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ONE DIMENSIONAL WEIGHTED HARDY'S INEQUALITIES AND APPLICATION

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Abstract. Let Ω be a C^2 class bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$. In the present paper we shall improve one dimensional weighted Hardy inequalities with one-sided boundary condition by adding sharp remainders. As an application, we shall establish n dimensional weighted Hardy inequalities with weight functions being powers of the distance function $\delta(x)$ to the boundary $\partial\Omega$. Our results will be applicable to variational problems in a coming paper [3].

1. Introduction

Let $1 < p < \infty$ and $C_c^\infty((0,1])$ denote the set of all C^∞ functions with compact supports in $(0,1]$. One dimensional Hardy inequality with one-sided boundary condition is represented by

$$\int_0^1 |u'(t)|^p \, dt \geqslant \left(1 - \frac{1}{p}\right)^p \int_0^1 \frac{|u(t)|^p}{t^p} \, dt + \left(1 - \frac{1}{p}\right)^{p-1} |u(1)|^p \tag{1.1}$$

for every $u \in C_c^\infty((0,1])$. When $u(1) = 0$, this is a well-known Hardy inequality (see [11]. To see the optimality of coefficient of the second term in the right hand side, by the density argument it suffices to employ $u_\varepsilon(t) = t^{1-1/p+\varepsilon}$ as a test function and make $\varepsilon \downarrow 0$.

Our first purpose in this paper is not only to establish a weighted version of (1.1) but also improve it by adding sharp remainder terms. As weight functions we consider power type weights $t^{\alpha p}$ for $t \in [0,1]$. Surprisingly our result on this matter is essentially dependent on the range of parameter α. Let us explain with symbolic and most simple cases as examples. To this end we classify the range of the parameter α into two cases and define the best constant $\Lambda_{\alpha,p}$ as follows:

DEFINITION 1.1. The parameter α is said to be noncritical and critical if α satisfies $\alpha < 1 - 1/p$ and $\alpha \geqslant 1 - 1/p$ respectively.

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DEFINITION 1.2. For $1 < p < +\infty$ we set

$$\Lambda_{\alpha,p} = \begin{cases} \left|1 - \frac{1}{p} - \alpha\right|^p, & \text{if } \alpha \neq 1 - \frac{1}{p}, \\ \left(1 - \frac{1}{p}\right)^p, & \text{if } \alpha = 1 - \frac{1}{p}. \end{cases} \tag{1.2}$$

When α is noncritical under this definition, as a corollary to Theorem 2.1 we have a sharp Hardy type inequality:

$$\int_0^1 |u'(t)|^p t^{\alpha p}\, dt \geqslant \Lambda_{\alpha,p} \int_0^1 \frac{|u(t)|^p}{t^p} t^{\alpha p}\, dt + (\Lambda_{\alpha,p})^{1-1/p} |u(1)|^p, \tag{1.3}$$

for every $u \in C_c^\infty((0,1])$. To see the optimality of coefficient of the second term in the right hand side, one can employ $u_\varepsilon(t) = t^{1-\alpha-1/p+\varepsilon}$ as a test function as before. When α is critical, it follows from Proposition 2.1 that

$$\inf_{u \in W} \int_0^1 |u'(t)|^p t^{\alpha p} = 0, \tag{1.4}$$

where $W = \{u \in C^1([0,1]) : u(0) = 0, u(1) = 1\}$. Nevertheless we will have a sharp Hardy type inequalities (2.6) and (2.7) as a corollary to Theorem 2.2.

In Section 2.2, as an important application, we will establish n dimensional weighted Hardy inequalities with weight function being powers of the distance function $\delta(x) = \text{dist}(\text{x}, \partial\Omega)$ to the boundary $\partial\Omega$. In this task it is crucial to establish sharp weighted Hardy inequalities in the tubler neighborhood Ω_η of Ω, which are reduced to the one dimensional inequalities in Section 2.1. To this end Ω is assumed to be a bounded domain of $\mathbf{R}^N$ $(N \geqslant 1)$ whose boundary $\partial\Omega$ is a C^2 compact manifolds in the present paper. We prepare more notations to describe our results. For $\alpha \in \mathbf{R}$, by $L^p(\Omega, \delta^{p\alpha})$ we denote the space of Lebesgue measurable functions with weight $\delta^{\alpha p}$, for which

$$||u||_{L^p(\Omega,\delta^{p\alpha})} = \left(\int_\Omega |u|^p \delta^{\alpha p}\, dx\right)^{1/p} < +\infty. \tag{1.5}$$

$W_{\alpha,0}^{1,p}(\Omega)$ is given by the completion of $C_c^\infty(\Omega)$ with respect to the norm defined by

$$||u||_{W_{\alpha,0}^{1,p}(\Omega)} = |||\nabla u|||_{L^p(\Omega,\delta^{p\alpha})} + ||u||_{L^p(\Omega,\delta^{p\alpha})}. \tag{1.6}$$

Then $W_{\alpha,0}^{1,p}(\Omega)$ becomes a Banach space with the norm $||\cdot||_{W_{\alpha,0}^{1,p}(\Omega)}$. Under these preparation we will state the noncritical weighted Hardy inequality as Theorem 2.3, which is the counter-part to Theorem 2.1. In particular as its corollary, we have the simplest one:

$$\int_\Omega |\nabla u|^p \delta^{\alpha p} \geqslant \mu \int_\Omega |u|^p \delta^{p(\alpha-1)}, \qquad \forall u \in W_{\alpha,0}^{1,p}(\Omega), \tag{1.7}$$

where $\alpha < 1 - \frac{1}{p}$ and μ is a positive constant essentially depending on the boundary $\partial\Omega$. If $\alpha = 0$ and $p = 2$, then (1.7) is a well-known Hardy inequality and valid for a bounded domain Ω of $\mathbf{R}^N$ with Lipschitz boundary (c.f. [5, 6, 12]). Further if Ω is convex and $\alpha = 0$, then $\mu = \Lambda_{0,p}$ holds for arbitrary $1 < p < \infty$ (see [13]).

It is worthy to remark that (1.7) is never valid in the critical case that $\alpha \geqslant 1 - 1/p$ by (1.4) (see also Proposition 2.2). Nevertheless, we will establish in this case a variant of weighted Hardy's inequalities as Theorem 2.5 which correspond to those in Theorem 2.2. As its corollary we describe Hardy's inequalities with a compact perturbation which are closely relating to the so-called weak Hardy property of Ω. We remark that a constant γ^{-1} in (2.14) and (2.15) concerns the weak Hardy constant, but in this case the strong Hardy constant is $+\infty$ (see [6] for the detail). In [2], two of the authors have improved the weighted Hardy inequalities adopting $|x|^{\alpha p}$ (powers of distance to the origin $O \in \Omega$) as weight functions instead of $\delta^{\alpha p}$. In the present paper, some inequalities of Hardy type in [2] are employed with minor modifications, especially when $1 < p < 2$ (see also [4, 7, 8, 9]). We note that our results will be further improved in [10] for non-doubling weights. Lastly we remark that our results will be applicable to variational problems in a coming paper [3].

This paper is organized in the following way: The main results are described in Section 2. Theorem 2.1 and Theorem 2.2 are established in Section 3. Theorem 2.3 and Theorem 2.5 together with their corollaries are proved in Section 4. The proof of Theorem 2.4 is given in Section 5 and the proofs of Proposition 2.1 and Proposition 2.2 are given in Section 6. In Appendix the proofs of Lemma 3.2 and Lemma 3.4 are provided for the sake of self-containedness.

2. Main results

DEFINITION 2.1. For $t \in (0,1)$ and $R > e$, we set

$$A_1(t) := \log \frac{R}{t}, \qquad A_2(t) := \log A_1(t). \tag{2.1}$$

2.1. Results in the one dimensional case

The proofs of Theorem 2.1 and Theorem 2.2 including corollaries will be given in Section 3 and Appendix.

THEOREM 2.1. (Noncritical case) *Assume that $\alpha < 1 - 1/p$, $1 < p < \infty$ and $R > e$. Then, there exist positive numbers $C_0 = C_0(\alpha, p, R)$, $C_1 = C_1(\alpha, p, R)$ and $L = L(\alpha, p, R)$ such that for every $u \in C_c^\infty((0,1])$, we have*

$$\int_0^1 \left(|u'|^p - \left|\frac{u}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t)^2} \right) \right) t^{\alpha p}\, dt$$
$$\geqslant C_1 \int_0^1 \left(|u'|^p + \left|\frac{u}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t)^2} \right) \right) t^{\alpha p+1}\, dt + L|u(1)|^p. \tag{2.2}$$

COROLLARY 2.1. *Assume that* $\alpha < 1 - 1/p$ *and* $1 < p < \infty$. *Then, for every* $u \in C_c^\infty((0,1])$

$$\int_0^1 |u'(t)|^p t^{\alpha p}\, dt \geqslant \Lambda_{\alpha,p} \int_0^1 \frac{|u(t)|^p}{t^p} t^{\alpha p}\, dt + (\Lambda_{\alpha,p})^{1-1/p} |u(1)|^p. \tag{2.3}$$

In the critical case we have somewhat more precise results.

THEOREM 2.2. (Critical case)

1. *Assume that* $\alpha > 1 - 1/p$, $1 < p < \infty$ *and* $R > e$. *Then there exist positive numbers* $C_0 = C_0(\alpha, p, R)$, $C_1 = C_1(\alpha, p, R)$ *and* $L = L(\alpha, p, R)$ *such that for every* $u \in C_c^\infty((0,1])$, *we have*

$$\int_0^1 \left(|u'|^p - \left|\frac{u}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t)^2} \right) \right) t^{\alpha p}\, dt + L|u(1)|^p$$
$$\geqslant C_1 \int_0^1 \left(|u'|^p + \left|\frac{u}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t)^2} \right) \right) t^{\alpha p + 1}\, dt. \tag{2.4}$$

2. *Assume that* $\alpha = 1 - 1/p$, $1 < p < \infty$ *and* $R > e^e$. *Then, there exist positive numbers* $C_0 = C_0(\alpha, p, R)$, $C_1 = C_1(\alpha, p, R)$*and and* $L = L(\alpha, p, R)$ *such that for every* $u \in C_c^\infty((0,1])$, *we have*

$$\int_0^1 \left(|u'|^p - \left|\frac{u}{t}\right|^p \frac{1}{A_1(t)^p} \left(\Lambda_{\alpha,p} + \frac{C_0}{A_2(t)^2} \right) \right) t^{p-1}\, dt + L|u(1)|^p$$
$$\geqslant C_1 \int_0^1 \left(|u'|^p + \left|\frac{u}{t}\right|^p \frac{1}{A_1(t)^p} \left(\Lambda_{\alpha,p} + \frac{C_0}{A_2(t)^2} \right) \right) t^p\, dt. \tag{2.5}$$

COROLLARY 2.2.

1. *If* $\alpha > 1 - 1/p$ *and* $1 < p < \infty$, *then for every* $u \in C_c^\infty((0,1])$

$$\int_0^1 |u'(t)|^p t^{\alpha p}\, dt + (\Lambda_{\alpha,p})^{1-1/p} |u(1)|^p \geqslant \Lambda_{\alpha,p} \int_0^1 \frac{|u(t)|^p}{t^p} t^{\alpha p}\, dt. \tag{2.6}$$

2. *If* $\alpha = 1 - 1/p$, $1 < p < \infty$ *and* $R > e$, *then for every* $u \in C_c^\infty((0,1])$

$$\int_0^1 |u'(t)|^p t^{p-1}\, dt + (\Lambda_{\alpha,p})^{\alpha} A_1(1)^{1-p} |u(1)|^p \geqslant \Lambda_{\alpha,p} \int_0^1 \frac{|u(t)|^p}{t A_1(t)^p}\, dt. \tag{2.7}$$

REMARK 2.1. We remark that Corollaries 2.1 and 2.2 follow direct from the arguments in the proofs of the corresponding theorems except for the optimality of the constant $L = (\Lambda_{\alpha,p})^{1-1/p}$. For the proofs of the optimality, one can employ as test functions $u_\varepsilon = t^{1-\alpha-1/p+\varepsilon}$ in (2.3), $u_\varepsilon = t^{1-\alpha-1/p+\varepsilon}$ in (2.6) and $u_\varepsilon = A_1(t)^{1-1/p-\varepsilon}$ in (2.7) respectively with ε being sufficiently small.

Further we remark an elementary result which will be useful in the subsequent.

PROPOSITION 2.1. (Critical case) *Assume that* $\alpha \geqslant 1-1/p$ *and* $1<p<\infty$. *Then we have*

$$\inf_{u \in W} \int_0^1 |u'(t)|^p t^{\alpha p} = 0, \tag{2.8}$$

where $W = \{u \in C^1([0,1]) : u(0)=0, u(1)=1\}$.

The proof will be given in Section 6.

2.2. Results in a domain of $\mathbf{R}^N$

The proofs of Theorem 2.3, Corollary 2.3, and Theorem 2.5 will be given in Section 4. Theorem 2.4 will be proved in Section 5. Let $\delta(x) = \mathrm{dist}(\mathrm{x}, \partial\Omega)$. We use the following notations:

$$\Omega_\eta = \{x \in \Omega : \delta(x) < \eta\}, \qquad \Sigma_\eta = \{x \in \Omega : \delta(x) = \eta\}. \tag{2.9}$$

THEOREM 2.3. (Noncritical case) *Assume that* Ω *is a bounded domain of class* C^2 *in* $\mathbf{R}^N$. *Assume that* $\alpha < 1-1/p$, $1<p<\infty$ *and* $R > e \cdot \sup_{x\in\Omega} \delta(x)$. *Assume that* η *is a sufficienty small positive number. Then, there exist positive numbers* $C_2 = C_2(\alpha,p,R,\eta)$ *and* $L = L(\alpha,p,R,\eta)$ *such that for every* $u \in W^{1,p}_{\alpha,0}(\Omega)$, *we have*

$$\int_{\Omega_\eta} \left(|\nabla u|^p - \Lambda_{\alpha,p}\left|\frac{u}{\delta}\right|^p\right)\delta^{\alpha p} \geqslant C_2 \int_{\Omega_\eta} \left|\frac{u}{\delta}\right|^p \frac{1}{A_1(\delta)^2}\delta^{p\alpha} + L\int_{\Sigma_\eta} |u|^p \delta^{\alpha p}. \tag{2.10}$$

COROLLARY 2.3. *Under the same assumptions as in Theorem 2.3, there exists a positive number* $\gamma = \gamma(\alpha,p,R)$ *such that for every* $u \in W^{1,p}_{\alpha,0}(\Omega)$, *we have*

$$\int_{\Omega} \left(|\nabla u|^p - \gamma\left|\frac{u}{\delta}\right|^p\right)\delta^{\alpha p} \geqslant 0. \tag{2.11}$$

Moreover for any bounded domain $\Omega \subset \mathbf{R}^N$ we can prove the following:

THEOREM 2.4. (Noncritical case) *Assume that* Ω *is a bounded domain of class* C^2 *in* $\mathbf{R}^N$. *Assume that* $\alpha < 1-1/p$, $1<p<\infty$ *and* $R > e \cdot \sup_{x\in\Omega}\delta(x)$. *Then, the followings are equivalent.*

1. *There exists positive a number* γ *such that the inequality (2.11) is valid for every* $u \in W^{1,p}_{\alpha,0}(\Omega)$.

2. *For a sufficiently small* $\eta > 0$, *there exist positive numbers* κ, C_2 *and* L *such that the inequality (2.10) with* $\Lambda_{\alpha,p}$ *replaced by* κ *is valid for every* $u \in W^{1,p}_{\alpha,0}(\Omega)$.

THEOREM 2.5. (Critical case) *Assume that Ω is a bounded domain of class C^2 in $\mathbf{R}^N$.*

1. *Assume that $\alpha > 1-1/p$, $1 < p < \infty$ and $R > e^e \cdot \sup_{x\in\Omega}\delta(x)$. Assume that η is a sufficienty small positive number. Then, there exist positive numbers $C_2 = C_2(\alpha,p,R,\eta)$ and $L = L(\alpha,p,R,\eta)$ such that for every $u \in W^{1,p}_{\alpha,0}(\Omega)$, we have*

$$\int_{\Omega_\eta}\left(|\nabla u|^p - \Lambda_{\alpha,p}\left|\frac{u}{\delta}\right|^p\right)\delta^{\alpha p} + L\int_{\Sigma_\eta}|u|^p\delta^{\alpha p} \geqslant C_2\int_{\Omega_\eta}\left|\frac{u}{\delta}\right|^p\frac{\delta^{p\alpha}}{A_1(\delta)^2}. \tag{2.12}$$

2. *Assume that $\alpha = 1-1/p, 1 < p < \infty$ and $R > e^e \cdot \sup_{x\in\Omega}\delta(x)$. Assume that η is a sufficienty small positive number. Then, there exist positive numbers $C_2 = C_2(\alpha,p,R,\eta)$ and $L = L(\alpha,p,R,\eta)$ such that for every $u \in W^{1,p}_{\alpha,0}(\Omega)$, we have*

$$\begin{aligned}\int_{\Omega_\eta}\left(|\nabla u|^p - \Lambda_{\alpha,p}\left|\frac{u}{\delta}\right|^p\frac{1}{A_1(\delta)^p}\right)\delta^{p-1} + L\int_{\Sigma_\eta}|u|^p\delta^{p-1} \\ \geqslant C_2\int_{\Omega_\eta}\left|\frac{u}{\delta}\right|^p\frac{1}{A_1(\delta)^p}\frac{1}{A_2(\delta)^2}\delta^{p-1}.\end{aligned} \tag{2.13}$$

COROLLARY 2.4. *Assume that Ω is a bounded domain of class C^2 in $\mathbf{R}^N$.*

1. *Assume that $\alpha > 1-1/p$, $1 < p < \infty$ and $\gamma = \gamma(\alpha,p,R,\eta)$. Then, there exists a positive number $L' = L'(\alpha,p,R,\eta)$ such that for every $u \in W^{1,p}_{\alpha,0}(\Omega)$, we have*

$$\int_{\Omega}\left(|\nabla u|^p - \gamma\left|\frac{u}{\delta}\right|^p\right)\delta^{\alpha p} + L'\int_{\Sigma_\eta}|u|^p\delta^{\alpha p} \geqslant 0. \tag{2.14}$$

2. *Assume that $\alpha = 1-1/p$, $1 < p < \infty$ and $R > e^e \cdot \sup_{x\in\Omega}\delta(x)$. Then, there exists positive numbers γ and $L' = L'(\alpha,p,R,\eta)$ such that for every $u \in W^{1,p}_{\alpha,0}(\Omega)$, we have*

$$\int_{\Omega}\left(|\nabla u|^p - \gamma\left|\frac{u}{\delta}\right|^p\frac{1}{A_1(\delta)^p}\right)\delta^{p-1} + L'\int_{\Sigma_\eta}|u|^p\delta^{p-1} \geqslant 0. \tag{2.15}$$

PROPOSITION 2.2. (Critical case) *Assume that Ω is a bounded domain of class C^2 in $\mathbf{R}^N$. Assume that $1 < p < \infty$ and $\alpha \geqslant 1-1/p$. Then for arbitrary $\eta \in (0,\sup_{x\in\Omega}\delta(x))$ we have*

$$\inf\left\{\int_{\Omega}|\nabla u|^p\delta^{\alpha p} : u \in C^1_c(\Omega), u = 1 \text{ on } \{\delta(x) = \eta\}\right\} = 0. \tag{2.16}$$

The proof will be given in Section 6. From this it is worthy to remark that Hardy's inequality (1.7) never holds in the critical case.

3. Proofs of Theorem 2.1 and Theorem 2.2

3.1. Auxiliary inequalities in the noncritical case

When $p=2$ and $\alpha=0$, the first lemma is established in [5].

LEMMA 3.1. (Noncritical case) *Assume that $f\in C([0,1])\cap C^1((0,1])$ is a monotone nondecreasing function such that $f(1)\leqslant 1$. Assume that $\alpha<1-1/p$ and $1<p<\infty$. Then for every $u\in C_c^\infty((0,1])$, we have*

$$\int_0^1\left(|u'|^p-\Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p\right)t^{\alpha p}\,dt\geqslant\int_0^1\left(|u'|^p-\Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p\right)t^{\alpha p}f\,dt. \tag{3.1}$$

In particular we have

$$\int_0^1|u'|^p t^{\alpha p}\,dt\geqslant\Lambda_{\alpha,p}\int_0^1\left|\frac{u}{t}\right|^p t^{\alpha p}\,dt. \tag{3.2}$$

Proof of Lemma 3.1. Without loss of generality we assume that $f\geqslant 0$, $f(1)=1$, and $u\geqslant 0$. Define $g=1-f$. Then $g\geqslant 0$ and $g'\leqslant 0$. By integration by parts we have

$$\begin{aligned}(1-\alpha-1/p)\int_0^1 u^p t^{p(\alpha-1)}g\,dt&=-1/p\left[u^p t^{p(\alpha-1)+1}g\right]_0^1\\&\quad+1/p\int_0^1 u^p t^{p(\alpha-1)+1}g'\,dt+\int_0^1 u^{p-1}u't^{p(\alpha-1)+1}g\,dt.\end{aligned}$$

Since $g'=-f'\leqslant 0$ and $g\geqslant 0$,

$$(1-\alpha-1/p)\int_0^1 u^p t^{p(\alpha-1)}g\,dt\leqslant\int_0^1 u^{p-1}u't^{p(\alpha-1)+1}g\,dt.$$

By Hölder's inequality, noting that $p(\alpha-1)+1=(p-1)(\alpha-1)+\alpha$, we have

$$(1-\alpha-1/p)\left(\int_0^1 u^p t^{p(\alpha-1)}g\,dt\right)^{1/p}\leqslant\left(\int_0^1|u'|^p t^{p\alpha}g\,dt\right)^{1/p}.$$

Using $g=1-f$ and the definition of $\Lambda_{\alpha,p}$, we have

$$\int_0^1(|u'|^p-\Lambda_{\alpha,p}(u/t)^p)t^{\alpha p}\,dt\geqslant\int_0^1(|u'|^p-\Lambda_{\alpha,p}(u/t)^p)t^{\alpha p}f\,dt.\quad\square$$

LEMMA 3.2. (Noncritical case) *Assume that $\alpha<1-1/p$, $1<p<\infty$ and $R>e$. Then, there exist positive numbers $C_3=C_3(\alpha,p,R)$ and $L=L(\alpha,p,R)$ such that for every $u\in C_c^\infty((0,1])$, we have*

$$\int_0^1\left(|u'|^p-\Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p\right)t^{\alpha p}\,dt\geqslant C_3\int_0^1\left|\frac{u}{t}\right|^p t^{\alpha p}\frac{1}{A_1(t)^2}\,dt+L|u(1)|^p. \tag{3.3}$$

The proofs of Lemma 3.2 together with Lemma 3.4 will be given in §6. It follows from Lenma 3.1 and Lemma 3.2 that we have

LEMMA 3.3. (Noncritical case) *Assume that* $\alpha < 1-1/p$, $1<p<\infty$ *and* $R>e$. *There exist positive numbers* $C_4 = C_4(\alpha,p,R)$ *and* $L = L(\alpha,p,R)$ *such that for every* $u \in C_c^\infty((0,1])$, *we have*

$$\int_0^1 \left(|u'|^p - \Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p\right) t^{\alpha p}\,dt \geqslant C_4 \int_0^1 \left(|u'|^p + \Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p\right) \frac{t^{\alpha p}}{A_1^2}\,dt + L|u(1)|^p. \quad (3.4)$$

In particular C_4 *is given by* $C_4 = C_3/(1+2\Lambda_{\alpha,p})$.

Proof of Lemma 3.3. We use Lemma 3.1 for $f = C_3 A_1^{-2}$ with C_3 being small. Multiplying (3.3) by $2\Lambda_{\alpha,p}$ and adding it to (3.1), we have (3.4) for $C_4 = C_3/(1+2\Lambda_{\alpha,p})$. □

3.2. Proof of Theorem 2.1

By adding

$$-C_0 \int_0^1 \left|\frac{u}{t}\right|^p \frac{1}{A_1(t)^2} t^{\alpha p}\,dt$$

to the both side of (3.4) we have

$$\begin{aligned}\int_0^1 \left(|u'|^p - \left|\frac{u}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t)^2}\right)\right) t^{\alpha p}\,dt \\ \geqslant C_4 \int_0^1 \left(|u'|^p + \left|\frac{u}{t}\right|^p \left(\Lambda_{\alpha,p} - \frac{C_0}{C_4}\right)\right) \frac{1}{A_1(t)^2} t^{\alpha p}\,dt + L|u(1)|^p.\end{aligned} \quad (3.5)$$

Now we set $C_0 = \Lambda_{\alpha,p} C_4/3$, $C_1' = C_4/3$. Assuming that $C_0 \leqslant \Lambda_{\alpha,p}(\log R)^2$, we have $C_0/A_1(t)^2 \leqslant C_0/(\log R)^2 \leqslant \Lambda_{\alpha,p}$, and hence

$$\Lambda_{\alpha,p} - \frac{C_0}{C_4} = \frac{2}{3}\Lambda_{\alpha,p} \geqslant \frac{1}{3}\left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t)^2}\right).$$

Then we have

$$\begin{aligned}\int_0^1 \left(|u'|^p - \left|\frac{u}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t)^2}\right)\right) t^{\alpha p}\,dt \\ \geqslant C_1' \int_0^1 \left(|u'|^p + \left|\frac{u}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t)^2}\right)\right) \frac{1}{A_1(t)^2} t^{\alpha p}\,dt + L|u(1)|^p.\end{aligned}$$

By a calculation we see that $tA_1^2 \leqslant 4R/e^2$ $(t \in [0,1])$. Thus the desired inequality holds for $C_1 = C_1' e^2/(4R)$. □

3.3. Auxiliary inequalities in the critical case

The following lemma will be established in Section 6 together with Lemma 3.2.

LEMMA 3.4. (Critical case)

1. *Assume that* $\alpha > 1-1/p$, $1<p<\infty$ *and* $R>e$. *There exist positive numbers* $C_5 = C_5(\alpha,p,R)$ *and* $L = L(\alpha,p,R)$ *such that for every* $u \in C_c^\infty((0,1])$, *we have*

$$\int_0^1 \left(|u'|^p - \Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p\right) t^{\alpha p}\,dt + L|u(1)|^p \geqslant C_5 \int_0^1 \left|\frac{u}{t}\right|^p \frac{t^{\alpha p}}{A_1^2}\,dt. \tag{3.6}$$

2. *Assume that* $\alpha = 1-1/p$, $1<p<\infty$ *and* $R>e$. *There exist positive numbers* $C_5 = C_5(\alpha,p,R)$ *and* $L = L(\alpha,p,R)$ *such that for every* $u \in C_c^\infty((0,1])$, *we have*

$$\int_0^1 \left(|u'|^p - \Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p \frac{1}{A_1^p}\right) t^{p-1}\,dt + L|u(1)|^p \geqslant C_5 \int_0^1 \left|\frac{u}{t}\right|^p \frac{t^{p-1}}{A_1^p A_2^2}\,dt. \tag{3.7}$$

LEMMA 3.5. (Critical case)

1. *Assume that* $\alpha > 1-1/p$, $1<p<\infty$ *and* $R>e$. *There exist positive numbers* $C_6 = C_6(\alpha,p,R)$ *and* $L = L(\alpha,p,R)$ *such that for every* $u \in C_c^\infty((0,1])$, *we have*

$$\int_0^1 \left(|u'|^p - \Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p\right) t^{\alpha p}\,dt + L|u(1)|^p \geqslant C_6 \int_0^1 |u'|^p \frac{t^{\alpha p}}{A_1^2}\,dt. \tag{3.8}$$

2. *Assume that* $\alpha = 1-1/p$, $1<p<\infty$ *and* $R>e^e$. *There exist positive numbers* $C_6 = C_6(\alpha,p,R)$ *and* $L = L(\alpha,p,R)$ *such that for every* $u \in C_c^\infty((0,1])$, *we have*

$$\int_0^1 \left(|u'|^p - \Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p \frac{1}{A_1^p}\right) t^{p-1}\,dt + L|u(1)|^p \geqslant C_6 \int_0^1 |u'|^p \frac{t^{p-1}}{A_2^2}\,dt. \tag{3.9}$$

Proof of Lemma 3.5. Admitting Lemma 3.4 for the moment, we prove Lemma 3.5. Unfortunately we can not employ a counterpart of Lemma 3.1, hence we use a direct argument. We establish (3.9) (the assertion 2) only because the argument for (3.8) is quite similar. We prepare the following fundamental inequalities which are established in [1] as Lemma 2.1 for $X > -1$.

LEMMA 3.6. 1. *For* $p \geqslant 2$ *we have*

$$|1+X|^p - 1 - pX \geqslant c(p)|X|^q, \quad \text{for any } q \in [2,p] \text{ and } X \in \mathbf{R}. \tag{3.10}$$

2. *For* $1 < p \leqslant 2$ *and* $M \geqslant 1$, *we have*

$$|1+X|^p - 1 - pX \geqslant c(p)\begin{cases} M^{p-2}X^2, & |X| \leqslant M, \\ |X|^p, & |X| \geqslant M. \end{cases} \tag{3.11}$$

Here $c(p)$ *is a positive number independent of each* X, $M \geqslant 1$ *and* $q \in [2,p]$.

Proof. By Taylar expansion we have (3.10) with $q = 2$. For $p > 1$, we note that

$$\lim_{X\to 0}\frac{|1+X|^p - 1 - pX}{X^2} = \frac{p(p-1)}{2}, \quad \lim_{|X|\to\infty}\frac{|1+X|^p - 1 - pX}{|X|^p} = 1. \tag{3.12}$$

Therefore (3.10) is valid for any $q \in [2, p]$ for a small $c(p) > 0$. If $X > -1$, then (3.11) also follows from Taylar expansion and (3.12). If we choose $c(p)$ sufficiently small, then it remains valid for $X \leqslant -1$. □

First we assume that $p \geqslant 2$ and $\alpha = 1 - 1/p$. For any $u \in C_c^1((0,1])$, let us set $u = A_1^\alpha h$, where $A_1(t) = \log(R/t)$ and $h \in C_c^1((0,1])$. Without a loss of generality we assume $u \geqslant 0$. Letting $X = -tA_1 h'/(\alpha h)\,(h \neq 0); 0\,(h = 0)$, we have

$$\begin{aligned} |u'|^p t^{p-1} - \Lambda_{\alpha,p} u^p \frac{t^{p-1}}{t^p A_1^p} &= \Lambda_{\alpha,p}\frac{h^p}{tA_1}\left(|1+X|^p - 1\right) \\ &\geqslant -(\Lambda_{\alpha,p})^{1-\frac{1}{p}}(h^p)' + c(p)|h'|^p(A_1 t)^{p-1}. \end{aligned} \tag{3.13}$$

Here we used (3.10) with $q = p$. On the other hand we have

$$\begin{aligned} |u'|^p \frac{t^{p-1}}{A_2(t)^2} &= \alpha^p \frac{h^p}{tA_1A_2^2}|1+X|^p \leqslant 2^p\alpha^p\frac{h^p}{tA_1A_2^2}\left(1+|X|^p\right) \\ &= 2^p\alpha^p\frac{u^p}{tA_1^pA_2^2} + 2^p|h'|^p(tA_1)^{p-1}\frac{1}{A_2^2}. \end{aligned} \tag{3.14}$$

Here we used a trivial inequality: $|1+X|^p \leqslant 2^p(1+|X|^p)$. By integrating (3.13) and (3.14) on $(0,1)$ and employing Lemma 3.4, the desired inequality follows for a sufficiently small constant $C_6 > 0$.

Secondly we assume that $1 < p < 2$. If $|X| \geqslant M$, then (3.14) is valid. If $|X| \leqslant M$, again from (3.14) we immediately have

$$|u'|^p\frac{t^{p-1}}{A_2(t)^2} \leqslant 2^p\alpha^p\left(1+M^p\right)\frac{h^p}{tA_1A_2^2} = C(M)\frac{u^p}{tA_1^pA_2^2} \qquad (|X| \leqslant M). \tag{3.15}$$

Thus we have

$$|u'|^p\frac{t^{p-1}}{A_2(t)^2} \leqslant C(M)\frac{u^p}{tA_1^pA_2^2} + 2^p\chi_{|X|\geqslant M}(t)|h'|^p(tA_1)^{p-1}\frac{1}{A_2^2}. \tag{3.16}$$

Here $\chi_S(t)$ is a characteristic function of S. We have (3.13) provided that $|X| \geqslant M$. Since $A_2^{-2} \leqslant 1$, for a sufficiently small C_6 the desired inequality (3.9) follows from (3.13), (3.16) and Lemma 3.4 (2). □

3.4. Proof of Theorem 2.2

It follows from (3.7) and (3.9) that we have

$$\int_0^1 \left(|u'|^p - \Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p \frac{1}{A_1^p}\right) t^{p-1}\,dt + L|u(1)|^p \geqslant C_7 \int_0^1 \left(|u'|^p + \Lambda_{\alpha,p}\left|\frac{u}{t}\right|^p \frac{1}{A_1^p}\right)\frac{t^{p-1}}{A_2^2}\,dt \tag{3.17}$$

for every $u \in C_c^\infty((0,1])$. Here $C_7 = \min(C_5, \Lambda_{\alpha,p}C_6)/2$. By adding

$$-C_0 \int_0^1 \left|\frac{u}{t}\right|^p \frac{1}{A_1(t)^p}\frac{1}{A_2(t)^2} t^{p-1}\,dt$$

to the both side of (3.9) we have

$$\int_0^1 \left(|u'|^p - \left|\frac{u}{t}\right|^p \frac{1}{A_1(t)^p}\left(\Lambda_{\alpha,p} + \frac{C_0}{A_2(t)^2}\right)\right) t^{p-1}\,dt + L|u(1)|^p \geqslant C_7 \int_0^1 \left(|u'|^p + \left|\frac{u}{t}\right|^p \frac{1}{A_1(t)^p}\left(\Lambda_{\alpha,p} - \frac{C_0}{C_7}\right)\right)\frac{1}{A_1(t)^2} t^{p-1}\,dt,$$

Now we set $C_0 = \Lambda_{\alpha,p}C_7/3$ and $C_1' = C_7/3$. Assuming that $C_0 \leqslant \Lambda_{\alpha,p}(\log\log R)^2$, we have $C_0/A_2(t)^2 \leqslant C_0/(\log\log R)^2 \leqslant \Lambda_{\alpha,p}$. Then

$$\int_0^1 \left(|u'|^p - \left|\frac{u}{t}\right|^p \frac{1}{A_1(t)^p}\left(\Lambda_{\alpha,p} + \frac{C_0}{A_2(t)^2}\right)\right) t^{p-1}\,dt + L|u(1)|^p \geqslant C_1' \int_0^1 \left(|u'|^p + \left|\frac{u}{t}\right|^p \frac{1}{A_1(t)^p}\left(\Lambda_{\alpha,p} + \frac{C_0}{A_2(t)^2}\right)\right)\frac{1}{A_1(t)^2} t^{p-1}\,dt.$$

By a calculation we see that for some $C(R) > 0$, $tA_2^2 \leqslant C(R)$ for any $t \in [0,1]$. Therefore the desired inequality holds for $C_1 = C_1' C(R)^{-1}$. □

4. Proofs of Theorems 2.3, 2.5 and Corollaries 2.3, 2.4

We first establish Theorem 2.3 using Theorem 2.1. Theorem 2.5 is proved in a quite similar way using Theorem 2.2. Then we prove Corollary 2.3 and Corollary 2.4.

Proofs of Theorem 2.3 *and Theorem* 2.5. Let us prepare some notations and fundamental facts. Define $\Sigma = \partial\Omega$ and $\Sigma_t = \{x \in \Omega : \delta(x) = t\}$. Since Σ is is of class C^2, there exists an $\eta_0 > 0$ such that we have a C^2 diffeomorphism $G : \Omega_\eta \mapsto (0,\eta)\times\Sigma$ for any $\eta \in (0,\eta_0)$. By $G^{-1}(t,\sigma)$ $((t,\sigma) \in (0,\eta_0)\times\Sigma)$ we denote the inverse of G. Let H_t denote the mapping $G^{-1}(t,\cdot)$ of Σ onto Σ_t. This mapping is also a C^2 diffeomorphism and its Jacobian is close to 1 in $(0,\eta_0)\times\Sigma$. Therefore, for every non-negative continuous function u on $\overline{\Omega_\eta}$ with $\eta \in (0,\eta_0)$ we have

$$\int_{\Omega_\eta} u = \int_0^\eta dt \int_{\Sigma_t} u\,d\sigma_t = \int_0^\eta dt \int_\Sigma u(t, H_t(\sigma))(\mathrm{Jac}\,\mathrm{H_t})\,d\sigma, \tag{4.1}$$

$$|\mathrm{Jac}\,\mathrm{H_t}(\sigma) - 1| \leqslant \mathrm{ct}, \quad \text{for every } (\mathrm{t},\sigma) \in (0,\eta_0)\times\Sigma, \tag{4.2}$$

where c is a positive constant independent of each (t,σ), $d\sigma$ and $d\sigma_t$ denote surface elements on Σ and Σ_t respectively. Further we have

$$\int_0^\eta dt \int_\Sigma u(t,H_t(\sigma))(1-ct)\,d\sigma \leqslant \int_{\Omega_\eta} u \leqslant \int_0^\eta dt \int_\Sigma u(t,H_t(\sigma))(1+ct)\,d\sigma, \tag{4.3}$$

$$\int_\Sigma u(\eta,H_\eta(\sigma))(1-c\eta)\,d\sigma \leqslant \int_{\Sigma_\eta} u\,d\sigma_\eta \leqslant \int_\Sigma u(\eta,H_\eta(\sigma))(1+c\eta)\,d\sigma. \tag{4.4}$$

Then we immediately have

$$\int_\Sigma d\sigma \int_0^\eta \left|\frac{\partial u}{\partial t}\right|^p (1-ct)t^{\alpha p}\,dt \leqslant \int_{\Omega_\eta} |\nabla u|^p \delta^{\alpha p}$$

$$\int_\Sigma d\sigma \int_0^\eta \left|\frac{u}{t}\right|^p (1-ct)t^{\alpha p}\,dt \leqslant \int_{\Omega_\eta} |u|^p \delta^{p(\alpha-1)} \leqslant \int_\Sigma d\sigma \int_0^\eta \left|\frac{u}{t}\right|^p (1+ct)t^{\alpha p}\,dt.$$

Proof of (2.10). Under these consideration, (2.10) is reduced to one-dimensional Hardy's inequality. Setting $v(t) = u(t,\sigma)$ and $v' = \partial u/\partial t$ we have

$$\begin{aligned}\int_0^\eta &\left(|v'|^p - \left|\frac{v}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_2}{A_1(t)^2}\right)\right) t^{\alpha p}\,dt \\ &\geqslant c\int_0^\eta \left(|v'|^p + \left|\frac{v}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_2}{A_1(t)^2}\right)\right) t^{\alpha p+1}\,dt + L|v(\eta)|^p \eta^{\alpha p}(1+c\eta).\end{aligned} \tag{4.5}$$

By a change of variable $t = s\eta$, putting $w(s) = v(s\eta)$ with $v \in C_c^1((0,1])$,

$$\begin{aligned}&\int_0^1 \left(|w'|^p - \left|\frac{w}{s}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_2}{A_1(s\eta)^2}\right)\right) s^{\alpha p}\,ds \\ &\geqslant c\eta \int_0^1 \left(|w'|^p + \left|\frac{w}{s}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_2}{A_1(s\eta)^2}\right)\right) s^{\alpha p+1}\,ds + L|w(1)|^p \eta^{p-1}(1+c\eta).\end{aligned} \tag{4.6}$$

On the other hand, by Theorem 2.1 with R changed to R/η, we have, for every $w \in C_c^1((0,1])$,

$$\begin{aligned}\int_0^1 &\left(|w'|^p - \left|\frac{w}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t\eta)^2}\right)\right) t^{\alpha p}\,dt \\ &\geqslant C_1 \int_0^1 \left(|w'|^p + \left|\frac{w}{t}\right|^p \left(\Lambda_{\alpha,p} + \frac{C_0}{A_1(t\eta)^2}\right)\right) t^{\alpha p+1}\,dt + L|w(1)|^p,\end{aligned} \tag{4.7}$$

where C_0 and C_1 may depend on η but independent of each function v. Now we take η and C_2 so that $C_1/\eta > c$, $C_2 \leqslant C_0$ and $\eta^{p-1}(1+c\eta) < 1$ respectively. Since w is an arbitrary function in $C_c^1((0,\eta])$, we get (2.10).

Proof of (2.12) *and* (2.13). In parallel to the verification of (2.10), (2.12) and (2.13) can be proved using (2.4) and (2.5) together with (4.4). Hence we omit the detail.

Proof of Corollary 2.3. Assume on the contrary that Hardy inequality (2.11) does not hold. Then there exists a sequence of functions $\{u_k\} \subset W^{1,p}_{\alpha,0}(\Omega)$ such that

$$\lim_{k\to\infty} \int_\Omega |\nabla u_k|^p \delta^{\alpha p}\, dx = 0, \quad \int_\Omega |u_k|^p \delta^{p(\alpha-1)}\, dx = 1 \quad (k = 1,2,\cdots). \tag{4.8}$$

By Theorem 2.3 we have

$$\begin{aligned}
&\int_\Omega |\nabla u_k|^p \delta^{\alpha p}\, dx = \int_{\Omega_\eta} |\nabla u_k|^p \delta^{\alpha p}\, dx + \int_{\Omega\setminus\Omega_\eta} |\nabla u_k|^p \delta^{\alpha p}\, dx \\
&\geqslant \Lambda_{\alpha,p}\left(1 - \int_{\Omega\setminus\Omega_\eta} |u_k|^p \delta^{p(\alpha-1)}\, dx\right) + L\int_{\Sigma_\eta} |u_k|^p \delta^{\alpha p} + \int_{\Omega\setminus\Omega_\eta} |\nabla u_k|^p \delta^{\alpha p}\, dx.
\end{aligned}$$

Since $\delta \geqslant \eta$ in $\Omega\setminus\Omega_\eta$, by the standard argument we have $u_k \to C(constant)$ in $W^{1,p}(\Omega\setminus\Omega_\eta)$ as $k\to\infty$. Since $L>0$, we have $C=0$. Hence we see $0 \geqslant \Lambda_{\alpha,p}$, and we reach to a contradiction. □

Proof of Corollary 2.4. Assume on the contrary that Hardy inequality (2.14) does not hold. Then there exists a sequence of functions $\{u_k\} \subset W^{1,p}_{\alpha,0}(\Omega)$ such that

$$\begin{cases} \lim_{k\to\infty} \int_\Omega |\nabla u_k|^p \delta^{\alpha p}\, dx = 0, \quad \lim_{k\to\infty} \int_{\Sigma_\eta} |u|^p \delta^{\alpha p} = 0, \\ \int_\Omega |u_k|^p \delta^{p(\alpha-1)}\, dx = 1 \quad (k=1,2,\cdots). \end{cases} \tag{4.9}$$

By Theorem 2.5 we have

$$\begin{aligned}
&\int_\Omega |\nabla u_k|^p \delta^{\alpha p}\, dx + L\int_{\Sigma_\eta} |u_k|^p \delta^{\alpha p} \\
&\geqslant \Lambda_{\alpha,p}\left(1 - \int_{\Omega\setminus\Omega_\eta} |u_k|^p \delta^{p(\alpha-1)}\, dx\right) + \int_{\Omega\setminus\Omega_\eta} |\nabla u_k|^p \delta^{\alpha p}\, dx.
\end{aligned}$$

Since $\delta \geqslant \eta$ in $\Omega\setminus\Omega_\eta$, as before we have $u_k \to 0$ in $W^{1,p}(\Omega\setminus\Omega_\eta)$ as $k\to\infty$. Hence we have $0 \geqslant \Lambda_{\alpha,p}$, and we get a contradiction. □

5. Proof of Theorem 2.4

It suffices to show the implication $1 \to 2$. Since $A_1(\delta)^{-1} \leqslant 1$ in Ω and the trace operator $T: W^{1,p}_{0,\alpha}(\Omega^c_\eta) \mapsto L^p(\Sigma_\eta;\delta^{\alpha p})$ is continuous for a small $\eta > 0$, one can assume that $C_2 = L = 0$. Now, we assume on the contrary that there exists a sequence of functions $\{u_k\} \subset W^{1,p}_{\alpha,0}(\Omega)$ such that

$$\lim_{k\to\infty} \int_{\Omega_\eta} |\nabla u_k|^p \delta^{\alpha p}\, dx = 0, \quad \int_{\Omega_\eta} |u_k|^p \delta^{p(\alpha-1)}\, dx = 1 \quad (k=1,2,\cdots). \tag{5.1}$$

Here we prepare a lemma on extension:

LEMMA 5.1. (Extension) *For any* $\eta > 0$ *there exists an extension operator* $E = E(\eta) : W^{1,p}_{\alpha,0}(\Omega_\eta) \mapsto W^{1,p}_{\alpha,0}(\Omega)$ *such that:*

1. $E(u) = u$ *a.e. in* Ω_η

2. *There exists some positive number* $C = C(\eta)$ *such that for any* $u \in W^{1,p}_{\alpha,0}(\Omega_\eta)$,

$$|||\nabla E(u)|||_{L^p(\Omega,\delta^{\alpha p})} \leqslant C\Big(|||\nabla u|||_{L^p(\Omega_{\eta/2},\delta^{\alpha p})} + ||u||_{W^{1,p}_{\alpha,0}(\Omega_\eta\setminus\Omega_{\eta/2})}\Big).$$

Admitting this for the moment, we prove Theorem 2.4. Let $v_k = E(u_k) \in W^{1,p}_{\alpha,0}(\Omega)$ for $k = 1,2,\ldots$. It follows from (3.15), the assumption 1 and the property of E that v_k becomes a Cauchy sequence and $v_k \to v$ in $W^{1,p}_{\alpha,0}(\Omega)$ for some $v \in W^{1,p}_{\alpha,0}(\Omega)$ as $k \to \infty$. On the other hand by choosing a subsequence if necessary, we see that $u_k \to c$ a.e. in Ω_η for some constant c as $k \to \infty$. Then, by the assumption 1

$$\begin{aligned}1 &\leqslant \int_\Omega |E(u_k)|^p \delta^{p(\alpha-1)} \leqslant \gamma^{-1}\int_\Omega |\nabla E(u_k)|^p \delta^{\alpha p}\\ &\leqslant \gamma^{-1} C\Big(|||\nabla u_k|^p||_{L^p(\Omega_{\eta/2},\delta^{\alpha p})} + ||u_k||_{W^{1,p}_{\alpha,0}(\Omega_\eta\setminus\Omega_{\eta/2})}\Big) < \infty.\end{aligned}$$

Since $(\alpha-1)p < -1$, we have $c = 0$. Thus $u_k \to 0$ in $L^p(\Omega_\eta \setminus \Omega_{\eta/2})$. Thus we see that $||u_k||_{W^{1,p}_{\alpha,0}(\Omega_\eta\setminus\Omega_{\eta/2})} \to 0$ as $k \to \infty$. From this together with (2.10) we have a contradiction. □

Proof of Lemma 5.1. Since δ is Lipschitz continuous, we see that $\partial\Omega_\eta$ and $\partial\Omega_{\eta/2}$ are Lipschitz compact manifolds. By the standard theory we have an extension operator $\tilde{E} : W^{1,p}(\Omega_\eta \setminus \Omega_{\eta/2}) \mapsto W^{1,p}(\Omega\setminus\Omega_{\eta/2})$ such that $\tilde{E}(u) = u$ a.e. in $\Omega_\eta \setminus \Omega_{\eta/2}$, and

$$|||\nabla\tilde{E}(u)|^p||_{L^p(\Omega\setminus\Omega_{\eta/2},\delta^{\alpha p})} \leqslant C(\eta)||u||_{W^{1,p}_{\alpha,0}(\Omega_\eta\setminus\Omega_{\eta/2})}.$$

Define for $u \in W^{1,p}_{\alpha,0}(\Omega_\eta)$

$$E(u) = u\ (x \in \Omega_{\eta/2},); \quad \tilde{E}(u)\ (x \in \Omega\setminus\Omega_{\eta/2}). \tag{5.2}$$

Then the assertion follows. □

6. Proofs of Propositions 2.1 and 2.2

Proposition 2.1 is known in a more general fashion. In fact a variant is seen in Maz'ya [14] (Lemma 2, p144). For the sake of reader's convenience we give a direct verification. We note that Proposition 2.2 is a consequence of Proposition 2.1.

Proof of Proposition 2.1. First we assume that $\alpha > 1 - \frac{1}{p}$. Define for $\varepsilon \in (0,1)$, $u_\varepsilon = t/\varepsilon\ (0 \leqslant t \leqslant \varepsilon);\ 1\ (t \geqslant \varepsilon)$. Then we immediately have $u_\varepsilon(0) = 0, u_\varepsilon(1) = 1$ and

$\int_0^1 |u'_\varepsilon|^p t^{\alpha p}\, dt \to 0$ as $\varepsilon \downarrow 0$. By using C^1 approximation of each u_ε, the assertion is proved. Further we note that $\int_0^1 |u_\varepsilon|^p t^{(\alpha-1)p}\, dt \to 1/(\alpha p - p + 1) > 0$ as $\varepsilon \downarrow 0$. In the critical case, define for $\varepsilon \in (0, 1/2)$

$$u_\varepsilon = 0\,(0 \leqslant t \leqslant \varepsilon); \quad \frac{A_1(\varepsilon) - A_1(t)}{A_1(\varepsilon) - A_1(1/2)}\,(\varepsilon \leqslant t \leqslant 1/2); \quad 1\,(1/2 \leqslant t \leqslant 1). \tag{6.1}$$

Then $\int_0^1 |u'_\varepsilon|^p t^{p-1}\, dt = (A_1(\varepsilon) - A_1(1/2))^{1-p} \to 0$ as $\varepsilon \downarrow 0$. On the other hand we have $u_\varepsilon(0) = 0, u_\varepsilon(1) = 1$ and hence the assertion is now clear. Further we note that

$$\int_\varepsilon^1 |u_\varepsilon|^p \frac{1}{tA_1(t)^p}\, dt \geqslant \frac{A_1(1)^{1-p} - A_1(1/2)^{1-p}}{p-1} > 0 \quad \text{as } \varepsilon \downarrow 0. \quad \square$$

Proof of Proposition 2.2. We give a proof when $\alpha > 1 - \frac{1}{p}$, because the argument is quite similar in the rest of the case. If a positive number η_0 is sufficiently small, then one can assume that $\delta \in C^2(\Omega_{\eta_0})$ and a manifolds $\{x \in \Omega; \delta = \eta\}$ is of C^2 class for $\eta \in (0, \eta_0]$. By virtue of (4.3) we have

$$\int_{\Omega_\eta} |u|^p \delta^{p(\alpha-1)} \leqslant \int_\Sigma d\sigma \int_0^\eta \left|\frac{u}{t}\right|^p (1 + ct) t^{\alpha p}\, dt,$$

hence the assertion follows from Proposition 2.1. $\square$

7. Appendix; Proofs of Lemma 3.2 and Lemma 3.4

7.1. Preliminary

In this section we prepare a series of one dimensional weighted Hardy's inequalities. The followings are given in [2] as Lemma 3.1 and Lemma 3.4 respectively.

LEMMA 7.1. *Assume that $R > e$. Then, for any $h \in C_c^1((0,1])$ we have*

$$\int_0^1 |h'(t)|^2 t dt \geqslant \frac{1}{4} \int_0^1 |h(t)|^2 A_1(t)^{-2} \frac{dt}{t} - \frac{1}{2} A_1(1)^{-1} h(1)^2. \tag{7.1}$$

Proof. Let $h(t) = A_1(t)^{\frac{1}{2}} w(t)$. Then we have

$$|h'(t)|^2 t = \frac{t}{A_1(t)} \left(-\frac{1}{2t} w(t) + w'(r) A_1(t) \right)^2 \geqslant \frac{|h(t)|^2}{4tA_1(t)^2} - \frac{1}{2} \left(\frac{d}{dt} w^2(t) \right). \tag{7.2}$$

Since $w(0) = 0$, we have (7.1) and the rest of the proof is clear. $\square$

LEMMA 7.2. *Assume that $R > e^e$. Then, for any $h \in C_c^1((0,1])$ we have*

$$\int_0^1 |h'(t)|^2 t A_1(t)\, dr \geqslant \frac{1}{4} \int_0^1 \frac{|h(t)|^2}{A_1(t) \cdot A_2(t)^2} \frac{dt}{t} - \frac{1}{2} A_2(1)^{-1} h(1)^2. \tag{7.3}$$

Proof. For $h(t) = A_2(t)^{\frac{1}{2}} w(t)$, we have in a similar way

$$|h'(t)|^2 t A_1(t) \geqslant \frac{1}{4} \frac{|h(t)|^2}{A_1(t) A_2(t)^2} - \frac{1}{2} \left(\frac{d}{dt} w^2(t) \right). \tag{7.4}$$

Then the rest of the proof is clear. □

DEFINITION 7.1. A function $\varphi \in C^1([0,1])$ is said to belong to $G([0,1])$ if and only if

$$\varphi(0) = 0, \quad \varphi'(0) \neq 0 \quad \text{and} \quad \varphi'(1) = 0. \tag{7.5}$$

DEFINITION 7.2. For $\varphi \in G([0,1])$ and $M > 1$ we define three subsets of $[0,1]$ as follows:

$$\begin{cases} A(\varphi, M) = \left\{ t \in [0,1] \,\middle|\, |\varphi'(t)| \leqslant M \frac{|\varphi(t)|}{t} \right\}, \\ B(\varphi, M) = \left\{ t \in [0,1] \,\middle|\, |\varphi'(t)| > M \frac{|\varphi(t)|}{t} \right\}, \\ C(\varphi, M) = \left\{ t \in [0,1] \,\middle|\, |\varphi'(t)| = M \frac{|\varphi(t)|}{t} \right\}. \end{cases} \tag{7.6}$$

Clearly $[0,1] = A(\varphi, M) \cup B(\varphi, M)$. From (7.5) we see $0, 1 \in A(\varphi, M)$. We note that the set $C(\varphi, M)$ coincides with the set of critical points of $\log(|\varphi| r^{\pm M})$. By a standard argument we have the following approximation lemma (cf. Lemma 3.5 in [2]).

LEMMA 7.3. *Let $M > 1$ and $\varphi \in G([0,1]) \cap C^2([0,1])$. Assume that $\varphi \geqslant 0$. Then there exists a sequence of functions $\varphi_k \in G([0,1]) \cap C^2([0,1])$ such that $\varphi_k > 0$ in $(0,1)$, $\varphi_k \to \varphi$ in $C^1([0,1])$ as $k \to +\infty$ and $C(\varphi_k, M)$ consists of finite points for any k.*

We prepare some estimates for the proofs of Lemma 3.2 and Lemma 3.4.

LEMMA 7.4. *Assume that $1 < p < 2$ and $R > e$. Then for any $\varepsilon > 0$ there is a positive number M such that we have for any $\varphi \in G([0,1])$*

$$\int_{B(\varphi,M)} \frac{|\varphi||\varphi'|}{A_1(t)} dt \leqslant \varepsilon \int_{B(\varphi,M)} |\varphi|^{2-p} |\varphi'|^p t^{p-1} dt. \tag{7.7}$$

Proof. We may assume that $\varphi > 0$. Then by the definition we have $t|\varphi'|/\varphi > M$ on $B(\varphi, M)$. Hence we immediately have

$$\varphi^{2-p} |\varphi'|^p t^{p-1} = \varphi |\varphi'| \cdot \left(t \frac{|\varphi'|}{\varphi} \right)^{p-1} \geqslant M^{p-1} \varphi |\varphi'|, \qquad \text{on } B(\varphi, M). \tag{7.8}$$

Therefore it suffices to choose M so that $M^{1-p} (\log R)^{-1} \leqslant \varepsilon$. □

LEMMA 7.5. *Assume that* $1 < p < 2$ *and* $R > e$. *Then we have for any* $\varphi \in G([0,1])$

$$\int_{A(\varphi,M)} |\varphi'(t)|^2 t\,dt \geqslant \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{tA_1(t)^2}\,dt - \frac{1}{2}\frac{\varphi(1)^2}{A_1(1)} + \frac{1}{2}\int_{B(\varphi,M)} \frac{|\varphi|^2}{tA_1(t)^2}\,dt - \int_{B(\varphi,M)} \frac{|\varphi||\varphi'|}{A_1(t)}\,dt. \tag{7.9}$$

Proof. By Lemma 7.3 we can assume that $C(\varphi,M)$ consists of finitely many points. Recall that $0,1 \in A(\varphi,M)$. From the argument of Lemma 7.1 we have

$$\begin{aligned}\int_{A(\varphi,M)} |\varphi'(t)|^2 t\,dt &\geqslant \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{tA_1(t)^2}\,dt - \frac{1}{2}\int_{A(\varphi,M)} \frac{d}{dt}\left(\frac{\varphi(t)^2}{A_1(t)}\right)dt \\ &= \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{tA_1(t)^2}\,dt + \frac{1}{2}\int_{B(\varphi,M)} \frac{d}{dt}\left(\frac{\varphi(t)^2}{A_1(t)}\right)dt - \frac{1}{2}\frac{\varphi(1)^2}{A_1(1)}.\end{aligned} \tag{7.10}$$

Thus we have the desired estimate. □

In a quite similar way we have

LEMMA 7.6. *Assume that* $1 < p < 2$ *and* $R > e^e$. *Then we have for any* $\varphi \in G([0,1])$

$$\int_{A(\varphi,M)} |\varphi'(t)|^2 rA_1(t)\,dt \geqslant \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{tA_1(t)A_2(t)^2}\,dt - \frac{1}{2}\frac{\varphi(1)^2}{A_2(1)} + \frac{1}{2}\int_{B(\varphi,M)} \frac{|\varphi|^2}{tA_1(t)A_2(t)^2}\,dt - \int_{B(\varphi,M)} \frac{|\varphi||\varphi'|}{A_2(t)}\,dt. \tag{7.11}$$

7.2. Proof Lemma 3.2

Assume that $\alpha < 1 - 1/p$. For $u \in C_c^1((0,1])$, we define

$$u(t) = h(t)t^{\beta}, \quad \beta = 1 - \frac{1}{p} - \alpha, \quad (\beta^p = \Lambda_{\alpha,p}). \tag{7.12}$$

Without the loss of generarity we assume that $u \geqslant 0$ in $(0,1)$, then we have

$$\int_0^1 |u'|^p t^{\alpha p}\,dt - \Lambda_{\alpha,p}\int_0^1 \frac{|u|^p}{t^p} t^{\alpha p}\,dt = \Lambda_{\alpha,p}\int_0^1 h^p \left\{\left|1 + \frac{rh'}{\beta h}\right|^p - 1\right\}\frac{dt}{t}. \tag{7.13}$$

For the moment we assume $p \geqslant 2$. By the fundamental inequality (3.10) with $q = 2$, we obtain

$$\begin{aligned}\text{(R.H.S.) of (7.13)} &\geqslant \frac{\Lambda_{\alpha,p}}{\beta}\int_0^1 ph^{p-1}h'\,dt + c(p)\frac{\Lambda_{\alpha,p}}{\beta^2}\int_0^1 h^{p-2}(h')^2 t\,dt \\ &= \beta^{p-1}h(1)^p + c(p)\beta^{p-2}\frac{4}{p^2}\int_0^1 \left|\left(h^{\frac{p}{2}}(t)\right)'\right|^2 t\,dt. \quad \text{(Note that } h(0) = 0.)\end{aligned} \tag{7.14}$$

Using Lemma 7.1 we get

$$\int_0^1 \left|\left(h^{\frac{p}{2}}(t)\right)'\right|^2 tdt \geqslant \frac{1}{4}\int_0^1 \frac{|u(t)|^p}{t^p} A_1(r)^{-2} t^{\alpha p} dt - \frac{1}{2} A_1(1)^{-1} h(1)^p.$$

Combining this with (7.14), we get the inequality (3.3) with $C_3 = c(p)\beta^{p-2}/p^2$ and $L = \beta^{p-1} - 2C_3 A_1(1)^{-1}$, making $c(p)$ smaller if necessary.

We proceed to the case that $1 < p < 2$. For $u \in C_c^1((0,1])$, we retain the notation (7.12). Suppose that M is sufficiently large. In Definition 7.2 we replace φ and M by h and βM respectively, and assume that $h \in G([0,1])$ again. Lemma 3.6 (2) implies

$$\begin{aligned} &\int_0^1 |u'|^p t^{\alpha p} dt - \Lambda_{\alpha,p} \int_0^1 \frac{|v|^p}{t^p} t^{\alpha p} dt = \Lambda_{\alpha,p} \int_0^1 h^p(t) \left\{ \left|1 + \frac{th'}{\beta h}\right|^p - 1 \right\} \frac{dt}{t} \\ &\geqslant \beta^{p-1} h(1)^p + \frac{4c(p)(M\beta)^{p-2}}{p^2} \int_{A(h,\beta M)} ((h^{\frac{p}{2}})')^2 t\, dt + c(p) \int_{B(h,\beta M)} |h'|^p t^{p-1} dt. \end{aligned} \tag{7.15}$$

Using Lemma 7.5 with $A(h,\beta M) = A(h^{\frac{p}{2}}, p\beta M/2)$ and $B(h,\beta M) = B(h^{\frac{p}{2}}, p\beta M/2)$,

$$\begin{aligned} &\int_{A(h,\beta M)} ((h^{\frac{p}{2}})')^2 t\, dt \\ &\geqslant \frac{1}{4} \int_{A(h,\beta M)} \frac{h^p}{tA_1(t)^2} dt - \frac{1}{2}\frac{h(1)^p}{A_1(1)} + \frac{1}{2}\int_{B(h,\beta M)} \frac{h^p}{tA_1(t)^2} dt - \frac{p}{2}\int_{B(h,\beta M)} \frac{h^{p-1}|h'|}{A_1(t)} dt. \end{aligned} \tag{7.16}$$

We can estimate the last term to obtain

$$\frac{p}{2}\int_{B(h,\beta M)} \frac{h^{p-1}|h'|}{A_1(t)} dt \leqslant \frac{p}{2} \frac{1}{(\beta M)^{p-1}\log R} \int_{B(h,\beta M)} |h'|^p t^{p-1} dt. \tag{7.17}$$

Here we simply used the fact that $t|h'| > \beta M h$ holds on the set $B(h,\beta M)$. Combining this with (7.15) and (7.16) for sufficiently large M, we have the desired inequality. □

7.3. Proof of Lemma 3.4

We treat the case $\alpha = 1 - 1/p$ only, because the argument for $\alpha > 1 - 1/p$ is similar to the previous subsection. For $u \in C_c^1((0,1])$ we define

$$u(t) = A_1(t)^\beta h(t), \quad \beta = 1 - \frac{1}{p}. \tag{7.18}$$

Without loss of generality we assume $u, h \geqslant 0$. First we assume $p \geqslant 2$. Using (3.10) with $q = 2$ and $X = -\beta^{-1} t A_1(t) h'(t) h(t)^{-1}$, we obtain

$$\begin{aligned} &\int_0^1 |u'|^p t^{p-1} dt - \beta^p \int_0^1 \frac{|u(t)|^p}{tA_1(t)^p} dt \\ &= \beta^p \int_0^1 \frac{h(t)^p}{tA_1(t)} \left(\left|1 - \frac{tA_1(t)h'(t)}{\beta h(t)}\right|^p - 1 \right) dt \\ &\geqslant -\beta^{p-1} h(1)^p + \frac{4c(p)\beta^{p-2}}{p^2} \int_0^1 \left|\left(h(t)^{\frac{p}{2}}(t)\right)'\right|^2 tA_1(t)\, dt. \end{aligned} \tag{7.19}$$

Using Lemma 7.2 we get

$$\int_0^1 \left| \left(h(t)^{\frac{p}{2}} \right)' \right|^2 tA_1(t)\,dt \geqslant \frac{1}{4}\int_0^1 \frac{|u(t)|^p}{tA_1(t)^p A_2(t)^2}dt - \frac{1}{2}A_2^{-1}(1)h(1)^p. \tag{7.20}$$

Combining this with (7.19) we get the desired inequality where $C_5 = c(p)\beta^{p-2}p^{-2}$ and $L = A_1^{1-p}(\beta^{p-1} + 2C_5A_2^{-1})$. Then we proceed to the case that $1 < p < 2$. Since the argument is quite similar, we give a sketch of proof. Suppose that M is sufficiently large. We retain the notation (7.18) and modify Definition 7.2 as follows:

DEFINITION 7.3. For $\varphi \in G([0,1])$ and $M > 1$ we define three subsets of $[0,1]$ as follows:

$$\begin{cases} A(\varphi, M) = \left\{ t \in [0,1] \,\middle|\, |\varphi'(t)| \leqslant M\frac{|\varphi(t)|}{tA_1(t)} \right\}, \\ B(\varphi, M) = \left\{ t \in [0,1] \,\middle|\, |\varphi'(t)| > M\frac{|\varphi(t)|}{tA_1(t)} \right\}, \\ C(\varphi, M) = \left\{ t \in [0,1] \,\middle|\, |\varphi'(t)| = M\frac{|\varphi(t)|}{tA_1(t)} \right\}. \end{cases} \tag{7.21}$$

Again we replace φ and M by h and βM respectively and assume $h \geqslant 0$ in $(0,1)$. By Lemma 3.7, the first line of (7.19) is estimated from below by the following:

$$\begin{aligned} &-\beta^{p-1}\int_0^1 ph^{p-1}h'\,dt + \quad \beta^p \int_{A(h,\beta M)} h^p c(p) M^{p-2} \left(\frac{tA_1(t)h'}{\beta h} \right)^2 \frac{A_1(t)^{-1}}{t}dt \quad (7.22) \\ &+\beta^p \int_{B(h,\beta M)} h^p c(p) \left| \frac{tA_1(t)h'}{\beta h} \right|^p \frac{A_1(t)^{-1}}{t}dt \\ &= -\beta^{p-1}h(1)^p + c(p)(M\beta)^{p-2}\int_{A(h,\beta M)} h^{p-2}|h'|^2 tA_1(t)\,dt \\ &+ c(p)\int_{B(h,\beta M)} |h'|^p t^{p-1} A_1(t)^{p-1}\,dt. \end{aligned}$$

Here we note that $A(h,\beta M) = A(h^{\frac{p}{2}}, p\beta M/2)$ and $B(h,\beta M) = B(h^{\frac{p}{2}}, p\beta M/2)$. Then applying Lemma 7.6 for $\varphi = h^{\frac{p}{2}}$, $A(h^{\frac{p}{2}}, p\beta M/2)$ and $B(h^{\frac{p}{2}}, p\beta M/2)$ we have

$$\begin{aligned} \int_{A(h,\beta M)} h^{p-2}(h')^2 tA_1(t)\,dt &= \frac{4}{p^2}\int_{A(h,\beta M)} ((h^{\frac{p}{2}})')^2 tA_1(t)\,dt \qquad (7.23) \\ &\geqslant \frac{4}{p^2}\left(\frac{1}{4}\int_{A(h,\beta M)} \frac{h(t)^p}{tA_1(t)A_2(t)^2}dt - \frac{1}{2}A_2(1)^{-1}h(1)^p \right. \\ &\left. + \frac{1}{2}\int_{B(h,\beta M)} \frac{h(t)^p}{tA_1(t)A_2(t)^2}dt - \frac{p}{2}\int_{B(h,\beta M)} \frac{h(t)^{p-1}|h'(t)|}{A_2(t)}dt \right). \end{aligned}$$

From an easy variant of Lemma 7.4 we can estimate the last term to obtain

$$\frac{p}{2}\int_{B(h,\beta M)} \frac{h^{p-1}|h'|}{A_2(t)}dt \leqslant \frac{p}{2}\frac{1}{(\beta M)^{p-1}\log(\log R)}\int_{B(h,\beta M)} |h'|^p A_1(t)^{p-1}t^{p-1}\,dt. \tag{7.24}$$

Here we simply used the fact that $tA_1(t)|h'| > \beta M h$ holds on the set $B(h,\beta M)$. Combining this with (7.22) and (7.23), for a large M, we have the desired inequality. $\square$

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