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COMPOSITION OPERATORS AND CLOSURES OF DIRICHLET TYPE SPACES IN LOGARITHMIC BLOCH–TYPE SPACES

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Abstract. Closures of Dirichlet type spaces in logarithmic Bloch-type spaces are investigated in the paper. Moreover, the boundedness and compactness of composition operators from logarithmic Bloch-type spaces to closures of Dirichlet type spaces in logarithmic Bloch-type spaces are characterized.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of all functions analytic in $\mathbb{D}$. For $0 < p < \infty$, H^p denotes the Hardy space, which consists of all functions $f \in H(\mathbb{D})$ for which (see [6])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

As usual, H^∞ denotes the space of bounded analytic functions in $\mathbb{D}$.

Recall that the Bloch space $\mathscr{B}$ is the set of all functions $f \in H(\mathbb{D})$ which satisfies

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

It is well known that $\mathscr{B}$ is a Banach space if it is equipped with the norm $\|\cdot\|_{\mathscr{B}}$. Notice that $H^\infty \subset \mathscr{B}$. The little Bloch space, denoted by $\mathscr{B}_0$, is the subspace of $\mathscr{B}$ consisting of all $f \in H(\mathbb{D})$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. It is well known that $\mathscr{B}_0$ is the closure of polynomials in $\mathscr{B}$.

Let $0 < \alpha < \infty$. The logarithmic Bloch-type space $\mathscr{B}_{\log}^\alpha$ is the space of functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathscr{B}_{\log}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \log\frac{e}{1-|z|^2} |f'(z)| < \infty.$$

It is easy to check that $\mathscr{B}_{\log}^\alpha$ is a Banach space under the norm $\|\cdot\|_{\mathscr{B}_{\log}^\alpha}$. Obviously, the space $\mathscr{B}_{\log}^\alpha$ turns into the logarithmic Bloch space $\mathscr{L}\mathscr{B}$ when $\alpha = 1$. The little logarithmic Bloch-type space $\mathscr{B}_{\log,0}^\alpha$, is a subspace of $\mathscr{B}_{\log}^\alpha$ consisting of all $f \in H(\mathbb{D})$

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such that

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\log\frac{e}{1-|z|^2}|f'(z)|=0.$$

For $0<p<\infty$ and $s>-1$, the Dirichlet type space, denoted by $\mathscr{D}_s^p$, consists of those $f\in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{D}_s^p}=|f(0)|+\left((s+1)\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^s dA(z)\right)^{1/p}<\infty.$$

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ, induced by φ, is defined on $H(\mathbb{D})$ as follows.

$$C_\varphi(f)(z)=f(\varphi(z)),\ \ z\in\mathbb{D}.$$

A standard introductory reference for studying composition operators on various analytic function spaces is [5].

It is a well-known consequence of the Schwartz-Pick lemma that composition operators are typically bounded on the Bloch space. The compactness of composition operators on the Bloch space is studied by Madigan and Matheson in [17]. Wulan, Zheng and Zhu in [25] obtained a new compactness criterion for the composition operator C_φ on $\mathscr{B}$ in terms of the norm of φ^n. The boundedness and compactness of the composition operator on the logarithmic Bloch space was studied by Yoneda in [26]. For example, Yoneda showed that C_φ is compact on $\mathscr{LB}$ if and only if

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)\log\frac{e}{1-|\varphi(z)|^2}}|\varphi'(z)|=0.$$

More characterizations for the boundedness and compactness of composition operators between different Bloch type spaces are given in [7, 12, 13, 14, 15, 23, 28].

Let X and Y be two Banach analytic function spaces. It is natural to denote the closure of $X\cap Y$ in the norm of Y by $\mathscr{C}_Y(X\cap Y)$ for simplicity. Given $f\in\mathscr{B}$ and $\varepsilon>0$, we define

$$\Gamma_\varepsilon(f)=\{z\in\mathbb{D}:(1-|z|^2)|f'(z)|\geqslant\varepsilon\}.$$

The question of characterizing $\mathscr{C}_{\mathscr{B}}(H^\infty)$ is still open. Readers can refer to [2]. Anderson in [1] mentioned that Jones gave an description of the closure of $BMOA$ in $\mathscr{B}$. For example, Jones showed that if $f\in\mathscr{B}$, then $f\in\mathscr{C}_{\mathscr{B}}(BMOA)$ if and only if for every $\varepsilon>0$,

$$\sup_{a\in\mathbb{D}}\int_{\Gamma_\varepsilon(f)}\frac{1-|\sigma_a(z)|^2}{(1-|z|^2)^2}dA(z)<\infty.$$

Jones didn't publish this result, while Ghatage and Zheng in [9] provided a complete proof for it. Zhao studied $\mathscr{C}_{\mathscr{B}}(F(p,p-2,s))$ when $1\leqslant p<\infty$ and $0<s\leqslant 1$ in [27]. Monreal Galán and Nicolau in [18] characterized the closure in the Bloch norm of $H^p\cap\mathscr{B}$ for $1<p<\infty$. Bao and Göğüş in [4] characterized the closure of $\mathscr{D}_\alpha^2\cap\mathscr{B}(-1<\alpha\leqslant 1)$ in the Bloch space, where $\mathscr{D}_\alpha^2$ is the Dirichlet type space. In 2019, Galanopoulos and Girela studied $\mathscr{C}_{\mathscr{B}}(\mathscr{D}_\alpha^p\cap\mathscr{B})$ in [8]. Among others, they obtained the following results.

THEOREM A. *Suppose that* $1 \leqslant p < \infty, p-2 < \alpha \leqslant p-1$, *and let* f *be a Bloch function. Then* $f \in \mathscr{C}_{\mathscr{B}}(\mathscr{D}^p_\alpha \cap \mathscr{B})$ *if and only if for any* $\varepsilon > 0$,

$$\int_{\Gamma_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}} < \infty.$$

THEOREM B. *Suppose that* $0 < p < 1, -1 < \alpha \leqslant p-1$, *and let* f *be a Bloch function.*

(i) If $f \in \mathscr{C}_{\mathscr{B}}(\mathscr{D}^p_\alpha \cap \mathscr{B})$, *then for any* $\varepsilon > 0$, $\int_{\Gamma_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}} < \infty$.

(ii) If there exists $\gamma > 2 - \frac{1+\alpha}{p}$ *such that for any* $\varepsilon > 0$, $\int_{\Gamma_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{\gamma}} < \infty$, *then* $f \in \mathscr{C}_{\mathscr{B}}(\mathscr{D}^p_\alpha \cap \mathscr{B})$.

See [3, 16, 20, 21, 24] for more results of closures of some function spaces in Bloch type spaces.

The purpose of this paper is to study the closure of $\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log}$ in the logarithmic Bloch-type space $\mathscr{B}^\beta_{\log}$. We give a complete characterization for $\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$. Moreover, we study the boundedness and compactness of composition operators $C_\varphi : \mathscr{B}^\alpha_{\log}(\mathscr{B}^\alpha_{\log,0}) \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ and $C_\varphi : \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log}) \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$.

Throughout this paper, we say that $f \lesssim h$ if there exists a constant $C > 0$ such that $f \leqslant Ch$. The symbol $f \approx h$ means that $f \lesssim h \lesssim f$.

2. characterization of $\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$

To state and prove our main results in this paper, we need some lemmas. The following well-known estimate can be found in [10, Lemma 3].

LEMMA 1. *Suppose* $s > -1$ *and* $t > 0$. *Then there exists a positive constant* C *such that*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-\bar{z}w|^{2+s+t} \log \frac{e}{1-|w|^2}} dA(w) \leqslant \frac{C}{(1-|z|^2)^t \log \frac{e}{1-|z|^2}}$$

for all $z \in \mathbb{D}$.

LEMMA 2. *[10] Let* $\alpha > 0$ *and* n *be a positive integer. Then* $f \in \mathscr{B}^\alpha_{\log}$ *if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n-1} \log \frac{e}{1-|z|^2} |f^{(n)}(z)| < \infty.$$

Moreover, $\|f\|_{\mathscr{B}^\alpha_{\log}}$ *is equivalent to* $\|f\|_{\mathscr{B}^{\alpha,n}_{\log}}$. *Here*

$$\|f\|_{\mathscr{B}^{\alpha,n}_{\log}} = |f(0)| + |f'(0)| + \ldots + |f^{(n-1)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n-1} \log \frac{e}{1-|z|^2} |f^{(n)}(z)|.$$

We also need the following estimate (cf. [19, Lemma 3.5]).

LEMMA 3. *Suppose that $0 \leqslant t_1 < s < t_0$. Then there exists a positive constant C such that*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-w\bar{z}|^{2+t_0}|1-a\bar{z}|^{t_1}} \log^k \frac{e}{1-|z|^2} dA(z) \leqslant \frac{C}{(1-|w|^2)^{t_0-s}|1-\bar{a}w|^{t_1}} \log^k \frac{e}{1-|w|^2}.$$

THEOREM 1. *Let n be a positive integer, $1 < p < \infty$ and $p-1 < s < \infty$. Let $1 < \beta < \frac{s+n-1}{p-1}$. Suppose that $f \in \mathscr{B}^{\beta}_{\log}$. Then $f \in \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$ if and only if for any $\varepsilon > 0$,*

$$\int_{\Omega_{n,\beta,\varepsilon}(f)} \frac{(1-|z|^2)^{s-p\beta}}{\log^p \frac{e}{1-|z|^2}} dA(z) < \infty, \tag{1}$$

where

$$\Omega_{n,\beta,\varepsilon}(f) = \left\{ z \in \mathbb{D} : (1-|z|^2)^{\beta+n-1} \log \frac{e}{(1-|z|^2)} |f^{(n)}(z)| \geqslant \varepsilon \right\}.$$

Proof. Take a function $f \in \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$ and $\varepsilon > 0$. Then there exists a $g \in \mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log}$ such that $\|f-g\|_{\mathscr{B}^{\beta,n}_{\log}} \leqslant \frac{\varepsilon}{2}$. Note that

$$\begin{aligned}(1-|z|^2)^{\beta+n-1} \log \frac{e}{1-|z|^2} |f^{(n)}(z)| &\leqslant \sup_{w\in\mathbb{D}} (1-|w|^2)^{\beta+n-1} \log \frac{e}{1-|w|^2} |f^{(n)}(w) - g^{(n)}(w)| \\ &\quad + (1-|z|^2)^{\beta+n-1} \log \frac{e}{1-|z|^2} |g^{(n)}(z)| \\ &\leqslant \frac{\varepsilon}{2} + (1-|z|^2)^{\beta+n-1} \log \frac{e}{1-|z|^2} |g^{(n)}(z)|, \quad z \in \mathbb{D}.\end{aligned}$$

This implies that $\Omega_{n,\beta,\varepsilon}(f) \subseteq \Omega_{n,\beta,\frac{\varepsilon}{2}}(g)$. Then it follows that

$$\begin{aligned}&\int_{\mathbb{D}} |g^{(n)}(z)|^p (1-|z|^2)^{(n-1)p+s} dA(z) \\ &\quad\geqslant \int_{\Omega_{n,\beta,\frac{\varepsilon}{2}}(g)} |g^{(n)}(z)|^p (1-|z|^2)^{(n-1)p+s} dA(z) \\ &\quad= \int_{\Omega_{n,\beta,\frac{\varepsilon}{2}}(g)} \frac{|g^{(n)}(z)|^p (1-|z|^2)^{(\beta+n-1)p} \log^p \frac{e}{1-|z|^2}}{(1-|z|^2)^{(\beta+n-1)p-(n-1)p-s} \log^p \frac{e}{1-|z|^2}} dA(z) \\ &\quad\geqslant (\frac{\varepsilon}{2})^p \int_{\Omega_{n,\beta,\frac{\varepsilon}{2}}(g)} \frac{(1-|z|^2)^{s-p\beta}}{\log^p \frac{e}{1-|z|^2}} dA(z) \\ &\quad\geqslant (\frac{\varepsilon}{2})^p \int_{\Omega_{n,\beta,\varepsilon}(f)} \frac{(1-|z|^2)^{s-p\beta}}{\log^p \frac{e}{1-|z|^2}} dA(z).\end{aligned}$$

Since $g \in \mathscr{D}_s^p$, we get the desired result.

Conversely, we assume that (1) holds. Fix $\varepsilon > 0$ and let f satisfy (1). Without loss of generality, we may assume that $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$. For any $z \in \mathbb{D}$, using Proposition 4.27 in [29], we have

$$f(z) = \frac{1}{(\gamma+2)\cdots(\gamma+n)} \int_{\mathbb{D}} \frac{f^{(n)}(w)(1-|w|^2)^{n+\gamma}}{(1-z\overline{w})^{2+\gamma}\overline{w}^n} dA(w),$$

where $\gamma \geqslant 0$. Following [27], we set $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{(\gamma+2)\cdots(\gamma+n)} \int_{\Omega_{n,\beta,\varepsilon}(f)} \frac{f^{(n)}(w)(1-|w|^2)^{n+\gamma}}{(1-z\overline{w})^{2+\gamma}\overline{w}^n} dA(w)$$

and

$$f_2(z) = \frac{1}{(\gamma+2)\cdots(\gamma+n)} \int_{\mathbb{D}\setminus\Omega_{n,\beta,\varepsilon}(f)} \frac{f^{(n)}(w)(1-|w|^2)^{n+\gamma}}{(1-z\overline{w})^{2+\gamma}\overline{w}^n} dA(w).$$

Obviously,

$$f_1^{(n)}(z) = (\gamma+n+1) \int_{\Omega_{n,\beta,\varepsilon}(f)} \frac{f^{(n)}(w)(1-|w|^2)^{n+\gamma}}{(1-z\overline{w})^{n+2+\gamma}} dA(w)$$

and

$$f_2^{(n)}(z) = (\gamma+n+1) \int_{\mathbb{D}\setminus\Omega_{n,\beta,\varepsilon}(f)} \frac{f^{(n)}(w)(1-|w|^2)^{n+\gamma}}{(1-z\overline{w})^{n+2+\gamma}} dA(w).$$

Let $h(z) = f_1(z) - \sum_{k=1}^{n-1} \frac{f_1^{(k)}(0)}{k!} z^k$. Then $h(0) = h'(0) = \ldots = h^{(n-1)}(0) = 0$, and $(f - h)^{(n)}(z) = f_2^{(n)}(z)$. If we choose $\gamma > \max\{0, \beta-2\}$ and using Lemma 1, we obtain

$$\begin{aligned}
\|f-h\|_{\mathscr{B}_{\log}^{\beta,n}} &= \sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta+n-1} \log\frac{e}{1-|z|^2} |f_2^{(n)}(z)| \\
&\lesssim \sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta+n-1} \log\frac{e}{1-|z|^2} \int_{\mathbb{D}\setminus\Omega_{n,\beta,\varepsilon}(f)} \frac{|f^{(n)}(w)|(1-|w|^2)^{n+\gamma}}{|1-z\overline{w}|^{n+2+\gamma}} dA(w) \\
&\lesssim \sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta+n-1} \log\frac{e}{1-|z|^2} \int_{\mathbb{D}} \frac{(1-|w|^2)^{\gamma-\beta+1}}{|1-z\overline{w}|^{n+2+\gamma} \log\frac{e}{1-|w|^2}} dA(w) \\
&\lesssim \varepsilon.
\end{aligned}$$

This means that $h \in \mathscr{B}^{\beta}_{\log}$. Then

$$\begin{aligned}
&\int_{\mathbb{D}} |h^{(n)}(z)|^p (1-|z|^2)^{(n-1)p+s} dA(z) \\
&= \int_{\mathbb{D}} |f_1^{(n)}(z)|^p (1-|z|^2)^{(n-1)p+s} dA(z) \\
&\leqslant \|f_1\|^{p-1}_{\mathscr{B}^{\beta,n}_{\log}} \int_{\mathbb{D}} |f_1^{(n)}(z)| \frac{(1-|z|^2)^{(n-1)p+s-(p-1)(\beta+n-1)}}{(\log \frac{e}{1-|z|^2})^{p-1}} dA(z) \\
&\lesssim \|f_1\|^{p-1}_{\mathscr{B}^{\beta,n}_{\log}} \int_{\mathbb{D}} \frac{(1-|z|^2)^{(n-1)p+s-(p-1)(\beta+n-1)}}{(\log \frac{e}{1-|z|^2})^{p-1}} \\
&\quad \Big(\int_{\Omega_{n,\beta,\varepsilon}(f)} \frac{|f^{(n)}(w)|(1-|w|^2)^{n+\gamma}}{|1-z\overline{w}|^{n+2+\gamma}} dA(w) \Big) dA(z) \\
&\lesssim \int_{\Omega_{n,\beta,\varepsilon}(f)} |f^{(n)}(w)|(1-|w|^2)^{n+\gamma} \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{(n-1)p+s-(p-1)(\beta+n-1)}}{|1-z\overline{w}|^{n+2+\gamma} (\log \frac{e}{1-|z|^2})^{p-1}} dA(z) \right) dA(w).
\end{aligned}$$

Observe that $1 < \beta < \frac{s+n-1}{p-1}$. If we choose $\gamma > \max\{0, \beta-2, s-p\beta+\beta-1\}$, using Fubini's theorem and Lemma 3 we obtain

$$\int_{\mathbb{D}} |h^{(n)}(z)|^p (1-|z|^2)^{(n-1)p+s} dA(z) \lesssim \int_{\Omega_{n,\beta,\varepsilon}(f)} \frac{(1-|w|^2)^{s-p\beta}}{\log^p \frac{e}{1-|w|^2}} dA(w) < \infty.$$

Therefore, $h \in \mathscr{D}^p_s$. Then for any $\varepsilon > 0$, there exists a function $h \in \mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log}$ such that $\|f-h\|_{\mathscr{B}^{\beta}_{\log}} \lesssim \varepsilon$, which means that $f \in \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$. The proof is complete.

3. Composition operators on $\mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$

Next, we characterize the boundedness and compactness of composition operators from the logarithmic Bloch-type space $\mathscr{B}^{\alpha}_{\log}(\mathscr{B}^{\alpha}_{\log,0})$ to $\mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$ and on $\mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$. We denote $\Omega_{n,\beta,\varepsilon}(f)$ by $\Omega_{\beta,\varepsilon}(f)$ when $n = 1$.

THEOREM 2. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that $1 < p < \infty, \alpha > 0$ and $1 < \beta < \frac{s}{p-1} < \infty$. Then $C_\varphi : \mathscr{B}^{\alpha}_{\log} \to \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$ is bounded if and only if for any $\varepsilon > 0$,*

$$\int_{\Lambda_\varepsilon(\varphi)} \frac{(1-|z|)^{s-p\beta}}{\log^p \frac{e}{1-|z|^2}} dA(z) < \infty, \tag{2}$$

where

$$\Lambda_\varepsilon(\varphi) = \left\{ z \in \mathbb{D} : \frac{(1-|z|^2)^\beta \log \frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha \log \frac{e}{1-|\varphi(z)|^2}} |\varphi'(z)| \geqslant \varepsilon \right\}.$$

Proof. For the sufficiency we assume that (2) holds for any $\varepsilon > 0$. Let $f \in \mathscr{B}^{\alpha}_{\log}$. Then

$$\begin{aligned}
&|(f\circ\varphi)'(z)|(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2}\\
=&|f'(\varphi(z))|(1-|\varphi(z)|^2)^{\alpha}\log\frac{e}{1-|\varphi(z)|^2}\frac{|\varphi'(z)|(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha}\log\frac{e}{1-|\varphi(z)|^2}}\\
\leqslant&\|f\|_{\mathscr{B}^{\alpha}_{\log}}\frac{(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha}\log\frac{e}{1-|\varphi(z)|^2}}|\varphi'(z)|.
\end{aligned}$$

Thus, for any fixed $\delta > 0$, if $|(f\circ\varphi)'(z)|(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2} > \delta$, then we have

$$\frac{(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha}\log\frac{e}{1-|\varphi(z)|^2}}|\varphi'(z)| \geqslant \frac{\delta}{\|f\|_{\mathscr{B}^{\alpha}_{\log}}} = \varepsilon.$$

Therefore,

$$\infty > \int_{\Lambda_{\varepsilon}(\varphi)}\frac{(1-|z|^2)^{s-p\beta}}{\log^{p}\frac{e}{1-|z|^2}}dA(z) \gtrsim \int_{\Omega_{\beta,\delta}(f\circ\varphi)}\frac{(1-|z|^2)^{s-p\beta}}{\log^{p}\frac{e}{1-|z|^2}}dA(z).$$

According to Theorem 1, we get that

$$f\circ\varphi \in \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^{p}_{s}\cap\mathscr{B}^{\beta}_{\log}).$$

This means that $C_{\varphi} : \mathscr{B}^{\alpha}_{\log} \to \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^{p}_{s}\cap\mathscr{B}^{\beta}_{\log})$ is bounded.

In order to prove the necessity, we suppose that $C_{\varphi} : \mathscr{B}^{\alpha}_{\log} \to \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^{p}_{s}\cap\mathscr{B}^{\beta}_{\log})$ is bounded. It is well known that there exists two functions $f_1,\ f_2 \in \mathscr{B}^{\alpha}_{\log}$ such that (see [11, Lemma 2.2])

$$|f_1'(z)|+|f_2'(z)| \geqslant \frac{1}{(1-|z|^2)^{\alpha}\log\frac{e}{1-|z|^2}}.$$

Due to our assumption, we get that

$$f_1\circ\varphi,\ f_2\circ\varphi \in \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^{p}_{s}\cap\mathscr{B}^{\beta}_{\log}).$$

Thus, for any $\varepsilon > 0$, we have

$$\int_{\Omega_{\beta,\frac{\varepsilon}{2}}(f_1\circ\varphi)}\frac{(1-|z|^2)^{s-p\beta}}{\log^{p}\frac{e}{1-|z|^2}}dA(z) < \infty$$

and

$$\int_{\Omega_{\beta,\frac{\varepsilon}{2}}(f_2\circ\varphi)}\frac{(1-|z|^2)^{s-p\beta}}{\log^{p}\frac{e}{1-|z|^2}}dA(z) < \infty.$$

If $z \in \Lambda_\varepsilon(\varphi)$, then we have

$$\begin{aligned}
&(|(f_1\circ\varphi)'(z)|+|(f_2\circ\varphi)'(z)|)(1-|z|^2)^\beta \log\frac{e}{1-|z|^2}\\
=&(|f_1'(\varphi(z))|+|f_2'(\varphi(z))|)|\varphi'(z)|(1-|z|^2)^\beta \log\frac{e}{1-|z|^2}\\
\geqslant&\frac{(1-|z|^2)^\beta\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{e}{1-|\varphi(z)|^2}}|\varphi'(z)|\geqslant\varepsilon.
\end{aligned}$$

This means that, either

$$|(f_1\circ\varphi)'(z)|(1-|z|^2)^\beta\log\frac{e}{1-|z|^2}\geqslant\frac{\varepsilon}{2}$$

or

$$|(f_2\circ\varphi)'(z)|(1-|z|^2)^\beta\log\frac{e}{1-|z|^2}\geqslant\frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned}
&\int_{\Lambda_\varepsilon(\varphi)}\frac{(1-|z|^2)^{s-p\beta}}{\log^p\frac{e}{1-|z|^2}}dA(z)\leqslant\int_{\Omega_{\beta,\frac{\varepsilon}{2}}(f_1\circ\varphi)\cup\Omega_{\beta,\frac{\varepsilon}{2}}(f_2\circ\varphi)}\frac{(1-|z|^2)^{s-p\beta}}{\log^p\frac{e}{1-|z|^2}}dA(z)\\
\leqslant&\int_{\Omega_{\beta,\frac{\varepsilon}{2}}(f_1\circ\varphi)}\frac{(1-|z|^2)^{s-p\beta}}{\log^p\frac{e}{1-|z|^2}}dA(z)+\int_{\Omega_{\beta,\frac{\varepsilon}{2}}(f_2\circ\varphi)}\frac{(1-|z|^2)^{s-p\beta}}{\log^p\frac{e}{1-|z|^2}}dA(z)\\
<&\infty.
\end{aligned}$$

The proof is complete.

THEOREM 3. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that $1<p<\infty, \alpha>1$ and $1<\beta<\frac{s}{p-1}<\infty$. Then $C_\varphi:\mathscr{B}^\alpha_{\log,0}\to\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s\cap\mathscr{B}^\beta_{\log})$ is bounded if and only if $\varphi\in\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s\cap\mathscr{B}^\beta_{\log})$ and*

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{e}{1-|\varphi(z)|^2}}|\varphi'(z)|<\infty.$$

Proof. The necessity of the conditions can be proved immediately. In fact, we suppose that $C_\varphi:\mathscr{B}^\alpha_{\log,0}\to\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s\cap\mathscr{B}^\beta_{\log})$ is bounded. Notice that $f(z)=z\in\mathscr{B}^\alpha_{\log,0}$, then we have

$$\varphi=C_\varphi f\in\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s\cap\mathscr{B}^\beta_{\log}).$$

Since $C_\varphi:\mathscr{B}^\alpha_{\log,0}\to\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s\cap\mathscr{B}^\beta_{\log})$ is bounded and $\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s\cap\mathscr{B}^\beta_{\log})\subseteq\mathscr{B}^\beta_{\log}$, then $C_\varphi:\mathscr{B}^\alpha_{\log,0}\to\mathscr{B}^\beta_{\log}$ is bounded. According to [22, Theorem 3.1], we otain

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{e}{1-|\varphi(z)|^2}}|\varphi'(z)|<\infty.$$

To prove the sufficiency, we assume that $\varphi \in \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$ and

$$Q := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha}\log\frac{e}{1-|\varphi(z)|^2}}|\varphi'(z)| < \infty.$$

Let $f \in \mathscr{B}^{\alpha}_{\log,0}$. Then for any $\varepsilon > 0$, there is a constant r $(0<r<1)$ such that

$$|f'(z)|(1-|z|^2)^{\alpha}\log\frac{e}{1-|z|^2} < \frac{\varepsilon}{Q}, \quad \text{whenever} \quad |z|>r.$$

Let $z \in \Omega_{\beta,\varepsilon}(f\circ\varphi)$. Then we have

$$\begin{aligned}
&Q|f'(\varphi(z))|(1-|\varphi(z)|^2)^{\alpha}\log\frac{e}{1-|\varphi(z)|^2}\\
\geqslant&|f'(\varphi(z))|(1-|\varphi(z)|^2)^{\alpha}\log\frac{e}{1-|\varphi(z)|^2}\frac{(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha}\log\frac{e}{1-|\varphi(z)|^2}}|\varphi'(z)|\\
\geqslant&|f'(\varphi(z))||\varphi'(z)|(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2} \geqslant \varepsilon.
\end{aligned}$$

That means $|\varphi(z)| \leqslant r$. Thus,

$$\begin{aligned}
&\frac{\|f\|_{\mathscr{B}^{\alpha}_{\log}}}{(1-r^2)^{\alpha}\log\frac{e}{1-r^2}}|\varphi'(z)|(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2}\\
\geqslant&\|f\|_{\mathscr{B}^{\alpha}_{\log}}\frac{|\varphi'(z)|(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha}\log\frac{e}{1-|\varphi(z)|^2}}\\
\geqslant&|f'(\varphi(z))||\varphi'(z)|(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2} \geqslant \varepsilon.
\end{aligned}$$

Let

$$\delta = \frac{\varepsilon(1-r^2)^{\alpha}\log\frac{e}{1-r^2}}{\|f\|_{\mathscr{B}^{\alpha}_{\log}}}.$$

Then $|\varphi'(z)|(1-|z|^2)^{\beta}\log\frac{e}{1-|z|^2} \geqslant \delta$. Hence, $\Omega_{\beta,\varepsilon}(f\circ\varphi) \subseteq \Omega_{\beta,\delta}(\varphi)$. Due to $\varphi \in \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$, we obtain

$$\infty > \int_{\Omega_{\beta,\delta}(\varphi)} \frac{(1-|z|^2)^{s-p\beta}}{\log^p\frac{e}{1-|z|^2}}dA(z) \geqslant \int_{\Omega_{\beta,\varepsilon}(f\circ\varphi)} \frac{(1-|z|^2)^{s-p\beta}}{\log^p\frac{e}{1-|z|^2}}dA(z).$$

According to Theorem 1, we get that $f\circ\varphi \in \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$. Therefore, $C_{\varphi} : \mathscr{B}^{\alpha}_{\log} \to \mathscr{C}_{\mathscr{B}^{\beta}_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^{\beta}_{\log})$ is bounded. The proof is complete.

THEOREM 4. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that $1 < p < \infty, \alpha > 1$ and $1 < \beta < \frac{s}{p-1} < \infty$. Then the following statements are equivalent.*

(i) $C_\varphi : \mathscr{B}^\alpha_{\log} \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ *is compact;*

(ii) $C_\varphi : \mathscr{B}^\alpha_{\log,0} \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ *is compact;*

(iii) $\varphi \in \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta \log \frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha \log \frac{e}{1-|\varphi(z)|^2}} |\varphi'(z)| = 0. \tag{3}$$

Proof. $(i) \Rightarrow (ii)$. The implication is obvious because $\mathscr{B}^\alpha_{\log,0} \subseteq \mathscr{B}^\alpha_{\log}$.

$(ii) \Rightarrow (iii)$. Assume that $C_\varphi : \mathscr{B}^\alpha_{\log,0} \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is compact. Obviously, $C_\varphi : \mathscr{B}^\alpha_{\log,0} \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is bounded. According to Theorem 3, we obtain $\varphi \in \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$. On the other hand, it is obvious that $\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log}) \subseteq \mathscr{B}^\beta_{\log}$. Then $C_\varphi : \mathscr{B}^\alpha_{\log,0} \to \mathscr{B}^\beta_{\log}$ is compact. This clearly implies that (3) holds by [22, Theorem 3.2].

$(iii) \Rightarrow (i)$. According to the assumed condition, we see that there exists an r $(0 < r < 1)$, such that

$$\frac{(1-|z|^2)^\beta \log \frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha \log \frac{e}{1-|\varphi(z)|^2}} |\varphi'(z)| < \frac{\varepsilon}{2}, \quad \text{whenever} \quad |\varphi(z)| > r.$$

Let $z \in \Lambda_\varepsilon(\varphi)$. Then $|\varphi(z)| \leqslant r$. Therefore,

$$\frac{(1-|z|^2)^\beta \log \frac{e}{1-|z|^2}}{(1-r^2)^\alpha \log \frac{e}{1-r^2}} |\varphi'(z)| \geqslant \frac{(1-|z|^2)^\beta \log \frac{e}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha \log \frac{e}{1-|\varphi(z)|^2}} |\varphi'(z)| \geqslant \varepsilon.$$

Thus

$$(1-|z|^2)^\beta \log \frac{e}{1-|z|^2} |\varphi'(z)| \geqslant \varepsilon (1-r^2)^\alpha \log \frac{e}{1-r^2}.$$

Set $\delta = \varepsilon(1-r^2)^\alpha \log \frac{e}{1-r^2}$. Then $z \in \Omega_{\beta,\delta}(\varphi)$. Since $\varphi \in \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$, we have

$$\infty > \int_{\Omega_{\beta,\delta}(\varphi)} \frac{(1-|z|^2)^{s-p\beta}}{\log^p \frac{e}{1-|z|^2}} dA(z) \gtrsim \int_{\Gamma_\varepsilon(\varphi)} \frac{(1-|z|^2)^{s-p\beta}}{\log^p \frac{e}{1-|z|^2}} dA(z).$$

According to Theorem 2, $C_\varphi : \mathscr{B}^\alpha_{\log} \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is bounded. We know that $C_\varphi : \mathscr{B}^\alpha_{\log} \to \mathscr{B}^\beta_{\log}$ is compact by [22, Theorem 3.2]. Therefore, $C_\varphi : \mathscr{B}^\alpha_{\log} \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is compact. The proof is complete.

THEOREM 5. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that $1 < p < \infty$ and $1 < \beta < \frac{s}{p-1} < \infty$. Then $C_\varphi : \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log}) \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is compact if and only if $\varphi \in \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ and*

$$\lim_{|\varphi(z)|\to 1} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\beta \frac{\log \frac{e}{1-|z|^2}}{\log \frac{e}{1-|\varphi(z)|^2}} |\varphi'(z)| = 0. \tag{4}$$

Proof. Assume that $C_\varphi : \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log}) \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is compact. Thus $C_\varphi : \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log}) \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is bounded. So we obtain $\varphi \in \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ since $z \in \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$. It is well known that $\mathscr{B}^\beta_{\log,0}$ is the closure of all polynomials in $\mathscr{B}^\beta_{\log}$ and the space $\mathscr{D}^p_s$ contains all polynomials. Therefore, $C_\varphi : \mathscr{B}^\beta_{\log,0} \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is compact. According to Theorem 4, we see that (4) holds.

Conversely, we suppose that $\varphi \in \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ and (4) holds. By [22, Theorem 3.2], we see that $\mathscr{B}^\beta_{\log} \to \mathscr{B}^\beta_{\log}$ is compact. By Theorem 4, we know that $C_\varphi : \mathscr{B}^\beta_{\log} \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is compact. Since $\mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log}) \subseteq \mathscr{B}^\beta_{\log}$, we obtain that $C_\varphi : \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log}) \to \mathscr{C}_{\mathscr{B}^\beta_{\log}}(\mathscr{D}^p_s \cap \mathscr{B}^\beta_{\log})$ is compact. The proof is complete.

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