Journal of Mathematical Inequalities
Volume 14, Number 4 (2020), 1223–1235
doi:10.7153/jmi-2020-14-79

SOME GENERALIZATIONS OF RETARDED NONLINEAR INTEGRAL INEQUALITIES AND ITS APPLICATIONS

ABDUL SHAKOOR, ILYAS ALI, SAMAD WALI AND ABDUR REHMAN

(*Communicated by J. Matkowski*)

Abstract. In this article, we state and prove several new retarded nonlinear integral inequalities for Gronwall-Bellman-Pachpatte type and these inequalities will be used as applications to study the boundedness, global existence, qualitative and quantitative behavior for the solutions of initial value problems of retarded nonlinear integro-differential equations. These inequalities extend some present inequalities to the current literature. An application is also presented to demonstrate the strength of our results.

1. Introduction

Throughout this article, the set of real numbers is denoted by $\mathbb{R}$, whereas $H = [0,\infty)$ is the subset of $\mathbb{R}$ and derivative is presented through $'$. Moreover, the sets of all continuous functions and continuously differentiable functions from H into H are denoted by $C(H,H)$ and $C'(H,H)$, respectively.

A significant role is played by differential and integral inequalities in the development of theory of differential and integral equations. Integral inequalities have significant applications in the study of existence, boundedness, quantitative and qualitative properties of solutions of nonlinear differential equations (such as [1–3] and references therein). Firstly, we will introduce Gronwall inequality that has many applications in the field of differential and integral equations.

THEOREM 1.1. (Gronwall inequality [4]) *Let $m : [\theta, \theta+k] \to \mathbb{R}$ be a continuous function and*

$$0 \leqslant m(r) \leqslant \int_{\theta}^{r} (bm(\lambda)+a)\,d\lambda, \quad \forall\, r \in [\theta, \theta+k], \tag{1.1}$$

where θ, k, a, and b are nonnegative constants. Then

$$0 \leqslant m(r) \leqslant ak \exp(bk), \quad \forall\, r \in [\theta, \theta+k]. \tag{1.2}$$

Since the establishment of above inequality, many mathematicians and scientists have shown their interest and gave many generalizations of Gronwall inequality (see for instance [5–12]).

The important generalization of above inequality is given below:

Mathematics subject classification (2010): 39B72, 26D10, 34A34.
Keywords and phrases: Integral inequality, retarded nonlinear integral inequality, retarded nonlinear integro-differential equations.

© Ele-Math, Zagreb
Paper JMI-14-79

THEOREM 1.2. (Gronwall-Bellman inequality [5]) *Let m and z be nonnegative continuous functions defined on the interval $E_1 = [0,k]$, and suppose m_0 and k are positive constants for which the inequality*

$$m(r) \leqslant m_0 + \int_0^r z(\lambda)m(\lambda)d\lambda, \quad \forall\, r \in E_1, \tag{1.3}$$

holds, then

$$m(r) \leqslant m_0 \exp\left(\int_0^r z(\lambda)d\lambda\right), \ \forall\, r \in E_1. \tag{1.4}$$

In [1], Pachpatte introduced the following linear integral inequality:

THEOREM 1.3. ([1]) *Let $m,h,q,p \in C(H,H)$ be nonnegative functions and m_0 be positive constant for which the inequality*

$$m(r) \leqslant m_0 + \int_0^r (h(\lambda)m(\lambda) + q(\lambda))d\lambda + \int_0^r h(\lambda)\left(\int_0^\lambda p(\mu)m(\mu)d\mu\right)d\lambda, \tag{1.5}$$

for all $r \in H$, holds. Then

$$m(r) \leqslant m_0 + \int_0^r q(\lambda)d\lambda + \int_0^r h(\lambda)\{m_0 \exp\left(\int_0^\lambda (h(\mu)+p(\mu))d\mu\right) + \int_0^\lambda q(\mu)\exp\left(\int_\mu^\lambda (h(\tau)+p(\tau))d\tau\right)d\mu\}d\lambda, \ \forall\, r \in H. \tag{1.6}$$

Pachpatte [1] established the following nonlinear integral inequality:

THEOREM 1.4. ([1]) *Let $m,h,q,p \in C(H,H)$ be nonnegative functions, and $\varphi \in C'(H,H)$ be nondecreasing function with $\varphi(r) > 0$ for all $r \in H$ and m_0 be a positive constant. If the inequality*

$$m(r) \leqslant m_0 + \int_0^r p(\lambda)\varphi(m(\lambda))d\lambda + \int_0^r h(\lambda)m(\lambda)d\lambda + \int_0^r h(\lambda)\left(\int_0^\lambda q(\mu)m(\mu)d\mu\right)d\lambda, \ \forall\, r \in H, \tag{1.7}$$

holds, then for $0 \leqslant r \leqslant R_1$,

$$m(r) \leqslant \alpha(r)\Psi^{-1}\left[\Psi(m_0) + \int_0^r p(\lambda)\varphi(\alpha(\lambda))d\lambda\right], \tag{1.8}$$

where

$$\alpha(r) = 1 + \int_0^r h(\lambda)\exp\left(\int_0^\lambda (h(\mu)+q(\mu))d\mu\right)d\lambda, \tag{1.9}$$

$$\Psi(y) = \int_{y_0}^y \frac{d\lambda}{\varphi(\lambda)}, \ \ y > 0, \ \ y_0 > 0, \tag{1.10}$$

Ψ^{-1} *is the inverse function of* Ψ, *and* $R_1 \in H$ *is chosen such that*

$$\Psi(m_0) + \int_0^r p(\lambda)\varphi(\alpha(\lambda))d\lambda \in Dom(\Psi^{-1}), \tag{1.11}$$

for all $r \in H$ *lying in the interval* $0 \leqslant r \leqslant R_1$.

The rest of this paper is organized as follows: Section 2 presents new retarded linear integral inequality with differentiable function instead of a constant function outside the integral sign for Gronwall-Bellman-Pachpatte type and some new retarded nonlinear integral inequalities for nonlinear function $\varphi(m(r))$ instead of linear function $m(r)$ while an application will be established in Section 3 to show the boundedness and global existence of the solution to integro-differential equation.

2. Main results

In this section, first we state and prove new retarded linear integral inequality with differentiable function instead of a constant function outside the integral sign for Gronwall-Bellman-Pachpatte type which will extend some existing results in [1, 5, 6]. Moreover, this inequality can used in analysis techniques of various problems in the field of retarded nonlinear differential and integral equations.

THEOREM 2.1. *Let* m, h, q, $p \in C(H,H)$ *be nonnegative functions and* l, $\theta \in C'(H,H)$ *be nondecreasing with* $l(r) \geqslant 1$, $\theta(r) \leqslant r$ *on* H. *If the inequality*

$$m(r) \leqslant l(r) + \int_0^{\theta(r)} (h(\lambda)m(\lambda) + q(\lambda))d\lambda + \int_0^{\theta(r)} h(\lambda)\left(\int_0^{\lambda} p(\mu)m(\mu)d\mu\right)d\lambda, \ \forall\, r \in H, \tag{2.1}$$

holds. Then

$$m(r) \leqslant l(r) + \int_0^{\theta(r)} q(\lambda)d\lambda + \int_0^{\theta(r)} h(\lambda)K(\theta^{-1}(\lambda))d\lambda, \quad \forall\, r \in H, \tag{2.2}$$

where

$$K(r) = l(0)\exp\left(\int_0^{\theta(r)} (h(\mu)+p(\mu))d\mu\right) + \int_0^{\theta(r)} \left(l'(\theta^{-1}(\lambda)) + q(\lambda)\right) \times \exp\left(\int_{\lambda}^{\theta(r)} (h(\mu)+p(\mu))d\mu\right)d\lambda, \quad \forall\, r \in H. \tag{2.3}$$

Proof. Let $M(r)$ be the right hand side of (2.1), then we have $M(0) = l(0)$, and

$$m(r) \leqslant M(r), \ \ m(\theta(r)) \leqslant M(\theta(r)) \leqslant M(r), \quad \forall\, r \in H. \tag{2.4}$$

Differentiating $M(r)$ with respect to r and using (2.4), we obtain

$$\begin{aligned} M'(r) &= l'(r)+\theta'(r)(h(\theta(r))m(\theta(r))+q(\theta(r)))+\theta'(r)h(\theta(r)) \\ &\quad\times\int_0^{\theta(r)} p(\lambda)m(\lambda)d\lambda \\ &\leqslant l'(r)+\theta'(r)q(\theta(r))+\theta'(r)h(\theta(r))(M(r) \\ &\quad+\int_0^{\theta(r)} p(\lambda)M(\lambda)d\lambda) \\ &\leqslant l'(r)+\theta'(r)q(\theta(r))+\theta'(r)h(\theta(r))N(r),\ \ \forall\, r\in H, \end{aligned} \tag{2.5}$$

where

$$N(r)=M(r)+\int_0^{\theta(r)} p(\lambda)M(\lambda)d\lambda,\ \ \forall\, r\in H. \tag{2.6}$$

Thus, we have $N(0)=M(0)=l(0)$, $M(r)\leqslant N(r)$ and $M(\theta(r))\leqslant N(\theta(r))\leqslant N(r)$. Now, differentiating (2.6) with respect to r and using (2.5), we get

$$\begin{aligned} N'(r) &= M'(r)+\theta'(r)p(\theta(r))M(\theta(r)) \\ &\leqslant l'(r)+\theta'(r)q(\theta(r))+\theta'(r)\left(h(\theta(r))+p(\theta(r))\right)N(r),\ \ \forall\, r\in H. \end{aligned}$$

The above inequality gives an estimation for $N(r)$ as follows

$$\begin{aligned} N(r) \leqslant l(0)\exp\left(\int_0^{\theta(r)}(h(\mu)+p(\mu))d\mu\right)+\int_0^{\theta(r)}\left(l'(\theta^{-1}(\lambda))+q(\lambda)\right) \\ \times\exp\left(\int_\lambda^{\theta(r)}(h(\mu)+p(\mu))d\mu\right)d\lambda,\ \ \forall\, r\in H. \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.5), we get

$$M'(r)\leqslant l'(r)+\theta'(r)q(\theta(r))+\theta'(r)h(\theta(r))K(r),\ \ \forall\, r\in H, \tag{2.8}$$

where $K(r)$ as defined in (2.3). After applying integration on inequality (2.8) from 0 to r, we obtain

$$M(r)\leqslant l(r)+\int_0^{\theta(r)} q(\lambda)d\lambda+\int_0^{\theta(r)} h(\lambda)K(\theta^{-1}(\lambda))d\lambda,\ \ \forall\, r\in H. \tag{2.9}$$

Substituting (2.9) into (2.4), we obtain the required inequality in (2.2). Proof is completed. □

REMARK 2.1. It is interesting to note that when we apply some additional conditions in Theorem 2.1, we get the following results:

1. When we put extra conditions $l(r)=m_0$ (a constant) and $\theta(r)=r$ in Theorem 2.1, then it is converted to the Theorem 1.3 [1].

2. If we take $l(r) = m_0$ (a constant), $\theta(r) = r$, $q(r) = 0$ and $p(r) = 0$ in Theorem 2.1, then Theorem 1.2 [5] becomes a corollary of our Theorem 2.1.

3. If we assume that $l(r) = m_0$ (a constant) in Theorem 2.1, then we obtain Theorem 2.1 in [6].

Now, we state and prove few new retarded nonlinear integral inequalities for nonlinear function $\varphi(m(r))$ instead of linear function $m(r)$ which yields another analysis technique to study explicit bounds, existence, qualitative and quantitative behavior for the solutions of retarded nonlinear differential and integral equations. These inequalities generalize certain former results in [1, 5].

Here we give the following theorem:

THEOREM 2.2. *Let m, h, q, $p \in C(H,H)$ be nonnegative functions and φ, l, $\theta \in C'(H,H)$ be increasing functions with $\theta(r) \leqslant r$, $l(r) \geqslant 1$, $\varphi(r) > 0$ for all $r \in H$. If the inequality*

$$m(r) \leqslant l(r) + \int_0^{\theta(r)} h(\lambda)m(\lambda)d\lambda + \int_0^{\theta(r)} h(\lambda)\left(\int_0^{\lambda} q(\mu)m(\mu)d\mu\right)d\lambda + \int_0^{\theta(r)} p(\lambda)\varphi(m(\lambda))d\lambda, \quad \forall\, r \in H, \tag{2.10}$$

holds, then

$$m(r) \leqslant \alpha(r)\Psi^{-1}\left(\Psi(l(0)) + \int_0^{\theta(r)} \left(l'(\theta^{-1}(\lambda)) + p(\lambda)\varphi(\alpha(\lambda))\right)d\lambda\right), \tag{2.11}$$

for all $r \in H$, where

$$\alpha(r) = 1 + \int_0^{\theta(r)} h(\lambda)\,\exp\left(\int_0^{\lambda}(h(\mu)+q(\mu))d\mu\right)d\lambda, \quad \forall\, r \in H, \tag{2.12}$$

$$\Psi(y) = \int_{y_0}^{y} \frac{d\lambda}{\varphi(\lambda)}, \quad y > 0, \quad y_0 > 0, \tag{2.13}$$

Ψ^{-1} is the inverse functions of Ψ and $R_1 \in H$ is chosen so that

$$\Psi(l(0)) + \int_0^{\theta(r)} \left(l'(\theta^{-1}(\lambda)) + p(\lambda)\varphi(\alpha(\lambda))\right)d\lambda \in Dom(\Psi^{-1}), \forall\, r \in H, \tag{2.14}$$

lying in the interval $0 \leqslant r \leqslant R_1$.

Proof. Define a function

$$N_1(r) = l(r) + \int_0^{\theta(r)} p(\lambda)\varphi(m(\lambda))d\lambda, \quad \forall\, r \in H, \tag{2.15}$$

then inequality (2.10) can be written as

$$m(r) \leqslant N_1(r) + \int_0^{\theta(r)} h(\lambda)m(\lambda)d\lambda + \int_0^{\theta(r)} h(\lambda)(\int_0^{\lambda} q(\mu)m(\mu)d\mu)d\lambda, \tag{2.16}$$

for all $r \in H$. If we let

$$m(r) \leqslant N_1(r)W(r), \quad \forall\, r \in H, \tag{2.17}$$

then

$$W(r) \leqslant 1 + \int_0^{\theta(r)} h(\lambda)W(\lambda)d\lambda + \int_0^{\theta(r)} h(\lambda)(\int_0^{\lambda} q(\mu)W(\mu)d\mu)d\lambda, \tag{2.18}$$

for all $r \in H$. Assume that $W_1(r)$ is equal to the right hand side of (2.18), then $W_1(0) = 1$, and

$$W(r) \leqslant W_1(r),\ W(\theta(r)) \leqslant W_1(\theta(r)) \leqslant W_1(r), \quad \forall\, r \in H. \tag{2.19}$$

Differentiating $W_1(r)$ with respect to r and using (2.19), we obtain

$$\begin{aligned} W_1'(r) &= \theta'(r)h(\theta(r))W(\theta(r)) + \theta'(r)h(\theta(r)) \int_0^{\theta(r)} q(\lambda)W(\lambda)d\lambda \\ &\leqslant \theta'(r)h(\theta(r))[W_1(r) + \int_0^{\theta(r)} q(\lambda)W_1(\lambda)d\lambda] \\ &\leqslant \theta'(r)h(\theta(r))W_2(r),\ \forall\, r \in H, \end{aligned} \tag{2.20}$$

where

$$W_2(r) = W_1(r) + \int_0^{\theta(r)} q(\lambda)W_1(\lambda)d\lambda,\ \forall\, r \in H, \tag{2.21}$$

which implies that $W_1(r) \leqslant W_2(r)$ and $W_2(0) = W_1(0) = 1$. Now, differentiating (2.21) with respect to r and using (2.20), we get

$$\begin{aligned} W_2'(r) &= W_1'(r) + \theta'(r)q(\theta(r))W_1(\theta(r)) \\ &\leqslant [\theta'(r)h(\theta(r)) + \theta'(r)q(\theta(r))]W_2(r), \quad \forall\, r \in H. \end{aligned} \tag{2.22}$$

The inequality (2.22) implies an estimation for $W_2(r)$ as

$$W_2(r) \leqslant \exp\left(\int_0^{\theta(r)} (h(\lambda) + q(\lambda))d\lambda\right),\ \forall\, r \in H. \tag{2.23}$$

Substituting (2.23) into (2.20), we have

$$W_1'(r) \leqslant \theta'(r)h(\theta(r))\exp\left(\int_0^{\theta(r)} (h(\lambda) + q(\lambda))d\lambda\right),\ \forall\, r \in H.$$

Integrating above inequality from 0 to r, we get

$$W_1(r) \leqslant 1 + \int_0^{\theta(r)} h(\lambda)\exp\left(\int_0^{\lambda} (h(\mu) + q(\mu))d\mu\right)d\lambda, \quad \forall\, r \in H.$$

Substituting above inequality into (2.19) then substituting (2.19) into (2.17), we get

$$m(r) \leqslant N_1(r)\alpha(r), \quad \forall\, r \in H, \tag{2.24}$$

where $\alpha(r)$ is defined in (2.12). Differentiating (2.15) and using (2.24), we obtain

$$\begin{aligned} N_1'(r) &= l'(r) + \theta'(r)p(\theta(r))\varphi(m(\theta(r))) \\ &\leqslant l'(r) + \theta'(r)p(\theta(r))\varphi(N_1(r)\alpha(r)) \\ &\leqslant l'(r) + \theta'(r)p(\theta(r))\varphi(N_1(r))\varphi(\alpha(r)), \;\; \forall\, r \in H, \end{aligned}$$

but $l(r) \geqslant 1$ and $\varphi(N_1(r)) \geqslant 1$, which implies that $\frac{l'(r)}{\varphi(N_1(r))} \leqslant l'(r)$, so

$$\frac{N_1'(r)}{\varphi(N_1(r))} \leqslant l'(r) + \theta'(r)p(\theta(r))\varphi(\alpha(r)), \;\; \forall\, r \in H.$$

By setting $r = \lambda$ in above inequality, integrating it from 0 to r and with the help of (2.13), we have

$$\Psi(N_1(r)) \leqslant \Psi(N_1(0)) + \int_0^{\theta(r)} \left(l'(\theta^{-1}(\lambda)) + p(\lambda)\varphi(\alpha(\lambda))\right) d\lambda, \; \forall\, r \in H,$$

or equivalently

$$N_1(r) \leqslant \Psi^{-1}\left(\Psi(l(0)) + \int_0^{\theta(r)} \left(l'(\theta^{-1}(\lambda)) + p(\lambda)\varphi(\alpha(\lambda))\right) d\lambda\right), \; \forall\, r \in H.$$

Substituting above inequality into (2.24), we get required inequality in (2.11). That completes the proof. □

REMARK 2.2. It is interesting to note that when we change the given assumptions in Theorem 2.2, we obtain the following inequalities:

1. If we assume $l(r) = m_0$ (a constant) and $\theta(r) = r$, then Theorem 1.4 [1] directly becomes corollary of our Theorem 2.2.
2. When we take $l(r) = m_0$ (a constant), $\theta(r) = r$, $h(r) = 0$, $q(r) = 0$, and $\varphi(m(r)) = m(r)$ then Theorem 2.2 reduce to Theorem 1.2 [5].

Here we present another retarded nonlinear integral inequality which help us to find the global existence and boundedness for the solutions of initial value problems of retarded nonlinear integro-differential equations.

THEOREM 2.3. *Let m, h, q, $p \in C(H,H)$ be nonnegative functions and φ, φ', θ, $l \in C'(H,H)$ be increasing functions with $\theta(r) \leqslant r$, $l(r) \geqslant 1$, $\varphi(r) > 0$, $\varphi'(r) \leqslant k$, for all $r \in H$, k is positive constant. If the inequality*

$$\begin{aligned} m(r) \leqslant l(r) &+ \int_0^{\theta(r)} \varphi(m(\lambda))[h(\lambda)\varphi(m(\lambda)) + q(\lambda)]d\lambda \\ &+ \int_0^{\theta(r)} \varphi(m(\lambda))h(\lambda)\left(\int_0^{\lambda} p(\mu)\varphi(m(\mu))d\mu\right) d\lambda, \end{aligned} \tag{2.25}$$

for all $r \in H$ *holds, then*

$$m(r) \leqslant \Psi^{-1}\left(\Psi(l(0)) + \int_0^{\theta(r)} \left(l(\theta^{-1}(\lambda)) + q(\lambda) + h(\lambda)\alpha_1(\theta^{-1}(\lambda))\right) d\lambda\right), \quad (2.26)$$

for all $r \in H$*, where*

$$\alpha_1(r) = \frac{\exp\left(\int_0^{\theta(r)} (kq(\lambda) + p(\lambda)) d\lambda\right)}{\varphi^{-1}(l(0)) - \int_0^{\theta(r)} \left(k(l'(\theta^{-1}(\lambda)) + h(\lambda)) \exp\left(\int_0^{\lambda} (kq(\mu) + p(\mu)) d\mu\right)\right) d\lambda}, \quad (2.27)$$

$$\Psi(y) = \int_{y_0}^{y} \frac{d\mu}{\varphi(\mu)}, \quad y > 0, \quad y_0 > 0, \quad (2.28)$$

and Ψ^{-1} *is the inverse of* Ψ*.*

Proof. Let $M_1(r)$ is equal to the right hand side of (2.25), then $M_1(0) = l(0)$, and

$$m(r) \leqslant M_1(r), \ m(\theta(r)) \leqslant M_1(\theta(r)) \leqslant M_1(r), \quad \forall\, r \in H. \quad (2.29)$$

Differentiating $M_1(r)$ with respect to r and using (2.29), we have

$$\begin{aligned} M_1'(r) &= l'(r) + \theta'(r)\varphi(m(\theta(r)))[h(\theta(r))\varphi(m(\theta(r))) + q(\theta(r))] \\ &\quad + \theta'(r)\varphi(m(\theta(r)))h(\theta(r)) \int_0^{\theta(r)} p(\lambda)\varphi(m(\lambda)) d\lambda \\ &\leqslant l'(r) + \theta'(r)\varphi(M_1(r))q(\theta(r)) + \theta'(r)\varphi(M_1(r))h(\theta(r))[\varphi(M_1(r)) \\ &\quad + \int_0^{\theta(r)} p(\lambda)\varphi(M_1(\lambda)) d\lambda \\ &\leqslant l'(r) + \theta'(r)\varphi(M_1(r))q(\theta(r)) + \theta'(r)\varphi(M_1(r))h(\theta(r))N_2(r), \end{aligned} \quad (2.30)$$

for all $r \in H$, where

$$N_2(r) = \varphi(M_1(r)) + \int_0^{\theta(r)} p(\lambda)\varphi(M_1(\lambda)) d\lambda, \ \forall\, r \in H, \quad (2.31)$$

thus we have $N_2(0) = \varphi(M_1(0)) = \varphi(l(0))$, and $\varphi(M_1(r)) \leqslant N_2(r)$. Now, differentiating (2.31) with respect to r and using (2.30), we obtain

$$\begin{aligned} N_2'(r) &= \varphi'(M_1(r))M_1'(r) + \theta'(r)p(\theta(r))\varphi(M_1(\theta(r))) \\ &\leqslant kl'(r) + [k\theta'(r)q(\theta(r)) + \theta'(r)p(\theta(r))]N_2(r) \\ &\quad + k\theta'(r)h(\theta(r))N_2^2(r), \quad \forall\, r \in H. \end{aligned}$$

As $l(r) \geqslant 1$, $N_2^2(r) \geqslant 1$ which implies that $\frac{l'(r)}{N_2^2(r)} \leqslant l'(r)$, so dividing above inequality by $N_2^2(r)$, then we have

$$\begin{aligned} N_2^{-2}(r)N_2'(r) &\leqslant kl'(r) + [k\theta'(r)q(\theta(r)) + \theta'(r)p(\theta(r))]N_2^{-1}(r) \\ &\quad + k\theta'(r)h(\theta(r)), \ \forall\, r \in H. \end{aligned} \quad (2.32)$$

If we let $N_2^{-1}(r) = X(r)$, $X(0) = N_2^{-1}(0) = \varphi^{-1}(l(0))$, and $N_2^{-2}(r)N_2'(r) = -X'(r)$, then inequality (2.32) can be written as

$$-X'(r) \leqslant kl'(r) + [k\theta'(r)q(\theta(r)) + \theta'(r)p(\theta(r))]X(r) + k\theta'(r)h(\theta(r))$$

or equivalently

$$X'(r) + [k\theta'(r)q(\theta(r)) + \theta'(r)p(\theta(r))]X(r) \geqslant -kl'(r) - k\theta'(r)h(\theta(r)),$$

for all $r \in H$. The above inequality implies an estimation for $X(r)$ as follows

$$X(r) \geqslant \frac{\varphi^{-1}(l(0)) - \int_0^{\theta(r)}[k(l'(\theta^{-1}(\lambda)) + h(\lambda))\exp(\int_0^{\lambda}(kq(\mu) + p(\mu))d\mu)]d\lambda}{\exp(\int_0^{\theta(r)}(k(q(\lambda)) + p(\lambda))d\lambda)},$$

for all $r \in H$. Thus $N_2(r) = X^{-1}(r) \leqslant \alpha_1(r)$, where $\alpha_1(r)$ is defined in (2.27). Substituting $N_2(r) \leqslant \alpha_1(r)$ in (2.30), we get

$$M_1'(r) \leqslant l'(r) + \theta'(r)\varphi(M_1(r))q(\theta(r)) + \theta'(r)\varphi(M_1(r))h(\theta(r))\alpha_1(r), \quad (2.33)$$

for all $r \in H$. Since $l(r) \geqslant 1$ and $\varphi(M_1(r)) \geqslant 1$, which implies that $\frac{l'(r)}{\varphi(M_1(r))} \leqslant l'(r)$ then we can write the inequality (2.33) as follows

$$\frac{M_1'(r)}{\varphi(M_1(r))} \leqslant l'(r) + \theta'(r)q(\theta(r)) + \theta'(r)h(\theta(r))\alpha_1(r), \ \forall\, r \in H.$$

With the help of inequality (2.28) and integrating above inequality from 0 to r, we obtain

$$\Psi(M_1(r)) \leqslant \Psi(M_1(0)) + \int_0^{\theta(r)} \left(l'(\theta^{-1}(\lambda)) + q(\lambda) + h(\lambda)\alpha_1(\theta^{-1}(\lambda))\right) d\lambda,$$

or equivalently

$$M_1(r) \leqslant \Psi^{-1}\left(\Psi(l(0)) + \int_0^{\theta(r)} \left(l'(\theta^{-1}(\lambda)) + q(\lambda) + h(\lambda)\alpha_1(\theta^{-1}(\lambda))\right) d\lambda\right),$$

for all $r \in H$. Substituting above inequality in (2.29), we obtain required inequality (2.26). That completes the proof. □

Now we present another nonlinear retarded integral inequality of this section.

THEOREM 2.4. *Let m, p, $q \in C(H,H)$ be non negative functions and φ_1, φ_2, θ, l $\in C'(H,H)$ be increasing functions with $\varphi_1'(r) = \varphi_2(r)$, $l(r) \geqslant 1$, $\theta(r) \leqslant r$, $\varphi_e(r) > 0$; $e = 1,2$, for all $r \in H$. If the inequality*

$$\varphi_1(m(r)) \leqslant l(r) + \int_0^{\theta(r)} p(\lambda)\varphi_1(m(\lambda))d\lambda + \int_0^{\theta(r)} q(\lambda)\varphi_2(m(\lambda))d\lambda, \quad (2.34)$$

for all $r \in H$*, holds, then*

$$m(r) \leqslant [\varphi_1^{-1}(l(0)) + \varphi_1^{-1}\left(\int_0^{\theta(r)} l'(\theta^{-1}(\lambda))\exp\left(-\int_0^{\lambda} p(\mu)d\mu\right)d\lambda\right) + \int_0^{\theta(r)} q(\lambda)d\lambda]\varphi_1^{-1}\left(\exp\left(\int_0^{\theta(r)} p(\lambda)d\lambda\right)\right), \ \forall\, r \in H, \tag{2.35}$$

where φ_1^{-1} *is the inverse function of* φ_1 *and* $R_1 \in H$ *is the largest number such that*

$$\varphi_1^{-1}(l(0)) + \int_0^{\theta(r)} q(\lambda)d\lambda + \varphi_1^{-1}(\int_0^{\theta(r)} l'(\theta^{-1}(\lambda))\exp(-\int_0^{\lambda} p(\mu)d\mu)d\lambda) > 0, \tag{2.36}$$

for all $r \in H$ *lying in the interval* $0 \leqslant r \leqslant R_1$*.*

Proof. Let $\varphi_1(M_2(r))$ is equal to the right hand side of (2.34), then we have $M_2(0) = \varphi^{-1}(l(0)$, and

$$m(r) \leqslant M_2(r),\ m(\theta(r)) \leqslant M_2(\theta(r)) \leqslant M_2(r),\ \forall\, r \in H. \tag{2.37}$$

Differentiating $\varphi_1(M_2(r))$ with respect to r and using (2.37), we obtain

$$\begin{aligned}\varphi_1'(M_2(r))(M_2'(r)) &= l'(r) + \theta'(r)p(\theta(r))\varphi_1(m(\theta(r)) + \theta'(r)q(\theta(r))\varphi_2(m(\theta(r))\\ &\leqslant l'(r) + \theta'(r)p(\theta(r))\varphi_1(M_2(r)) + \theta'(r)q(\theta(r))\varphi_2(M_2(r)).\end{aligned}$$

Using the relation $\varphi_1'(M_2(r)) = \varphi_2(M_2(r))$ and dividing both sides by $\varphi_1'(M_2(r))$, so we have

$$M_2'(r) \leqslant \frac{l'(r)}{\varphi_1'(M_2(r))} + \frac{\theta'(r)p(\theta(r))\varphi_1(M_2(r))}{\varphi_1'(M_2(r))} + \theta'(r)q(\theta(r)),\ \ \forall\, r \in H.$$

By setting $r = \lambda$ in above inequality and applying integration from 0 to r , we get

$$\begin{aligned}M_2(r) &\leqslant \varphi_1^{-1}(l(0)) + \int_0^r \frac{l'(\lambda)}{\varphi_1'(M_2(\lambda))}d\lambda + \int_0^{\theta(r)} p(\lambda)\frac{\varphi_1(M_2(\lambda))}{\varphi_1'(M_2(\lambda))}d\lambda\\ &\quad + \int_0^{\theta(r)} q(\lambda)d\lambda\\ &\leqslant \varphi_1^{-1}(l(0)) + \int_0^r \frac{l'(\lambda)}{\varphi_1'(M_2(\lambda))}d\lambda + \int_0^{\theta(r)} p(\lambda)\frac{\varphi_1(M_2(\lambda))}{\varphi_1'(M_2(\lambda))}d\lambda\\ &\quad + \int_0^{\theta(R)} q(\lambda)d\lambda, \quad \forall\, r \in H,\end{aligned} \tag{2.38}$$

where $0 \leqslant R \leqslant R_1$ was picked arbitrary and R_1 is defined in (2.36). Now, we let $M_3(r)$ be the right hand side of (2.38), then

$$M_3(0) = \varphi_1^{-1}(l(0)) + \int_0^{\theta(R)} q(\lambda)d\lambda \tag{2.39}$$

and

$$M_2(r) \leqslant M_3(r), \quad \forall\, r \leqslant R. \tag{2.40}$$

Differentiating $M_3(r)$ with respect to r and using (2.40), we have

$$\begin{aligned} M_3'(r) &= \frac{l'(r)}{\varphi_1'(M_2(r))} + \theta'(r)P(\theta(r))\frac{\varphi_1(M_2(\theta(r))}{\varphi_2(M_2(\theta(r)))}, \\ &\leqslant \frac{l'(r)}{\varphi_1'(M_3(r))} + \theta'(r)P(\theta(r))\frac{\varphi_1(M_3(r)}{\varphi_1'(M_3(r))}, \quad \forall\, r \leqslant R, \end{aligned}$$

which implies that

$$\varphi_1'(M_3(r))M_3'(r) - \theta'(r)p(\theta(r))\varphi_1(M_3(r)) \leqslant l'(r), \quad \forall\, r \leqslant R.$$

The above inequality gives an estimation for $M_3(r)$ as follows

$$\begin{aligned} M_3(r) \leqslant{}& [\varphi_1^{-1}(l(0)) + \int_0^{\theta(R)} q(\lambda)d\lambda + \varphi_1^{-1}(\int_0^{\theta(r)} l'(\theta^{-1}(\lambda)) \\ & \times \exp(-\int_0^{\lambda} p(\mu)d\mu)d\lambda]\varphi_1^{-1}(\exp(\int_0^{\theta(r)} p(\lambda)d\lambda), \quad \forall\, r \leqslant R. \end{aligned}$$

As $0 < R \leqslant R_1$ was picked arbitrary , if we let $r = R$, then above inequality can be written as

$$\begin{aligned} M_3(r) \leqslant{}& [\varphi_1^{-1}(l(0)) + \int_0^{\theta(r)} q(\lambda)d\lambda + \varphi_1^{-1}(\int_0^{\theta(r)} l'(\theta^{-1}(\lambda)) \\ & \times \exp(-\int_0^{\lambda} p(\mu)d\mu)d\lambda]\varphi_1^{-1}(\exp(\int_0^{\theta(r)} p(\lambda)d\lambda), \quad \forall\, r \leqslant R. \end{aligned}$$

Substituting above inequality into (2.40), then substituting (2.40) into (2.37), that gives the required inequality in (2.35). Proof is completed. □

3. Applications

This section presents an application for showing the strength of our derived results in Section 2. The derived inequalities of previous section are applying to examine the boundedness and global existence for the solution of initial value problem of retarded nonlinear integro-differential equation. Consider the following initial value problem for retarded nonlinear integro-differential equation:

$$\begin{cases} m'(r) = l'(r) + F\left(r,\, m(\theta(r)),\, \int_0^r G(\lambda, m(\theta(r)))\right), \ \forall\, r \in H, \\ m(0) = l(0), \end{cases} \tag{3.1}$$

where $F \in C(H^3, \mathbb{R}), G \in C(H^2, \mathbb{R})$ and $l(0)$ is positive constant. We suppose that

$$\begin{aligned} \int_0^r |l'(r) + F(\lambda, m(\theta(\lambda)), V)|d\lambda \leqslant{}& \int_0^r (l'(r) + \varphi(|m(\lambda)|)[h(\lambda)\varphi(|m(\lambda)|) \\ & + q(\lambda)] + \varphi(|m(\lambda)|h(\lambda)|V|)d\lambda, \end{aligned} \tag{3.2}$$

$$V = |G(r,m(r))| \leqslant p(r)(\varphi(|m(r)|)), \tag{3.3}$$

where the functions q, h, l, θ and p are already defined as in Theorem 2.3. If we take m as a solution of (3.1), then it can be written as

$$m(r) = l(r) + \int_0^r F\left(\lambda,\ m(\theta(\lambda)),\ \int_0^\lambda h(\mu,m(\theta(\mu))d\mu)\right)d\lambda,\ \forall\, r \in H. \tag{3.4}$$

Applying (3.2) and (3.3) in (3.4), we obtain that

$$\begin{aligned}|m(r)| &\leqslant l(r) + \int_0^r \varphi(|m(\theta(\lambda))|)[h(\lambda)\varphi(|m(\theta(\lambda))|) + q(\lambda)]d\lambda \\ &\quad + \int_0^r \varphi(|m(\theta(\lambda))|)h(\lambda)\left(\int_0^\lambda p(\mu)\varphi(|m(\theta(\mu))|)d\mu\right)d\lambda \\ &\leqslant l(r) + \int_0^{\theta(r)} \frac{\varphi(|m(\lambda)|)}{\theta'(\theta^{-1}(\lambda))}[h(\lambda)\varphi(|m(\lambda)|) + q(\lambda)]d\lambda \\ &\quad + \int_0^{\theta(r)} \frac{h(\theta^{-1}(\lambda))}{\theta'(\theta^{-1}(\lambda))}\varphi(m(\lambda))\left(\int_0^\lambda p(\mu)\varphi(|m(\mu)|)d\mu\right)d\lambda,\end{aligned} \tag{3.5}$$

for all $r \in H$. If (3.5) holds, then as an application of Theorem 2.3, we get

$$m(r) \leqslant \Psi^{-1}\left(\Psi(l(0)) + \int_0^{\theta(r)} \frac{(l(\theta^{-1}(\lambda)) + q(\lambda) + h(\lambda)\alpha_1(\theta^{-1}(\lambda)))}{\theta'(\theta^{-1}(\lambda))}d\lambda\right),$$

for all $r \in H$, which gives boundedness and global existence for m, where Ψ is defined in Theorem 2.3 and

$$\alpha_1(r) \leqslant \frac{\exp\left(\int_0^{\theta(r)} \frac{[kq(\lambda)+p(\lambda)]}{\theta'(\theta^{-1}(\lambda))}d\lambda\right)}{\varphi^{-1}(l(0)) - \int_0^{\theta(r)} \frac{k[l'(\theta^{-1}(\lambda))+h(\theta^{-1}(\lambda))]}{\theta'(\theta^{-1}(\lambda))}\exp\left(\int_0^\lambda [kq(\mu)+p(\mu)]d\mu\right)d\lambda},$$

for all $r \in H$.

REMARK 3.1. In some situations, the bounds and existence given by the other inequalities are not directly fit, and not possible to examine the stability and asymptotic behavior of solutions of classes of more general retarded nonlinear differential and integral equations. But the integral inequalities derived in this article allow us to study the global existence, uniqueness, stability, boundedness and asymptotic behavior and other properties of solutions of classes of more general retarded nonlinear differential and integral equations.

REFERENCES

[1] B. G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, London, 1998.

[2] D. D. BAINOV, P. S. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Academic Publishers, New York, 1992.

[3] R. P. AGARWAL, S. DENG, W. ZHENG, *Generalization of a retarded Gronwall-like inequality and its applications*, Appl. Math. Comput. 165 (2005) 599–612.

[4] T. H. GRONWALL, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann Math. 20 (1919) 292–296.

[5] R. BELLMAN, *The stability of solutions of linear differential equations*, Duke Math J. 10 (1943) 643–647.

[6] A. ABDELDAIM, A. A. EL-DEEB, *On generalized of certain retarded nonlinear integral inequalities and its applications in retarded integro-differential equations*, Appl. Math. Comput. 256 (2015) 375–380.

[7] I. A. BIHARI, *A generalization of a lemma of Bellman and its application to uniqueness problem of differential equation*, Acta Math Acad Sci Hung. 7 (1956) 81–94.

[8] O. LIPOVAN, *A retarded Gronwall-like inequality and its applications*, J. Math Anal Appl. 252 (2000) 389–401.

[9] W. S. CHEUNG, *Some new nonlinear inequalities and applications to boundary value problems*, Nonlinear Anal. 64 (2006) 2112–2128.

[10] A. ABDELDAIM, *Nonlinear retarded integral inequalities of Gronwall-Bellman type and Applications*, J. Math. Inequal. 10 (2016) 285–299.

[11] Y. J. CHO, S. S. DRAGOMIR, Y. H. KIM, *On some integral inequalities with iterated integrals*, J. Korean Math. Soc. 43 (2006) 563–578.

[12] A. SHAKOOR, I. ALI, M. AZAM, A. REHMAN, M. Z. IQBAL, *Further Nonlinear Retarded Integral Inequalities for Gronwall-Bellman Type and Their Applications*, Iran J Sci Technol Trans Sci. 43 (2019) 2559–2568.

(Received August 8, 2019)

Abdul Shakoor
Department of Mathematics
Khwaja Fareed University of Engineering and Information Technology
Rahim Yar Khan 64200, Pakistan
e-mail: ashakoor313@gmail.com

Ilyas Ali
Basic Sciences and Humanities Department
University of Engineering and Technology Lahore
Faisalabad Campus 38000, Pakistan

Samad Wali
School of Information and Communication Engineering
University of Electronic Science and Technology of China
611731, China

Abdur Rehman
Basic Sciences and Humanities Department
University of Engineering and Technology Lahore
Faisalabad Campus 38000, Pakistan