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ON THE MULTIPLICATIVE SUM ZAGREB INDEX OF GRAPHS WITH SOME GIVEN PARAMETERS

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Abstract. The multiplicative sum Zagreb index of a graph G is the product of the sums of degrees of pairs of adjacent vertices. In this work, we obtain the maximum values of the multiplicative sum Zagreb indices with fixed number of cut edges, or cut vertices, or edge connectivity, or vertex connectivity of graphs. Furthermore, we characterize the corresponding extremal graphs.

1. Introduction

Topological indices are mathematical descriptors reflecting some structural characteristics of organic molecules on the molecular graph, and they play an important role in chemistry, pharmacology, etc. (see [6–8]). The famous Zagreb indices, first introduced by Gutman and Trinajstić [11], are used to examine the structure dependence of total π-electron energy on molecular orbital. The first Zagreb index M_1 and the second Zagreb index M_2 of a graph G are defined as:

$$M_1(G) = \sum_{v\in V(G)} d_G(v)^2, \quad M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v),$$

where $d_G(u)$ is the degree of vertex u.

These two classical topological indices (M_1 and M_2) and their versions have been applied in studying heterosystems, ZE-isomerism, chirality and complexity of molecule, etc. Todeschini et al. [12] had presented a version of Zagreb indices which nowadays are called multiplicative Zagreb indices, and they are expressed as:

$$\Pi_1(G) = \prod_{v\in V(G)} d_G(v)^2, \quad \Pi_2(G) = \prod_{uv\in E(G)} d_G(u)d_G(v).$$

Eliasi et al. [3] proposed another version of Zagreb indices known as the multiplicative sum Zagreb index (denoted by Π_1^*), which is defined as

$$\Pi_1^*(G) = \prod_{uv\in E(G)} (d_G(u)+d_G(v)).$$

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Eliasi, Iranmanesh and Gutman [3] proved that the path has the minimum multiplicative sum Zagreb index among all connected graphs, and determined the second minimum multiplicative sum Zagreb index of trees. In [2], Xu and Das characterized the trees, unicylcic graphs and bicyclic graphs with the minimum and maximum multiplicative sum Zagreb indices. Božović, Kovijanić and Popivoda [10] obtained an upper bound of multiplicative sum Zagreb index on chemical trees and characterized the corresponding extremal graphs. Azari and Iranmanesh [9] presented some inequalities of the multiplicative sum Zagreb index on some graph operations such as join, composition, union, corona product, etc.

In this work, we only deal with simple connected graphs. Let $G=(V(G),\ E(G))$ be the graph having vertex set $V(G)$ and edge set $E(G)$. Given a graph G, we use $G-x$ or $G-xy$ to denote the graph that arises from G by deleting the vertex $x\in V(G)$ or the edge $xy\in E(G)$. Similarly, $G+xy$ is a graph that arises from G by adding an edge $xy\notin E(G)$, where $x,y\in V(G)$. Let $E'\subseteq E(G)$, we use $G-E'$ to denote the subgraph of G obtained by deleting the edges of E'. $X\subseteq V(G)$, $G-X$ denotes the subgraph of G obtained by deleting the vertices of X and the edges incident with them. A block of a graph is a maximum connected subgraph with no cut vertex. If a block has at most one cut vertex in the graph as a whole, we call it an endblock. A clique of a graph G is a subset $W\subset V(G)$ such that $G[W]$ is complete. As usual, we use P_n, K_n and S_n to denote the paths, the complete graphs and the stars on n vertices, respectively.

Let $P_r=x_0x_1\cdots x_r$ $(r\geqslant 1)$ be a path of graph G with $d_G(x_1)=\cdots=d_G(x_{r-1})=2$ (unless $r=1$). If $d_G(x_0),d_G(x_r)\geqslant 3$, then P_r is called an internal path of G; if $d_G(x_0)\geqslant 3, d_G(x_r)=1$, then P_r is called a pendant path of G. Denoted by $G_1\cup G_2$ the vertex-disjoint union of the graphs G_1 and G_2, and $G_1\vee G_2$ the graph arisen from $G_1\cup G_2$ by adding all possible edges between the vertices of G_1 and the vertices of G_2. Denoted by $\gamma(G)=|E(G)|-|V(G)|+1$ the cyclomatic number of graph G. The k cyclic graph is the graph whose cyclomatic number is k. For $\gamma(G)=0$, G is a tree.

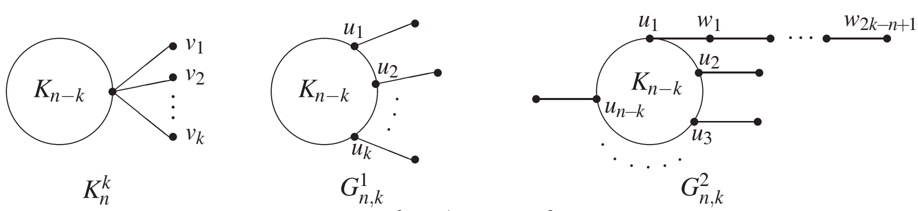

Fig. 1. K_n^k, $G^1_{n,k}$ and $G^2_{n,k}$.

Let K_n^k (as shown in Fig. 1) be the graph obtained by identifying one vertex of K_{n-k} with the central vertex of star S_{k+1}. Let $G^1_{n,k}$ (as shown in Fig. 1) be the graph arisen from K_{n-k} by attaching at most one pendant edge to its each vertex, where $0<k\leqslant\frac{n}{2}$. Let $G^2_{n,k}$ (as shown in Fig. 1) be the graph arisen from K_{n-k} by attaching exactly one pendant path to one vertex (u_1) of K_{n-k}, and attaching exactly one pendant edge to the other $n-k-1$ vertices $(u_2,\cdots,u_{n-k})$ of K_{n-k}, where $\frac{n}{2}<k\leqslant n-3$. We can see [1] for other terminologies and notations.

Inspired by [4, 5] and [13–18], we go on studying the mathematical properties of the connectivity and the multiplicative sum Zagreb index of graphs. In this work, we present the maximum values of the multiplicative sum Zagreb indices with fixed number of cut edges, or cut vertices, or edge connectivity, or vertex connectivity of a graph. Furthermore, we characterize the corresponding extremal graphs.

2. Preliminaries

By the definition of Π_1^*, the following Lemma 2.1 is obvious and can be found in [2].

LEMMA 2.1. ([2]) *Let $G=(V(G),E(G))$ be a simple connected graph. Then*
(i) If $e\in E(G)$, $\Pi_1^*(G)>\Pi_1^*(G-e)$;
(ii) If $e=uv\notin E(G)$, $u,v\in V(G)$, $\Pi_1^*(G)<\Pi_1^*(G+e)$.

LEMMA 2.2. *Let t,s,q be positive integers, where $t,s\geqslant 1$ and $q\geqslant 2$. Then*

$$\frac{(q+t+s+1)^{t+s}(2q)^q(2q+t+s)^q}{(q+t+1)^t(q+s+1)^s(2q+t)^q(2q+s)^q}>1.$$

Proof. Let $f(x)=\frac{x}{q+1+x}+\frac{q}{2q+x}+\ln(q+1+x)$ be a real function in x, where $x\geqslant 0$. Then

$$\begin{aligned}f'(x)&=\frac{q+1}{(q+1+x)^2}-\frac{q}{(2q+x)^2}+\frac{1}{q+1+x}\\&=\frac{1}{(q+1+x)^2}+\frac{1}{q+1+x}+q\left(\frac{1}{(q+1+x)^2}-\frac{1}{(2q+x)^2}\right)>0.\end{aligned}$$

So $f(x+s)>f(x)$ when $x\geqslant 0$. Let $g(x)=q\ln(2q+x)+x\ln(q+1+x)$ and $h(x)=g(x+s)-g(x)$ be two real functions in x, where $x\geqslant 0$. Thus, for $x\geqslant 0$, it can be concluded that

$$\begin{aligned}h'(x)=&\frac{q}{2q+x+s}+\frac{x+s}{q+1+x+s}+\ln(q+1+x+s)\\&-\frac{q}{2q+x}-\frac{x}{q+1+x}-\ln(q+1+x)\\=&f(x+s)-f(x)>0.\end{aligned}$$

So $h(t)>h(0)$, that is

$$\begin{aligned}&g(t+s)-g(t)>g(s)-g(0)\\\Longrightarrow&g(t+s)+g(0)>g(t)+g(s)\\\Longrightarrow&(t+s)\ln(q+1+t+s)+q\ln(2q+t+s)+q\ln(2q)\\&>r\ln(q+1+t)+q\ln(2q+t)+s\ln(q+1+s)+q\ln(2q+s)\end{aligned}$$

$$\begin{aligned}
\Longrightarrow &\ln\left((q+t+s+1)^{t+s}(2q)^q(2q+t+s)^q\right)\\
&\quad>\ln\left((q+t+1)^t(q+s+1)^s(2q+t)^q(2q+s)^q\right)\\
\Longrightarrow &(q+t+s+1)^{t+s}(2q)^q(2q+t+s)^q>(q+t+1)^t(q+s+1)^s(2q+t)^q(2q+s)^q.
\end{aligned}$$

We finish the proof.

LEMMA 2.3. *Let n_1,n_2,s be positive integers, where $n_2\geqslant n_1\geqslant 2$ and $s\geqslant 1$. Then*

$$\frac{(2n_1+2s-4)^{\binom{n_1-1}{2}}(2n_2+2s)^{\binom{n_2+1}{2}}}{(2n_1+2s-2)^{\binom{n_1}{2}}(2n_2+2s-2)^{\binom{n_2}{2}}}>1.$$

Proof. Let $f(x)=(x^2+x)\ln(2x+2s)-(x^2-x)\ln(2x+2s-2)$ be a real function in x, where $x\geqslant 1$. Then

$$\begin{aligned}
f'(x)&=(2x+1)\ln(2x+2s)-(2x-1)\ln(2x+2s-2)+\frac{x(2s+x-1)}{(s+x)(s+x-1)}\\
&>(2x-1)\ln\frac{2s+2x}{2s+2x-2}+\frac{x(2s+x-1)}{(s+x)(s+x-1)}>0.
\end{aligned}$$

Thus $f(n_2)>f(n_1-1)$, that is

$$\begin{aligned}
&(n_2^2+n_2)\ln(2n_2+2s)-(n_2^2-n_2)\ln(2n_2+2s-2)\\
&\quad>(n_1^2-n_1)\ln(2n_1+2s-2)-((n_1-1)^2-(n_1-1))\ln(2n_1+2s-4)\\
\Longrightarrow&\frac{(n_1-1)(n_1-2)}{2}\ln(2n_1+2s-4)+\frac{(n_2+1)n_2}{2}\ln(2n_2+2s)\\
&\quad>\frac{n_1(n_1-1)}{2}\ln(2n_1+2s-2)+\frac{n_2(n_2-1)}{2}\ln(2n_2+2s-2)\\
\Longrightarrow&\ln\left((2n_1+2s-4)^{\frac{(n_1-1)(n_1-2)}{2}}(2n_2+2s)^{\frac{(n_2+1)n_2}{2}}\right)\\
&\quad>\ln\left((2n_1+2s-2)^{\frac{n_1(n_1-1)}{2}}(2n_2+2s-2)^{\frac{n_2(n_2-1)}{2}}\right)\\
\Longrightarrow&(2n_1+2s-4)^{\frac{(n_1-1)(n_1-2)}{2}}(2n_2+2s)^{\frac{(n_2+1)n_2}{2}}\\
&\quad>(2n_1+2s-2)^{\frac{n_1(n_1-1)}{2}}(2n_2+2s-2)^{\frac{n_2(n_2-1)}{2}}.
\end{aligned}$$

This finishes the proof.

LEMMA 2.4. *Let n,a,n_1,n_2 be positive integers, where $n_2\geqslant n_1\geqslant 2$, $n_1+n_2<n$ and $a\geqslant n-1$. Then*

$$\frac{(a+n_1-1)^{n_1-1}(a+n_2+1)^{n_2+1}}{(a+n_1)^{n_1}(a+n_2)^{n_2}}>1.$$

Proof. Let $g(x) = x\ln(a+x) - (x-1)\ln(a+x-1)$ be a real function in x, where $a \geqslant n-1, x \geqslant 2$. Then

$$g'(x) = \ln\frac{a+x}{a+x-1} + \frac{a}{(a+x)(a+x-1)} > 0.$$

Thus $g(n_2+1) > g(n_1)$, that is

$$\begin{aligned}&(n_2+1)\ln(a+n_2+1) - n_2\ln(a+n_2) > n_1\ln(a+n_1) - (n_1-1)\ln(a+n_1-1)\\ \Longrightarrow&\ln\left((a+n_2+1)^{n_2+1}(a+n_1-1)^{n_1-1}\right) > \ln\left((a+n_1)^{n_1}(a+n_2)^{n_2}\right)\\ \Longrightarrow&(a+n_2+1)^{n_2+1}(a+n_1-1)^{n_1-1} > (a+n_1)^{n_1}(a+n_2)^{n_2}.\end{aligned}$$

This completes the proof.

3. Multiplicative sum Zagreb index of graphs with fixed number of cut edges

We use $\mathbf{G}_E(n,k)$ to denote the n-vertex graphs with k cut edges. $\mathbf{G}_E(n,n-1)$ are trees, and trees with extremal multiplicative sum Zagreb index had been obtained in [2] and [3]. Let $G \in \mathbf{G}_E(n,k)$, if $\gamma(G) \geqslant 1$, then $k \leqslant n-3$. Thus, in this section, we always discuss the case of $1 \leqslant k \leqslant n-3$ when $G \in \mathbf{G}_E(n,k)$.

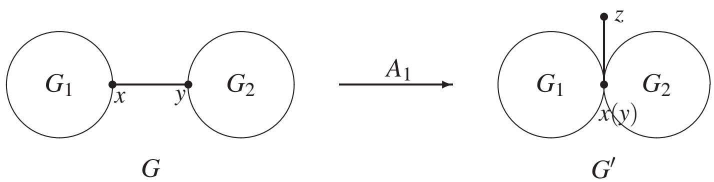

Fig. 2. Transformation A_1.

Transformation A_1: Suppose G_1 and G_2 are graphs with $n_1 \geqslant 3$ and $n_2 \geqslant 2$ vertices, respectively, where G_1 is 2-edge connected. Suppose G is a graph, as shown in Fig. 2, resulted from G_1 and G_2 by adding an edge from a vertex $x \in V(G_1)$ to a vertex $y \in V(G_2)$. Then xy is a non-pendant cut edge in G. Let G' be the graph obtained by identifying x of G_1 to y of G_2 and adding a pendant edge to $x(y)$, as shown in Fig. 2.

LEMMA 3.1. *Suppose G' and G are graphs in Fig. 2, then $\Pi_1^*(G') > \Pi_1^*(G)$.*

Proof. Denote $d_{G_1}(x) = d_1$ and $d_{G_2}(y) = d_2$. By the definition of Π_1^*, it follows that

$$\frac{\Pi_1^*(G')}{\Pi_1^*(G)} = \frac{(d_1+d_2+2)\left(\prod_{i=1}^{d_1}(d_{G_1}(x_i)+d_1+d_2+1)\right)\left(\prod_{j=1}^{d_2}(d_{G_2}(y_j)+d_1+d_2+1)\right)}{(d_1+d_2+2)\left(\prod_{i=1}^{d_1}(d_{G_1}(x_i)+d_1+1)\right)\left(\prod_{j=1}^{d_2}(d_{G_2}(y_j)+d_2+1)\right)} > 1.$$

The proof is completed.

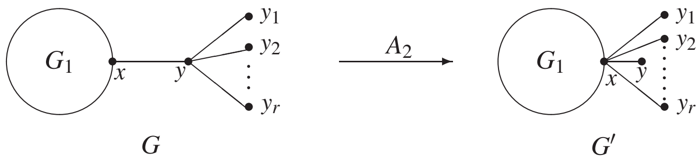

Fig. 3. Transformation A_2.

Transformation A_2: Suppose G is a graph as shown in Fig. 3, where xy is a non-pendant cut edge of G, G_1 is 2-edge connected, $d_G(x) \geqslant 2$, $N_G(y)/\{x\} = \{y_1, y_2, \cdots, y_r\}$ ($y_1, y_2, \cdots, y_r$ are pendant vertices). $G' = G - \{yy_1, yy_2, \cdots, yy_r\} + \{xy_1, xy_2, \cdots, xy_r\}$, as shown in Fig. 3.

LEMMA 3.2. ([2]) *Suppose G and G' are graphs in Fig. 3, then* $\Pi_1^*(G') > \Pi_1^*(G)$.

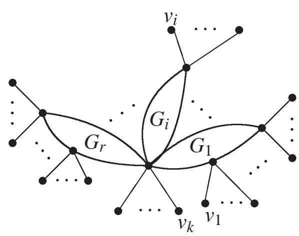

Fig. 4. The graph G^*.

REMARK 3.3. For any $G \in \mathbf{G}_E(n,k)$, if necessary, by repeating the graph transformation A_1 or A_2, any cut edges in G can changed into pendant edges. That is, if necessary, by a series of transformation A_1 or A_2, we can change G to G^* (as depicted in Fig. 4), where $G_1, G_2, \cdots, G_r$ are 2-edge connected graphs.

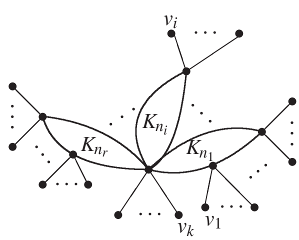

Fig. 5. The graph H.

By Lemma 3.1, 3.2 and Remark 3.3, the following Lemma 3.4 is obtained immediately.

LEMMA 3.4. *Suppose $G \in \mathbf{G}_E(n,k)$, then $\Pi_1^*(G) \leqslant \Pi_1^*(G^*)$, where G^* are graphs as depicted in Fig. 4.*

Denoted K_{n_i} $(1 \leqslant i \leqslant r)$ to be a clique which is obtained by adding edges in G_i $(1 \leqslant i \leqslant r)$ and changing G_i into complete subgraphs, where $G_1, G_2, \cdots, G_r$ in G^* are 2-edge connected graphs. By Lemma 2.1, we get the following Lemma 3.5.

LEMMA 3.5. *Suppose H is the graph as depicted in Fig. 5, where K_{n_i} $(1 \leqslant i \leqslant r)$ are cliques as above, then $\Pi_1^*(H) \geqslant \Pi_1^*(G^*)$.*

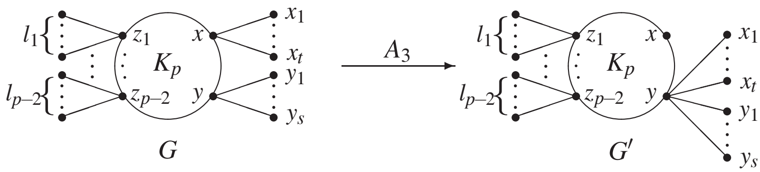

Fig. 6. Transformation A_3.

Transformation A_3: Suppose G is a graph as depicted in Fig. 6, $V(K_p) = \{x, y, z_1, \cdots, z_{p-2}\}$, each vertex on K_p either is of degree $p-1$ or has some pendant edges attached, where $p \geqslant 3$, $l_1, \cdots, l_{p-2} \geqslant 0$. $x_1, x_2, \cdots, x_t$ and $y_1, y_2, \cdots, y_s$ are pendant vertices adjacent to x and y, respectively, where $t, s \geqslant 1$. Let $G' = G - \{xx_1, xx_2, \cdots, xx_t\} + \{yx_1, yx_2, \cdots, yx_t\}$, as depicted in Fig. 6.

LEMMA 3.6. *Suppose G' and G are graphs in Fig. 6, then $\Pi_1^*(G') > \Pi_1^*(G)$.*

Proof. It is evident that $d_{G'}(x) = p-1$, $d_G(x) = p-1+t$, $d_{G'}(y) = p-1+t+s$, $d_G(y) = p-1+s$. By the definition of Π_1^*, we find that

$$\frac{\Pi_1^*(G')}{\Pi_1^*(G)} = \frac{(2p-2+t+s)(p+t+s)^{t+s}\prod\limits_{i=1}^{p-2}\Big((d_G(z_i)+p-1)(d_G(z_i)+p-1+t+s)\Big)}{(2p-2+t+s)(p+t)^t(p+s)^s\prod\limits_{i=1}^{p-2}\Big((d_G(z_i)+p-1+t)(d_G(z_i)+p-1+s)\Big)}$$

$$= \frac{(p+t+s)^{t+s}}{(p+t)^t(p+s)^s}\prod_{i=1}^{p-2}\frac{(d_G(z_i)+p-1)(d_G(z_i)+p-1+t+s)}{(d_G(z_i)+p-1+t)(d_G(z_i)+p-1+s)}.$$

Let $f(x) = \frac{(x+p-1)(x+p-1+t+s)}{(x+p-1+t)(x+p-1+s)}$ be a real function in x, where $x \geqslant p-1$. Then

$$f'(x) = \frac{ts(2p-2+2x+t+s)}{(x+p-1+t)^2(x+p-1+s)^2} > 0.$$

Thus

$$\frac{\Pi_1^*(G')}{\Pi_1^*(G)} \geqslant \frac{(p+t+s)^{t+s}}{(p+t)^t(p+s)^s}\left(\frac{(2p-2)(2p-2+t+s)}{(2p-2+t)(2p-2+s)}\right)^{p-2}$$
$$> \frac{(p+t+s)^{t+s}}{(p+t)^t(p+s)^s}\left(\frac{(2p-2)(2p-2+t+s)}{(2p-2+t)(2p-2+s)}\right)^{p-1},$$

since $(2p-2)(2p-2+t+s) - (2p-2+t)(2p-2+s) = -ts < 0$. Let $q = p-1$. By Lemma 2.2, the lemma holds immediately.

THEOREM 3.7. *Suppose $G \in \mathbf{G}_E(n,k)$, where $1 \leqslant k \leqslant n-3$, then*

$$\Pi_1^*(G) \leqslant n^k(2n-k-2)^{n-k-1}(2n-2k-2)^{\binom{n-k-1}{2}}$$

with equality if and only if $G \cong K_n^k$.

Proof. Assume that $G \in \mathbf{G}_E(n,k)$ has the maximum $\Pi_1^*(G)$. By Lemma 3.4 and 3.5, it follows that $\Pi_1^*(G) \leqslant \Pi_1^*(H)$.

Next, we prove that $r=1$. By contradiction. If $r \geqslant 2$, suppose without loss of generality that there exists an edge $e = xy \notin E(G)$, $x \in V(K_{n_i})$, $y \in V(K_{n_j})$, $1 \leqslant i < j \leqslant r$, and x,y is not the common vertex of K_{n_i} and K_{n_j}. By Lemma 2.1, it can be seen that $\Pi_1^*(G+e) > \Pi_1^*(G)$, a contradiction. So $r=1$. Thus G is a graph obtained from K_{n-k} by attaching some pendant edges to some vertices of K_{n-k} (the number of all pendant edges of G is k). By Lemma 3.6, $G \cong K_n^k$.

4. Multiplicative sum Zagreb index of graphs with fixed number of cut vertices

We use $\mathbf{G}_V(n,k)$ to denote the n-vertex graphs with k cut vertices. Since $\mathbf{G}_V(n,n-2)$ is a path, thus, in this section, we always discuss the case of $1 \leqslant k \leqslant n-3$ when $G \in \mathbf{G}_V(n,k)$.

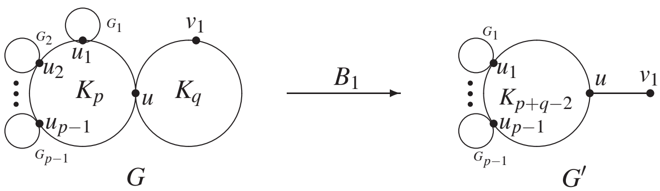

Fig. 7. Transformation B_1.

Transformation B_1: Suppose G is a graph as depicted in Fig. 7, K_p $(p \geqslant 2)$ and K_q $(q \geqslant 3)$ are two cliques of G, where K_q is an endblock. $V(K_p)$ and $V(K_q)$ have one cut vertex, say u, in common. $V(K_p) = \{u_1, u_2, \cdots, u_{p-1}, u\}$, $V(K_q) = \{v_1, v_2, \cdots, v_{q-1}, u\}$. G_i $(1 \leqslant i \leqslant p-1)$ is the subgraph attached to u_i $(1 \leqslant i \leqslant p-1)$ $(d_G(u_1) \geqslant 2$ when $p=2)$. Let $G' = G - \{v_1v_2, v_1v_3, \cdots, v_1v_{q-1}\} + \{u_1v_2, u_1v_3, \cdots, u_1v_{q-1}\} + \cdots + \{u_{p-1}v_2, u_{p-1}v_3, \cdots, u_{p-1}v_{q-1}\}$, as depicted in Fig. 7.

LEMMA 4.1. *Suppose G' and G are graphs in Fig. 7, then $\Pi_1^*(G') > \Pi_1^*(G)$.*

Proof. Observe that $d_G(u) = d_{G'}(u) = p+q-2$, $d_G(v_1) = q-1$, $d_{G'}(v_1) = 1$, $d_{G'}(u_i) = d_G(u_i) + q - 2$ $(i = 1,2,\cdots,p-1)$, $d_{G'}(v_j) = p+q-3$ $(j = 2,3,\cdots,q-1)$. For $x \in N_{G_i}(u_i)$, $d_{G'}(u_i) + d_{G'}(x) = d_G(u_i) + d_G(x) + q - 2 > d_G(u_i) + d_G(x)$, $i = 1,2,\cdots,p-1$. Then $\frac{d_{G'}(u_i)+d_{G'}(x)}{d_G(u_i)+d_G(x)} > 1$, $x \in N_{G_i}(u_i)$, $i = 1,2,\cdots,p-1$.

If $p=2$, $d_G(u_1)\geqslant 2$, by the definition of Π_1^*, it follows that

$$\begin{aligned}\frac{\Pi_1^*(G')}{\Pi_1^*(G)} =& \frac{(d_{G'}(u)+d_{G'}(v_1))(d_{G'}(u)+d_{G'}(u_1))\prod_{i=2}^{q-1}(d_{G'}(u_1)+d_{G'}(v_i))}{(d_G(u)+d_G(v_1))(d_G(u)+d_G(u_1))\prod_{i=2}^{q-1}(d_G(v_1)+d_G(v_i))} \\ &\cdot \frac{\prod_{i=2}^{q-1}(d_{G'}(u)+d_{G'}(v_i))}{\prod_{i=2}^{q-1}(d_G(u)+d_G(v_i))} \cdot \frac{\prod_{x\in N_{G_1}(u_1)}(d_{G'}(u_1)+d_{G'}(x))}{\prod_{x\in N_{G_1}(u_1)}(d_G(u_1)+d_G(x))} \\ >& \frac{(q+1)(d_G(u_1)+2q-2)(d_G(u_1)+2q-3)^{q-2}}{(2q-1)(d_G(u_1)+q)(2q-2)^{q-2}} \\ =& \frac{(q+1)(d_G(u_1)+2q-2)(d_G(u_1)+2q-3)}{(2q-1)(d_G(u_1)+q)(2q-2)} \cdot \left(\frac{d_G(u_1)+2q-3}{2q-2}\right)^{q-3} \\ \geqslant& \frac{(q+1)2q(2q-1)}{(2q-1)(2+q)(2q-2)} = \frac{(q+1)q}{(q-1)(q+2)} > 1,\end{aligned}$$

since the real function $f(x)=\frac{(x+2q-2)(x+2q-3)}{x+q}$ is monotonous increasing for $x\geqslant 2$ and $(q+1)q-(q-1)(q+2)=2>0$.

If $p\geqslant 3$, we have

$$\begin{aligned}\frac{\Pi_1^*(G')}{\Pi_1^*(G)} =& \frac{(d_{G'}(u)+d_{G'}(v_1))\prod_{i=1}^{p-1}(d_{G'}(u)+d_{G'}(u_i))\prod_{1\leqslant i<j\leqslant p-1}(d_{G'}(u_i)+d_{G'}(u_j))}{(d_G(u)+d_G(v_1))\prod_{i=1}^{p-1}(d_G(u)+d_G(u_i))\prod_{1\leqslant i<j\leqslant p-1}(d_G(u_i)+d_G(u_j))} \\ &\cdot \frac{\prod_{i=2}^{q-1}(d_{G'}(u)+d_{G'}(v_i))\prod_{2\leqslant i<j\leqslant q-1}(d_{G'}(v_i)+d_{G'}(v_j))}{\prod_{i=2}^{q-1}(d_G(u)+d_G(v_i))\prod_{2\leqslant i<j\leqslant q-1}(d_G(v_i)+d_G(v_j))} \\ &\cdot \frac{\prod_{i=1}^{p-1}\prod_{j=2}^{q-1}(d_{G'}(u_i)+d_{G'}(v_j))}{\prod_{i=2}^{q-1}(d_G(v_1)+d_G(v_i))} \cdot \frac{\prod_{i=1}^{p-1}\prod_{x\in N_{G_i}(u_i)}(d_{G'}(u_i)+d_{G'}(x))}{\prod_{i=1}^{p-1}\prod_{x\in N_{G_i}(u_i)}(d_G(u_i)+d_G(x))} \\ \geqslant& \frac{(p+q-1)\prod_{i=1}^{p-1}(d_G(u_i)+p+2q-4)\prod_{1\leqslant i<j\leqslant p-1}(d_G(u_i)+d_G(u_j)+2q-4)}{(p+2q-3)\prod_{i=1}^{p-1}(d_G(u_i)+p+q-2)\prod_{1\leqslant i<j\leqslant p-1}(d_G(u_i)+d_G(u_j))}\end{aligned}$$

$$\cdot \frac{\prod_{i=2}^{q-1}(2p+2q-5)\prod_{2\leqslant i<j\leqslant q-1}(2p+2q-6)}{\prod_{i=2}^{q-1}(p+2q-3)\prod_{2\leqslant i<j\leqslant q-1}(2q-2)}$$
$$\cdot \frac{\prod_{i=1}^{p-1}\prod_{j=2}^{q-1}(d_G(u_i)+p+2q-5)}{\prod_{i=2}^{q-1}(2q-2)}$$
$$> \frac{(p+q-1)\prod_{i=1}^{p-1}\prod_{j=2}^{q-1}(d_G(u_i)+p+2q-5)}{(p+2q-3)\prod_{i=2}^{q-1}(2q-2)}$$
$$\geqslant \frac{(p+q-1)(d_G(u_1)+p+2q-5)^{q-2}(d_G(u_2)+p+2q-5)^{q-2}}{(p+2q-3)(2q-2)^{q-2}}$$
$$\geqslant \frac{(p+q-1)(2p+2q-6)^{q-2}}{(p+2q-3)} \cdot \left(\frac{2p+2q-6}{2q-2}\right)^{q-2} > 1.$$

This completes the proof.

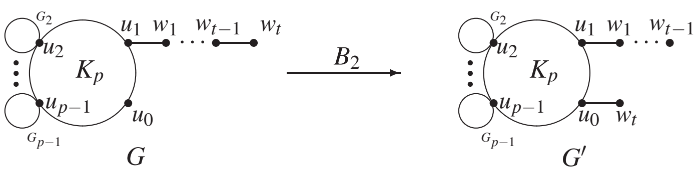

Fig. 8. Transformation B_2.

Transformation B_2: Suppose G is a graph as depicted in Fig. 8, K_p be a clique of G, where $p \geqslant 3$, $V(K_p) = \{u_0, u_1, \cdots, u_{p-1}\}$. $P = u_1 w_1 \cdots w_t$ $(t \geqslant 2)$ is a path attached to u_1. $N_G(u_0) = \{u_1, u_2, \cdots, u_{p-1}\}$, $N_G(u_1) = \{u_0, u_2, \cdots, u_{p-1}, w_1\}$. G_i $(2 \leqslant i \leqslant p-1)$ is the subgraph attached to u_i $(2 \leqslant i \leqslant p-1)$. Let $G' = G - w_{t-1}w_t + u_0 w_t$, as depicted in Fig. 8.

LEMMA 4.2. *Suppose G' and G are graphs in Fig. 8, then $\Pi_1^*(G') > \Pi_1^*(G)$.*

Proof. If $t = 2$, for $p \geqslant 3$, we have

$$\frac{\Pi_1^*(G')}{\Pi_1^*(G)} = \frac{(p+p)(p+1)(p+1)\prod_{i=2}^{p-1}(d_G(u_i)+p)}{(p+p-1)(p+2)(2+1)\prod_{i=2}^{p-1}(d_G(u_i)+p-1)} > 1.$$

If $t \geqslant 3$, then

$$\frac{\Pi_1^*(G')}{\Pi_1^*(G)} = \frac{(p+p)(2+1)(p+1)\prod_{i=2}^{p-1}(d_G(u_i)+p)}{(p+p-1)(2+2)(2+1)\prod_{i=2}^{p-1}(d_G(u_i)+p-1)} > 1.$$

The proof is completed.

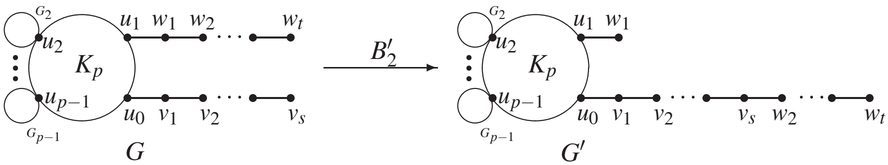

Fig. 9. Transformation B_2'.

Transformation B_2': Suppose G is a graph as depicted in Fig. 9, K_p be a clique of G, where $p \geqslant 3$. $V(K_p) = \{u_0, u_1, \cdots, u_{p-1}\}$. $P_1 = u_0 v_1 \cdots v_s$ $(s \geqslant 2)$ is a path attached to u_0 and $P_2 = u_1 w_1 \cdots w_t$ $(t \geqslant 2)$ is a path attached to u_1. $N_G(u_0) = \{u_1, u_2, \cdots, u_{p-1}, v_1\}$, $N_G(u_1) = \{u_0, u_2, \cdots, u_{p-1}, w_1\}$. G_i $(2 \leqslant i \leqslant p-1)$ is the subgraph attached to u_i $(2 \leqslant i \leqslant p-1)$. Let $G' = G - w_1 w_2 + v_s w_2$, as depicted in Fig. 9.

LEMMA 4.3. *Suppose G' and G are graphs in Fig. 9, then $\Pi_1^*(G') > \Pi_1^*(G)$.*

Proof. If $t = 2$, we notice that

$$\frac{\Pi_1^*(G')}{\Pi_1^*(G)} = \frac{(p+1)(2+2)(2+1)}{(p+2)(2+1)(2+1)} > 1.$$

If $t \geqslant 3$,

$$\frac{\Pi_1^*(G')}{\Pi_1^*(G)} = \frac{(p+1)(2+2)(2+2)}{(p+2)(2+2)(2+1)} > 1.$$

This finishes the proof.

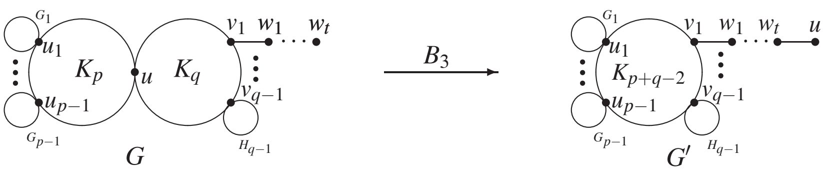

Fig. 10. Transformation B_3.

Transformation B_3: Suppose G is a graph as depicted in Fig. 10, K_p $(p \geqslant 3)$ and K_q $(q \geqslant 3)$ are two cliques of G. $V(K_p)$ and $V(K_q)$ have one cut vertex, say u, in common. $V(K_p) = \{u_1, u_2, \cdots, u_{p-1}, u\}$, $V(K_q) = \{v_1, v_2, \cdots, v_{q-1}, u\}$. $P = v_1 w_1 \cdots w_t$ $(t \geqslant$

$1)$ is a path attached to v_1 and $N_G(v_1)=\{u,v_2,\cdots,v_{q-1},w_1\}$. G_i $(1\leqslant i\leqslant p-1)$ is the subgraph attached to u_i $(1\leqslant i\leqslant p-1)$ and H_j $(2\leqslant j\leqslant q-1)$ is the subgraph attached to v_j $(2\leqslant j\leqslant q-1)$. Let $G'=G-\{uu_1,uu_2,\cdots,uu_{p-1},uv_1,uv_2,\cdots,uv_{q-1}\}+\{w_tu\}+\{u_1v_1,u_1v_2,\cdots,u_1v_{q-1}\}+\cdots+\{u_{p-1}v_1,u_{p-1}v_2,\cdots,u_{p-1}v_{q-1}\}$, as depicted in Fig. 10.

LEMMA 4.4. *Suppose G' and G are graphs in Fig. 10, then $\Pi_1^*(G')>\Pi_1^*(G)$.*

Proof. It can be seen that $d_G(u)=p+q-2$, $d_{G'}(u)=d_G(w_t)=1$, $d_{G'}(w_t)=2$, $d_G(v_1)=q$, $d_{G'}(v_1)=p+q-2$, $d_{G'}(u_i)=d_G(u_i)+q-2$ $(i=1,2,\cdots,p-1)$, $d_{G'}(v_j)=d_G(v_j)+p-2$ $(j=2,\cdots,q-1)$. For $x\in N_{G_i}(u_i)$, $d_{G'}(u_i)+d_{G'}(x)=d_G(u_i)+d_G(x)+q-2>d_G(u_i)+d_G(x)$, $i=1,2,\cdots,p-1$. For $y\in N_{H_j}(v_j)$, $d_{G'}(v_j)+d_{G'}(y)=d_G(v_j)+d_G(y)+p-2>d_G(v_j)+d_G(y)$, $j=2,3,\cdots,q-1$.

If $t=1$, by the definition of Π_1^*, it follows that

$$\begin{aligned}
\frac{\Pi_1^*(G')}{\Pi_1^*(G)} &\geqslant \frac{3(p+q)\prod\limits_{i=1}^{p-1}(d_G(u_i)+p+2q-4)\prod\limits_{1\leqslant i<j\leqslant p-1}(d_G(u_i)+d_G(u_j)+2q-4)}{(q+1)\prod\limits_{i=1}^{p-1}(d_G(u_i)+p+q-2)\prod\limits_{1\leqslant i<j\leqslant p-1}(d_G(u_i)+d_G(u_j))} \\
&\quad\cdot\frac{\prod\limits_{i=2}^{q-1}(d_G(v_i)+2p+q-4)\prod\limits_{2\leqslant i<j\leqslant q-1}(d_G(v_i)+d_G(v_j)+2p-4)}{\prod\limits_{i=2}^{q-1}(q+d_G(v_i))\prod\limits_{2\leqslant i<j\leqslant q-1}(d_G(v_i)+d_G(v_j))} \\
&\quad\cdot\frac{\prod\limits_{i=1}^{p-1}\prod\limits_{j=2}^{q-1}(d_G(u_i)+d_G(v_j)+p+q-4)}{(p+2q-2)\prod\limits_{i=2}^{q-1}(d_G(v_i)+p+q-2)} \\
&> \frac{3(p+q)}{(q+1)(p+2q-2)}\cdot\frac{\prod\limits_{i=1}^{p-1}\prod\limits_{j=2}^{q-1}(d_G(u_i)+d_G(v_j)+p+q-4)}{\prod\limits_{i=2}^{q-1}(d_G(v_i)+p+q-2)} \\
&> \frac{3(p+q)(d_G(u_1)+d_G(v_2)+p+q-4)}{(q+1)(p+2q-2)} \\
&\geqslant \frac{3(p+q)(2p+2q-6)}{(q+1)(p+2q-2)}>1.
\end{aligned}$$

The case of $t\geqslant 2$ can be proved similarly.

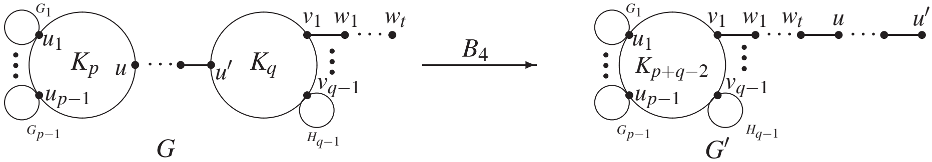

Fig. 11. Transformation B_4.

Transformation B_4: Suppose G is a graph as depicted in Fig. 11, K_p and K_q be two cliques of G, where $p,q \geqslant 3$. K_p connects K_q by an internal path $P = u \cdots u'$ with length $s \geqslant 1$. $V(K_p) = \{u_1, u_2, \cdots, u_{p-1}, u\}$, $V(K_q) = \{v_1, v_2, \cdots, v_{q-1}, u'\}$. $P_{t+1} = v_1 w_1 \cdots w_t$ $(t \geqslant 1)$ is a path attached to v_1 and $N_G(v_1) = \{u', v_2, \cdots, v_{q-1}, w_1\}$. G_i $(1 \leqslant i \leqslant p-1)$ is the subgraph attached to u_i $(1 \leqslant i \leqslant p-1)$ and H_j $(2 \leqslant j \leqslant q-1)$ is the subgraph attached to v_j $(2 \leqslant j \leqslant q-1)$. Let $G' = G - \{uu_1, uu_2, \cdots, uu_{p-1}, u'v_1, u'v_2, \cdots, u'v_{q-1}\} + \{w_t u\} + \{u_1v_1, u_1v_2, \cdots, u_1v_{q-1}\} + \cdots + \{u_{p-1}v_1, u_{p-1}v_2, \cdots, u_{p-1}v_{q-1}\}$, as depicted in Fig. 11.

LEMMA 4.5. *Suppose G' and G are graphs in Fig. 11, then* $\Pi_1^*(G') > \Pi_1^*(G)$.

Proof. We notice that $d_{G'}(u) = 2$, $d_G(u) = p$, $d_{G'}(u') = 1$, $d_G(u') = q$, $d_{G'}(w_t) = 2$, $d_G(w_t) = 1$, $d_{G'}(v_1) = p+q-2$, $d_G(v_1) = q$, $d_{G'}(u_i) = d_G(u_i) + q - 2$ $(i = 1, 2, \cdots, p-1)$, $d_{G'}(v_j) = d_G(v_j) + p - 2$ $(j = 2, \cdots, q-1)$. For $x \in N_{G_i}(u_i)$, $d_{G'}(u_i) + d_{G'}(x) > d_G(u_i) + d_G(x)$, $i = 1, 2, \cdots, p-1$. For $y \in N_{H_j}(v_j)$, $d_{G'}(v_j) + d_{G'}(y) > d_G(v_j) + d_G(y)$, $j = 2, 3, \cdots, q-1$.

If $t = s = 1$, in view of the definition of Π_1^*, it can be concluded that

$$\begin{aligned}
\frac{\Pi_1^*(G')}{\Pi_1^*(G)} \geqslant & \frac{4 \cdot 3(p+q) \prod_{i=1}^{p-1} (d_G(u_i) + p + 2q - 4) \prod_{1 \leqslant i < j \leqslant p-1} (d_G(u_i) + d_G(u_j) + 2q - 4)}{(p+q)(q+1)(q+q) \prod_{i=1}^{p-1} (d_G(u_i) + p) \prod_{1 \leqslant i < j \leqslant p-1} (d_G(u_i) + d_G(u_j))} \\
& \cdot \frac{\prod_{i=2}^{q-1} (d_G(v_i) + 2p + q - 4) \prod_{2 \leqslant i < j \leqslant q-1} (d_G(v_i) + d_G(v_j) + 2p - 4)}{\prod_{i=2}^{q-1} (q + d_G(v_i)) \prod_{2 \leqslant i < j \leqslant q-1} (d_G(v_i) + d_G(v_j))} \\
& \cdot \frac{\prod_{i=1}^{p-1} \prod_{j=2}^{q-1} (d_G(u_i) + d_G(v_j) + p + q - 4)}{\prod_{i=2}^{q-1} (d_G(v_i) + q)} \\
> & \frac{6}{q(q+1)} \cdot \frac{\prod_{i=1}^{p-1} \prod_{j=2}^{q-1} (d_G(u_i) + d_G(v_j) + p + q - 4)}{\prod_{i=2}^{q-1} (d_G(v_i) + q)}.
\end{aligned}$$

If $q=3$, then

$$\frac{\Pi_1^*(G')}{\Pi_1^*(G)} > \frac{6(d_G(u_1)+d_G(v_2)+p-1)}{3\times 4} > 1.$$

If $q>3$, then

$$\frac{\Pi_1^*(G')}{\Pi_1^*(G)} > \frac{6(d_G(u_1)+d_G(v_2)+p+q-4)(d_G(u_1)+d_G(v_3)+p+q-4)}{q(q+1)} > 1.$$

The case of $t,s>1$; $t=1,s>1$ or $t>1,s=1$ can be proved similarly as the case of $t=s=1$, omitted.

Let $G\in \mathbf{G}_V(n,k)$. In order to get the maximum $\Pi_1^*(G)$, we first provide a definition and a notation. Suppose K_p $(p\geqslant 3)$ and K_q $(q\geqslant 3)$ are two cliques in G, if K_p connects K_q by a path P (perhaps $|E(P)|=0$, namely K_p and K_q has a vertex in common which is a cut vertex of G) such that P doesn't intersect some other cliques K_r with $r\geqslant 3$, we call K_p and K_q are adjacent. Denote $\mathbf{G}_{n,k}=\{G \mid G\in \mathbf{G}_V(n,k)$ is the graph arisen from K_{n-k} by attaching at most one pendant path to its each vertex$\}$. Clearly, $\{G^1_{n,k},G^2_{n,k}\}\subset \mathbf{G}_{n,k}$.

THEOREM 4.6. *Suppose $G\in \mathbf{G}_V(n,k)$, where $1\leqslant k\leqslant n-3$, then*

(i) if $1\leqslant k\leqslant \frac{n}{2}$, $\Pi_1^*(G)\leqslant (n-k+1)^k(2n-2k)^{\binom{k}{2}}(2n-2k-2)^{\binom{n-2k}{2}}(2n-2k-1)^{k(n-2k)}$ with equality if and only if $G\cong G^1_{n,k}$;

(ii) if $\frac{n}{2}<k\leqslant n-3$, $\Pi_1^*(G)\leqslant 3(n-k+2)\cdot 4^{2k-n-1}(n-k+1)^{n-k-1}(2n-2k)^{\binom{n-k}{2}}$ with equality if and only if $G\cong G^2_{n,k}$.

Proof. Suppose $G\in \mathbf{G}_V(n,k)$ has the maximum Π_1^*. First some claims will be given.

CLAIM 1. Each cut vertex of G connects exactly two blocks, and all blocks of G are cliques.

Proof. By contradiction. Assume that x is a cut vertex in G, and $G-x=\bigcup_{i=1}^r G_i$, where $r\geqslant 3$. Choose $y\in V(G_2)\setminus\{x\}$ and $z\in V(G_r)\setminus\{x\}$, and let $G^*=G+yz$. Clearly, $G^*\in \mathbf{G}_V(n,k)$. By Lemma 2.1, it follows that $\Pi_1^*(G)<\Pi_1^*(G^*)$, a contradiction. Thus, we get that each cut vertex connects exactly two blocks of G. Moreover, by Lemma 2.1, we can conclude that all blocks in G are cliques.

By Claim 1, the following Claim 2 is obtained.

CLAIM 2. If two cliques K_p, K_q with $p,q\geqslant 3$ of G are adjacent, then the path, say P, connecting K_p and K_q is either $|E(P)|=0$ or an internal path.

CLAIM 3. Let K_q be an endblock of G. Then $q=2$.

Proof. We prove this claim by contradiction. Suppose that $q\geqslant 3$, let K_p $(p\geqslant 2)$ be a clique such that K_p connects K_q by a cut vertex, say u. By Claim 1, u is not the cut vertex of some other cliques. By Lemma 4.1, G can be changed to G' by transformation B_1 with a larger Π_1^*, which contradicts the choice of G. Hence, $q=2$.

By Claim 1, we suppose that $K_{n_1}, K_{n_2}, \cdots, K_{n_r}$ are all of the cliques in G.

CLAIM 4. Let $K_{n_1}, K_{n_2}, \cdots, K_{n_r}$ are all of the cliques in G. Then there is only one clique K_{n_i} with $n_i \geqslant 3$.

Proof. To the contrary, suppose that there are two cliques K_p, K_q ($K_p, K_q \in \{K_{n_1}, K_{n_2}, \cdots, K_{n_r}\}$ and $p \neq q$) such that K_p is adjacent to K_q, where $p, q \geqslant 3$. By Claim 3, it can be seen that K_p and K_q are not endblocks. Furthermore, by Claim 1, we can choose two such blocks such that at least one of them has a pendant path attached to one of its vertex. Suppose without loss of generality that K_q is one of such cliques which has a pendant path, say $P_{t+1} = v_1 w_1 \cdots w_t$ $(t \geqslant 1)$, attached on $v_1 \in V(K_q)$. By Claim 2 , we can see that K_p connects K_q by a cut vertex u or an internal path $P = u \cdots u'$ with length $s \geqslant 1$. By Lemma 4.4 or 4.5, G can be changed to G' by transformation B_3 or B_4 with a larger Π_1^*, a contradiction.

CLAIM 5. Suppose K_p is the only clique with $p \geqslant 3$, then $p = n - k$.

Proof. In view of Claim 1 and Claim 4, it can be concluded that there exist $k+1$ cliques in G and among them, k cliques are isomorphic to K_2. Furthermore, $G \in \mathbf{G}_V(n,k)$, and each cut vertex belongs to two cliques, we can get immediately that $2k + p - k = n$. As a result, $p = n - k$.

CLAIM 6. Let $H \in \mathbf{G}_{n,k}$. Then $\Pi_1^*(H) \leqslant \Pi_1^*(G_{n,k}^1)$ or $\Pi_1^*(H) \leqslant \Pi_1^*(G_{n,k}^2)$.

Proof. Let $H \in \mathbf{G}_{n,k}$ such that H has the maximum Π_1^*. If $H \cong G_{n,k}^1$ or $G_{n,k}^2$, the claim holds. Otherwise, $H \in \mathbf{G}_{n,k} \setminus \{G_{n,k}^1, G_{n,k}^2\}$. Then H satisfies the following (i) or (ii).

(i) There is a vertex of K_{n-k} with no pendant path attached in H, and H has a pendant path with length equal or more than 2;

(ii) H has at least two pendant paths with length equal or more than 2.

By Lemma 4.2 or 4.3, H can be changed to H' by transformation B_2 or B_2' with a larger Π_1^*, which contradicts the assumption of H.

By Claim 4 and 5, it follows that $G \in \mathbf{G}_{n,k}$. By Claim 6, it follows that $\Pi_1^*(G) \leqslant \Pi_1^*(G_{n,k}^1)$ when $1 \leqslant k \leqslant \frac{n}{2}$ and $\Pi_1^*(G) \leqslant \Pi_1^*(G_{n,k}^2)$ when $\frac{n}{2} < k \leqslant n-3$.

5. Multiplicative sum Zagreb index of graphs with fixed number of vertex connectivity or edge connectivity

LEMMA 5.1. *Let* $G \cong K_s \vee (K_{n_1} \cup K_{n_2})$ *and* $G' \cong K_s \vee (K_{n_1-1} \cup K_{n_2+1})$, *where* $n_1 + n_2 = n - s$, $n_2 \geqslant n_1 \geqslant 2$. *Then*

$$\Pi_1^*(G') > \Pi_1^*(G).$$

Proof. By the definition of Π_1^*, it follows that

$$\begin{aligned}\frac{\Pi_1^*(G')}{\Pi_1^*(G)} =& \frac{(2n_1+2n_2+2s-2)^{\binom{s}{2}}(2n_1+2s-4)^{\binom{n_1-1}{2}}(2n_2+2s)^{\binom{n_2+1}{2}}}{(2n_1+2n_2+2s-2)^{\binom{s}{2}}(2n_1+2s-2)^{\binom{n_1}{2}}(2n_2+2s-2)^{\binom{n_2}{2}}} \\ &\cdot \frac{(2n_1+n_2+2s-3)^{s(n_1-1)}(2n_2+n_1+2s-1)^{s(n_2+1)}}{(2n_1+n_2+2s-2)^{sn_1}(2n_2+n_1+2s-2)^{sn_2}} \\ =& \frac{(2n_1+2s-4)^{\frac{(n_1-1)(n_1-2)}{2}}(2n_2+2s)^{\frac{(n_2+1)n_2}{2}}}{(2n_1+2s-2)^{\frac{n_1(n_1-1)}{2}}(2n_2+2s-2)^{\frac{n_2(n_2-1)}{2}}} \\ &\cdot \left(\frac{(n+s-3+n_1)^{n_1-1}(n+s-1+n_2)^{n_2+1}}{(n+s-2+n_1)^{n_1}(n+s-2+n_2)^{n_2}}\right)^s.\end{aligned}$$

By Lemma 2.3 and 2.4 ($a=n+s-2$), we have $\frac{\Pi_1^*(G')}{\Pi_1^*(G)} > 1$.

THEOREM 5.2. *Suppose G is a graph of order $n \geqslant 4$ with vertex connectivity $\kappa < n-1$, then*

$$\Pi_1^*(G) \leqslant (\kappa+n-1)^{\kappa}(2n-2)^{\binom{\kappa}{2}}(2n-3)^{\kappa(n-\kappa-1)}(2n-4)^{\binom{n-\kappa-1}{2}}$$

with equality if and only if $G \cong K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$.

Proof. Choose G such that G has the maximum Π_1^* among all graphs of order n with vertex connectivity κ. Assume that X is a vertex cut with $|X|=\kappa$ of G such that $G-X$ has κ components, say $G_1, G_2, \cdots, G_\kappa$, where $\kappa \geqslant 2$. Let $n_1 = |V(G_1)|$ and $n_2 = |V(G_2 \cup \cdots \cup G_\kappa)|$. It is clear that G is a spanning sub-graph of $K_\kappa \vee (K_{n_1} \cup K_{n_2})$. By Lemma 2.1, $\Pi_1^*(G) \leqslant \Pi_1^*(K_\kappa \vee (K_{n_1} \cup K_{n_2}))$. Moreover, by Lemma 5.1, $G \cong K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$.

THEOREM 5.3. *Suppose G is a graph of order $n \geqslant 4$ with edge connectivity $\lambda < n-1$, then*

$$\Pi_1^*(G) \leqslant (\lambda+n-1)^{\lambda}(2n-2)^{\binom{\lambda}{2}}(2n-3)^{\lambda(n-\lambda-1)}(2n-4)^{\binom{n-\lambda-1}{2}}$$

with equality if and only if $G \cong K_\lambda \vee (K_1 \cup K_{n-\lambda-1})$.

Proof. Choose G such that G has the maximum Π_1^* among all graphs on n vertices with edge connectivity λ. Suppose the vertex connectivity of G is κ. It follows that $\kappa \leqslant \lambda < n-1$. By Theorem 5.2, we have $G \cong K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$. Furthermore, $K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$ is a spanning sub-graph of $K_\lambda \vee (K_1 \cup K_{n-\lambda-1})$ for $\kappa \leqslant \lambda$, in view of Lemma 2.1, the theorem holds immediately.

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