish Harnack inequality for various model, such as SDEs and SPDEs driven by Wiener process or Lévy process, see [15, 14, 10, 20, 21] and the reference. Recently, by using the method of coupling (see, for example [16]), Fan [6, 7] studied the Harnack inequality for SDEs driven by fractional Brownian motion, Yan and Yin [18, 19] proved the Harnack inequality for stochastic heat equation driven by fractional noise with Hurst index $0 < H < \frac{1}{2}$, while in the case of $\frac{1}{2} < H < 1$, the authors established Bismut formula firstly, then as an application, they get the Harnack inequality.

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The main motivation of this short note is to establish the Harnack inequality by coupling for the stochastic heat equation:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + b(u(t,x)) + \dot{W}^H(t,x),\ 0<t\leqslant T,\ 0<x<1, \\ u(t,0) = u(t,1) = 0,\ 0<t\leqslant T, \\ u(0,x) = f(x),\ x\in[0,1], \end{cases} \tag{1}$$

where $T>0$, $f\in L^2([0,1]) =: \mathbb{H}$ and $W^H(t,x)$ is the fractional noise with Hurst index $H>\frac{1}{2}$. In [19], in order to establish Bismut formula, the condition $f(x)\in\mathscr{C}^\gamma([0,1],\mathbb{R})$, $\gamma>2H-1$ is needed, compared to [19], in the short note we only need to assume that $f(x)\in\mathbb{H}$, and thus we can get the strong Feller property of the solution. For simplicity, we denote $u(t,x)=u(t,f,x)$ and $u(t,f)=u(t,f,\cdot)$ for $(t,x)\in[0,T]\times[0,1]$. We also define the operators $P_t, t>0$ by

$$(P_tG)(f) = \mathbb{E}[G(u(t,f))]$$

for all $G\in\mathscr{B}_b(\mathbb{H})$, where $\mathscr{B}_b(\mathbb{H})$ is the space of all bounded measurable functions on $\mathbb{H}$.

The rest of the paper is organized as follows. In Section 2, we recall some basic results about the fractional noise W^H. In section 3, we prove the main result.

2. Preliminaries

In this section, we briefly recall the definition of the stochastic integration with respect to W^H.

Let $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P})$ be a filtered probability space and $\frac{1}{2}<H<1$. A Gaussian process $W^H=\{W^H(t,x),(t,x)\in[0,T]\times[0,1]\}$ is called fractional noise if $W^H(0,x)=0$, $\mathbb{E}(W^H(t,x))=0$ and for $s,t\in[0,T]$,

$$\mathbb{E}[W^H(t,x)W^H(s,y)] = \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})\min(x,y),\ \forall x,y\in[0,1].$$

Let $\mathscr{E}$ be the set of step functions on $[0,T]\times[0,1]$. Denote by $\mathscr{H}$ the Hilbert space defined as the closure of $\mathscr{E}$ with respect to

$$\langle 1_{[0,t]\times[0,x]}, 1_{[0,s]\times[0,y]}\rangle_{\mathscr{H}} = \mathbb{E}(W^H(t,x)W^H(s,y))$$

for all $s,t\in[0,T]$ and $x,y\in[0,1]$. The linear mapping

$$W^H(\varphi) := \int_0^T\int_0^1 \varphi(t,x)W^H(dt,dx),\ \varphi\in\mathscr{E}$$

defined by $1_{[0,t]\times[0,x]}\mapsto W^H(t,x)$ can be extended as an isometry between $\mathscr{H}$ and the Gaussian spaces associated with W^H. This isometry is called the Wiener integral with respect to W^H and also denoted by

$$W^H(\varphi) = \int_0^T\int_0^1 \varphi(s,y)W^H(ds,dy),\ \forall\varphi\in\mathscr{H}.$$

Consider the kernel function

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2}-H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1-\left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du$$
$$= c_H\left(H-\frac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du$$

with $t > s > 0$, where $c_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}$, and define the linear operator K_H^* from $\mathscr{E}$ to $L^2([0,T]\times[0,1])$ as follows

$$(K_H^*\varphi)(s,x) = K_H(T,s)\varphi(s,x) + \int_s^T (\varphi(r,x)-\varphi(s,x))\frac{\partial K_H}{\partial r}(r,s)dr.$$

Then, we have

$$(K_H^* 1_{[0,t]\times A})(s,x) = K_H(t,s)1_{[0,t]\times A}$$

and

$$\langle K_H^*\varphi, K_H^*\phi\rangle_{L^2([0,T]\times[0,1])} = \langle\varphi,\phi\rangle_{\mathscr{H}}$$

for all $\varphi,\psi\in\mathscr{E}$, which show that the operator K_H^* provides an isometry between $\mathscr{E}$ and $L^2([0,T]\times[0,1])$, which can be extended to $\mathscr{H}$. Hence, the Gaussian family $W = \{W(t,A), t\in[0,T], A\in\mathscr{B}([0,1])\}$ defined by

$$W(t,A) = W^H((K_H^*)^{-1}1_{[0,t]\times A})$$

is a space-time noise, and

$$W^H(t,A) = \int_{[0,t]\times A} K_H(t,s)W(ds,dy)$$

for all $t\in[0,T]$ and $A\in\mathscr{B}([0,1])$.

LEMMA 2.1. *We have*

$$\int_0^T\int_0^1 \varphi(s,y)W^H(ds,dy) = \int_0^T\int_0^1 K_H^*\varphi(s,y)W(ds,dy)$$

and

$$\mathbb{E}[W^H(\psi)W^H(\varphi)] = \int_0^T\int_0^1 K_H^*\varphi(t,x)K_H^*\psi(t,x)dxdt,$$

Let $I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$ be the image of $L^2([0,T])$ by the operator $I_{0+}^{H+\frac{1}{2}}$, where I_{0+}^{α} is the α-order left Riemann-Liouville fractional integral on $[0,T]$. It is proved in [4] that the operator $K_H : L^2([0,T]) \to I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$ defined by $(K_H f)(t) = \int_0^t K_H(t,s)f(s)ds$, $f\in L^2([0,T])$ is an isomorphism and it has the following expression: $(K_H f)(t) = I_{0+}^1 s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f$.

The inverse operator K_H^{-1} is given by $(K_H^{-1}f)(s) = s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f'$ for all $f \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$, where D_{0+}^{α} is the α-order left Riemann-Liouville derivation.

Recall that W^H is an $\mathscr{F}_t$-fractional noise if it is a fractional noise such that the space-time white noise $W(t,A)$ defined above is $\mathscr{F}_t$-adapted and for each $t \in [0,T]$, $\{W(s,A) - W(t,A), A \in \mathscr{B}([0,1]), t \leqslant s \leqslant T\}$ are independent of $\mathscr{F}_t$. Given an $\mathscr{F}_t$-adapted process with integrable trajectories

$$\xi = \{\xi(t,x), t \in [0,T], x \in [0,1]\}.$$

Consider the transformation

$$\bar{W}^H(t,A) = W^H(t,A) + \int_0^t \int_A \xi(s,y)dyds.$$

Let $\bar{W}(ds,dy) = W(ds,dy) + K_H^{-1}\left(\int_0^{\cdot} \xi(r,y)dr\right)(s)dsdy$, then

$$\bar{W}^H(t,A) = \int_0^t \int_A K_H(t,s)\bar{W}(ds,dy)$$

and the following Girsanov theorem holds (see [11]).

THEOREM 2.2. *If the process ξ satisfies the next conditions:*

(i) $\int_0^{\cdot} \xi(r,y)dr \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T])) \otimes L^2([0,1])$, *almost surely.*

(ii) $\mathbb{E}L_T = 1$, *where*

$$L_T = \exp\Bigg[-\int_0^T \int_0^1 K_H^{-1}\left(\int_0^{\cdot} \xi(r,y)dr\right)(s)W(ds,dy) \\ -\frac{1}{2}\int_0^T \int_0^1 \left(K_H^{-1}\left(\int_0^{\cdot} \xi(r,y)dr\right)(s)\right)^2 dsdy\Bigg],$$

then $\bar{W}^H$ is an $\mathscr{F}_t$-fractional noise with Hurst index H under the new probability $d\bar{\mathbb{P}} = L_T d\mathbb{P}$.

3. Main results

In this section, we consider Harnack inequality for the operator P_t through the coupling by change of measure.

Assume that b satisfies the following assumption (A):

$$(x-y)(b(x)-b(y)) \leqslant K|x-y|^2, \;\; K > 0, \;\; x,y \in \mathbb{R}.$$

It has been showed in [11] that under assumption (A), the equation (1) has a unique solution.

For $\theta_0 \in (0,2)$. Let

$$\zeta(t) = \frac{2-\theta_0}{2K}\left(1 - e^{\frac{2}{3}K(t-T)}\right), \quad t \in [0,T]. \tag{1}$$

Then ζ is nonincreasing, smooth, positive on $[0,T)$ and satisfy

$$3\zeta'(t) - 2K\zeta(t) + 2 = \theta_0, \quad t \in [0,T].$$

The main result of the paper is

THEOREM 3.1. *Assume (A), let* $\frac{1}{2} < H < 1$, $\max\{1, \frac{2}{5-4H}\} < p < \frac{2}{3-2H}$, *then for any positive* $G \in \mathscr{B}_b(\mathbb{H})$, *we have*

$$(P_T G(f_2))^q \leqslant (P_T G^q)(f_1) \exp\left(C_p T^{2-2H} \frac{q}{(q-1)^2} \|f_1 - f_2\|_{\mathbb{H}}^2\right), \ \forall q > 1, \tag{2}$$

where $f_1, f_2 \in \mathbb{H}$, *and*

$$C_p = \frac{K^2 \Gamma(2H)\Gamma(3-2H) B^{\frac{2}{p}}\left(Hp - \frac{1}{2}p + 1, Hp - \frac{3}{2}p + 1\right)}{2H\Gamma^2(H - \frac{1}{2})\left|\frac{2p-2pH-1}{p-1}\right|^{\frac{2p-2}{p}}(1 - e^{-\frac{2}{3}K})^2} \times \left[\frac{1}{2H - 2 + \frac{2}{p}} + B\left(5 - 4H - \frac{2}{p}, 2H - 2 + \frac{2}{p}\right)\right].$$

Proof. We will divide our proofs into three steps.

Step I. Let $f_1, f_2 \in \mathbb{H}$, consider

$$\begin{cases} \partial_t v(t,x) = \frac{\partial^2}{\partial x^2} v(t,x) + b(v(t,x)) + \frac{u(t,f_1) - v(t,x)}{\zeta(t)} + \dot{W}^H(t,x), \ t \in (0,T), \ x \in (0,1) \\ v(t,0) = v(t,1) = 0, \ t \in (0,T), \\ v(0,x) = f_2(x), \ x \in [0,1], \end{cases}$$

According to [11], the above equation has a unique solution:

$$v(t,f_2) = f_2(x) + \int_0^t b(v(s,f_2))ds + \int_0^t \Delta v(s,f_2)ds + \int_0^t \frac{u(s,f_1) - v(s,f_2)}{\zeta(s)}ds + B^H(t),$$

where $\Delta = \frac{\partial^2}{\partial x^2}$, $e_n(x) = \sqrt{2}\sin(n\pi x)$, $n \geqslant 1$ and $B^H(t) = \sum_{n=1}^{\infty} \int_0^t \int_0^1 e_n(x) W^H(ds,dx) e_n$, thus we have

$$\begin{aligned} &\frac{\partial}{\partial t}\|u(t,f_1) - v(t,f_2)\|_{\mathbb{H}}^2 \\ &= 2\langle u(t,f_1) - v(t,f_2), \Delta(u(t,f_1) - v(t,f_2))\rangle_{\mathbb{H}} \\ &\quad + 2\langle u(t,f_1) - v(t,f_2), b(u(t,f_1)) - b(v(t,f_2))\rangle_{\mathbb{H}} - 2\frac{\|u(t,f_1) - v(t,f_2)\|_{\mathbb{H}}^2}{\zeta(t)} \\ &\leqslant 2\left(K - \frac{1}{\zeta(t)}\right)\|u(t,f_1) - v(t,f_2)\|_{\mathbb{H}}^2. \end{aligned}$$

Then the chain rule imply

$$\int_0^{s_0} \frac{\|u(s,f_1)-v(s,f_2)\|_{\mathbb{H}}^2}{\zeta^4(s)}ds + \frac{\|u(s_0,f_1)-v(s_0,f_2)\|_{\mathbb{H}}^2}{\theta_0\zeta^3(s_0)} \leqslant \frac{\|f_1-f_2\|_{\mathbb{H}}^2}{\theta_0\zeta^3(0)}, \tag{3}$$

where $s_0 \in [0,T)$.

Step II. For any $s \in [0,T)$, $x \in [0,1]$, let

$$\xi(s,x) := \frac{u(s,f_1)-v(s,f_2)}{\zeta(s)},$$

$$\widetilde{W}(ds,dy) := W(ds,dy) + K_H^{-1}\left(\int_0^{\cdot} \xi(r,y)dr\right)(s)dsdy,$$

and

$$\begin{aligned}\widetilde{W}^H(t,x) &= \int_0^t \int_0^x K_H(t,s)\widetilde{W}(ds,dy)\\ &= W^H(t,x) + \int_0^t \int_0^x \xi(s,y)dsdy, \quad t \in [0,T).\end{aligned}$$

Similar to [5, Corollary A.3.], one can get that

$$W^H(t,x) = \tilde{c_H}\int_0^t \int_0^x (t-s)^{2H-1} W^{1-H}(ds,dy), \tag{4}$$

where $\tilde{c_H} = \left(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\right)^{\frac{1}{2}}$ and W^{1-H} is the fractional noise with Hurst index $1-H$.

Thus we can rewrite

$$\begin{aligned}\widetilde{W}^H(t,x) &= \int_0^t \int_0^x K_H(t,s)\widetilde{W}(ds,dy)\\ &= W^H(t,x) + \int_0^t \int_0^x \xi(s,y)dsdy\\ &= \int_0^t \int_0^x \tilde{c_H}(t-s)^{2H-1}\left(\tilde{c_H}^{-1}(t-s)^{1-2H}\xi(s,y)dsdy + W^{1-H}(ds,dy)\right)\\ &= \int_0^t \int_0^x \tilde{c_H}(t-s)^{2H-1}\widetilde{W}^{1-H}(ds,dy), \quad t \in [0,T),\end{aligned}$$

where

$$\begin{aligned}\widetilde{W}^{1-H}(t,x) &= \int_0^t \int_0^x \tilde{c_H}^{-1}(t-s)^{1-2H}\xi(s,y)dsdy + W^{1-H}(t,x)\\ &= \int_0^t \int_0^x \tilde{c_H}^{-1}(t-s)^{1-2H}\xi(s,y)dsdy + \int_0^t \int_0^x K_{1-H}(t,s)W(ds,dy)\\ &= \int_0^t \int_0^x K_{1-H}(t,s)\left[K_{1-H}^{-1}\left(\int_0^{\cdot} \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\right)(s)dsdy + W(ds,dy)\right].\end{aligned}$$

We also define

$$L_t = \exp\Big[-\int_0^t\int_0^1 K_{1-H}^{-1}\Big(\int_0^{\cdot} \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\Big)(s)W(ds,dy) \\ -\frac{1}{2}\int_0^t\int_0^1\Big(K_{1-H}^{-1}\Big(\int_0^{\cdot} \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\Big)(s)\Big)^2 dsdy\Big],\ t\in[0,T).$$

Now we show $\{L_t, t\in[0,T)\}$ is a uniformly integrable martingale. In fact, according to [11, (17)], we see

$$\int_0^t\int_0^1\Big|K_{1-H}^{-1}\Big(\int_0^{\cdot} \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\Big)(s)\Big|^2 dyds \\ =\int_0^{s_0}\int_0^1\Big|s^{-H+\frac{1}{2}}I_{0+}^{H-\frac{1}{2}}s^{H-\frac{1}{2}}\xi(s,y)\Big|^2 dyds \\ =\frac{\tilde{c}_H^{-2}}{\Gamma^2(H-\frac{1}{2})}\int_0^t\int_0^1 s^{1-2H}\Big(\int_0^s r^{H-\frac{1}{2}}(t-r)^{1-2H}(s-r)^{H-\frac{3}{2}}\xi(r,y)dr\Big)^2 dsdy,\ 0\leqslant t<T.$$

Then, Minkowski's integral inequality implies that for any $0\leqslant t<T$,

$$\int_0^t\int_0^1\Big|K_{1-H}^{-1}\Big(\int_0^{\cdot} \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\Big)(s)\Big|^2 dyds \\ \leqslant\frac{\tilde{c}_H^{-2}}{\Gamma^2(H-\frac{1}{2})}\int_0^t s^{1-2H}\Big[\int_0^s r^{H-\frac{1}{2}}(t-r)^{1-2H}(s-r)^{H-\frac{3}{2}}\Big(\int_0^1\xi^2(r,y)dy\Big)^{\frac{1}{2}}dr\Big]^2 ds \\ \leqslant\frac{\tilde{c}_H^{-2}\|f_1-f_2\|_{\mathbb{H}}^2}{\Gamma^2(H-\frac{1}{2})\zeta^2(0)}\int_0^t s^{1-2H}\Big[\int_0^s r^{H-\frac{1}{2}}(t-r)^{1-2H}(s-r)^{H-\frac{3}{2}}dr\Big]^2 ds \\ \leqslant\frac{\tilde{c}_H^{-2}\|f_1-f_2\|_{\mathbb{H}}^2}{\Gamma^2(H-\frac{1}{2})\zeta^2(0)}\int_0^t s^{1-2H}\Big[\int_0^s r^{H-\frac{1}{2}}(t-r)^{1-2H}(s-r)^{H-\frac{3}{2}}dr\Big]^2 ds \tag{5}$$

In order to estimate (5), let $\max\{1,\frac{2}{5-4H}\}<p<\frac{2}{3-2H}$, then

$$\int_0^s r^{H-\frac{1}{2}}(t-r)^{1-2H}(s-r)^{H-\frac{3}{2}}dr \\ \leqslant\Big(\int_0^s(t-r)^{\frac{p-2Hp}{p-1}}dr\Big)^{\frac{p-1}{p}}\Big(\int_0^s r^{Hp-\frac{1}{2}p}(s-r)^{Hp-\frac{3}{2}p}dr\Big)^{\frac{1}{p}} \\ \leqslant\frac{t^{2-2H-\frac{1}{p}}+(t-s)^{2-2H-\frac{1}{p}}}{\Big|\frac{2p-2pH-1}{p-1}\Big|^{\frac{p-1}{p}}}B^{\frac{1}{p}}\Big(Hp-\frac{1}{2}p+1,Hp-\frac{3}{2}p+1\Big)s^{2H-2+\frac{1}{p}}.$$

Substitute this estimation into (5), we get

$$\begin{aligned}
&\int_0^t\int_0^1\left|K_{1-H}^{-1}\Big(\int_0^\cdot \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\Big)(s)\right|^2 dyds\\
&\leqslant 2\frac{\tilde{c}_H^{-2}B^{\frac{2}{p}}\Big(Hp-\frac{1}{2}p+1,Hp-\frac{3}{2}p+1\Big)\|f_1-f_2\|_{\mathbb{H}}^2}{\Gamma^2(H-\frac{1}{2})\left|\frac{2p-2pH-1}{p-1}\right|^{\frac{2p-2}{p}}\zeta^2(0)}\\
&\quad\times\int_0^t s^{2H-3+\frac{2}{p}}(t^{4-4H-\frac{2}{p}}+(t-s)^{4-4H-\frac{2}{p}})ds\\
&\leqslant 2\frac{\tilde{c}_H^{-2}B^{\frac{2}{p}}\Big(Hp-\frac{1}{2}p+1,Hp-\frac{3}{2}p+1\Big)\|f_1-f_2\|_{\mathbb{H}}^2}{\Gamma^2(H-\frac{1}{2})\left|\frac{2p-2pH-1}{p-1}\right|^{\frac{2p-2}{p}}\zeta^2(0)}\\
&\quad\times\left[\frac{T^{2-2H}}{2H-2+\frac{2}{p}}+T^{2-2H}B(5-4H-\frac{2}{p},2H-2+\frac{2}{p})\right]\\
&=: 2C_{p,\theta_0}\|f_1-f_2\|_{\mathbb{H}}^2T^{2-2H}.
\end{aligned}\tag{6}$$

So it follows that for $s_0\in[0,T)$,

$$\mathbb{E}\exp\Big[\frac{1}{2}\int_0^{s_0}\int_0^1\Big(K_{1-H}^{-1}\Big(\int_0^\cdot \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\Big)(s)\Big)^2 dsdy\Big]<\infty.$$

That is to say $\{L_t,t\in[0,s_0)\}$ is a martingale, so Theorem 2.2 ensures that $\{\widetilde{W}^{1-H},t\in[0,s_0)\}$ is a fractional noise with Hurst index $1-H$ under $d\mathbb{Q}:=L_{s_0}d\mathbb{P}$. Moreover, we have

$$\begin{aligned}
\log L_t=&-\int_0^t\int_0^1 K_{1-H}^{-1}\Big(\int_0^\cdot \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\Big)(s)\widetilde{W}(ds,dy)\\
&+\frac{1}{2}\int_0^t\int_0^1\Big(K_{1-H}^{-1}\Big(\int_0^\cdot \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\Big)(s)\Big)^2 dsdy,\ \ t\in[0,T).
\end{aligned}$$

Then it is easy to see that

$$\mathbb{E}(L_{s_0}\log L_{s_0})=\mathbb{E}_{s_0}\log L_{s_0}\leqslant C_{p,\theta_0}\|f_1-f_2\|_{\mathbb{H}}^2T^{2-2H},\ \ s_0\in[0,T).$$

Consequently, $\{L_t,t\in[0,T)\}$ is a uniformly integrable martingale, so $L_T:=\lim_{t\uparrow T}L_t$ exists and $\{L_t,t\in[0,T]\}$ is also a uniformly integrable martingale. Moreover, $\{\widetilde{W}^{1-H},t\in[0,T]\}$ is a fractional noise with Hurst index $1-H$ under $\mathbb{Q}$. Using (4), we know $\{\widetilde{W}^{H},t\in[0,T]\}$ is a fractional noise with Hurst index H under $\mathbb{Q}$.

Step III. A similar argument as [7] conclude that $u(T,f_1)=v(T,f_2)$. We rewrite $v(t,f_2)$ as

$$v(t,f_2)=f_2(x)+\int_0^t\Delta v(s,f_2)ds+\int_0^t b(v(s,f_2))+\widetilde{B}^H(t),$$

where $\widetilde{B}^H(t) = \sum_{n=1}^{\infty} \int_0^t \int_0^1 e_n(x) \widetilde{W}^H(ds,dx) e_n$, it follows that

$$P_T G(f_2) = \mathbb{E}(G(u(T,f_2))) = \mathbb{E}(L_T G(v(T,f_2))) = \mathbb{E}(L_T G(u(T,f_1))).$$

Let $M_T = -\int_0^T \int_0^1 K_{1-H}^{-1}\left(\int_0^{\cdot} \tilde{c}_H^{-1}(t-r)^{1-2H}\xi(r,y)dr\right)(s)\widetilde{W}(ds,dy)$, then for any $q>1$,

$$\begin{aligned}
\mathbb{E}\left(L_T^{\frac{q}{q-1}}\right) &= \mathbb{E}_T \exp\left[\frac{1}{q-1}M_T + \frac{1}{2(q-1)}\langle M\rangle_T\right] \\
&= \mathbb{E}_T \exp\left[\frac{1}{q-1}M_T - \frac{1}{2(q-1)^2}\langle M\rangle_T + \frac{q}{2(q-1)^2}\langle M\rangle_T\right] \\
&\leqslant \exp\left[\frac{C_{p,\theta_0} T^{2-2H} q \|f_1 - f_2\|_{\mathbb{H}}^2}{(q-1)^2}\right],
\end{aligned}$$

where E_T denotes the expectation under the probability $L_T d\mathbb{P}$. then Hölder inequality deduce that

$$[P_T G(f_2)]^q = [\mathbb{E}(L_T G(u(T,f_1)))]^q \leqslant P_T[G^q(f_1)] \exp\left[\frac{C_p q T^{2-2H} \|f_1 - f_2\|_{\mathbb{H}}^2}{(q-1)^2 \zeta^2(0)}\right].$$

The proof is completed. □

As a consequence of Theorem 3.1, we get the following proposition whose proof is very similar to Zhang [21] and we omit it.

PROPOSITION 3.2. *Let $\frac{1}{2} < H < 1$. Then the operator P_T is strong Feller, i.e., for each $G \in \mathscr{B}_b(\mathbb{H})$,*

$$\lim_{\|f_1 - f_2\|_{\mathbb{H}} \to 0} P_T(G(f_1)) = P_T(G(f_2)), \forall f_1, f_2 \in \mathbb{H}$$

holds.

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