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BEST BOUNDS FOR THE LAMBERT W FUNCTIONS

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Abstract. This paper is devoted to provide tractable closed-form upper and lower bounds for the two real branches of the Lambert W function $W(z(t))$ for all positive real variable t where $z(t)$ is increasing function on $(0,\infty)$ and bounded by zero and $-e^{-1}$.

1. Introduction

The Lambert W function which satisfies the exponential equation

$$W(z)e^{W(z)} = z, \qquad z \in \mathbb{C}, \tag{1.1}$$

has a rich variety of applications ranging from physics and computer science, to statistics and biology. It has many applications in pure and applied mathematics, some of which are briefly described in [1, 2]. It has also appeared in recent research in communications, such as relaying strategies [3], moment generating functions for modeling signal fading [4] and long-haul cooperative power allocation methods [5]. The table in [6] shows the applications of the real-valued W-function including the branch used. Also, an extensive study on approximations of Lambert function was published by Iacono and Boyd [7].

The Lambert W function is a multi-valued function defined in general for complex and assuming values $W(z)$ complex. If z is real and $z < -1/e$, then $W(z)$ is multi-valued complex. If z is real and $-1/e \leqslant z < 0$, there are two possible real values of $W(z)$: The branch satisfying $W(z) \geqslant -1$ is denoted by $W_0(z)$ and called the principal branch of the W function, and the other branch satisfying $W(z) \leqslant -1$ is denoted by $W_{-1}(z)$. If z is real and $z \geqslant 0$, there is a single real value for $W(z)$, which also belongs to the principal branch $W_0(z)$. The Lambert W-function has the derivative

$$W'(z) = \frac{W(z)}{z(W(z)+1)}, \qquad z \in \mathbb{C}, \tag{1.2}$$

Recently, sharp bounds for the real branch W_{-1} function established in [8]. Also, both branches $W_0(z)$ and $W_{-1}(z)$ of Lambert W function appeared as terms in best

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sharp bounds of the q-gamma function and some of related functions [9] where the variable function z is defined as

$$z(t) = -\frac{te^{\frac{1}{2}t-1}}{e^t - 1}, \qquad t > 0. \tag{1.3}$$

It has been proven in [9] that the function $z(t), t > 0$ is increasing on $(0,\infty)$ and satisfies the sharp inequality $-e^{-1} < z(t) < 0$ which declares that the Lambert W function $W(z(t))$ has two possible real branches $W_0(z(t)) \geqslant -1$ and $W_{-1}(z(t)) \leqslant -1$ for all positive real t.

In order to prove the results in [9], the author established the following functional inequalities

$$(W_0(z(t))+1)^2 > tg(t)W_0(z(t)), \qquad t > 0 \tag{1.4}$$

$$(W_{-1}^2(z(t))+1)^2 < tg(t)W_{-1}(z(t)), \qquad t > 0 \tag{1.5}$$

where $g(t) = z'(t)/z(t)$, for the both branches of the Lambert W function. Furthermore, according to Lemma 3.6 in [9], we can state the following Lemma:

LEMMA 1.1. *The function* $(W(z(t))+1)/t$ *is decreasing on* $(0,\infty)$ *for both branches and has the best bounds*

$$0 < \frac{W_0(z(t))+1}{t} < \frac{\sqrt{3}}{6}, \qquad t > 0 \tag{1.6}$$

$$-\frac{1}{2} < \frac{W_{-1}(z(t))+1}{t} < -\frac{\sqrt{3}}{6}, \qquad t > 0 \tag{1.7}$$

Inspired of the great important role of the function $z(t)$ in providing sharp bounds of the q-gamma function in terms of the Lambert W function, we devote this paper to provide sharp bounds for both branches $W_0(z(t))$ and $W_{-1}(z(t))$ of the Lambert W function.

It is worth remarkable to mention that the field of finding inequalities for the special functions has been drawn the attention of many contributors [10, 11, 12, 13, 14, 15, 16].

2. Bounds for the principal branch $W_0(z)$

This section is devoted to provide tractable closed-form upper and lower bounds for the principal branch $W_0(z(t))$ for all positive real variable t where $z(t)$ is defined in (1.3). To do these, we use functional analysis methods and the monotonicity properties for some certain functions involving the principal branch $W_0(z(t))$.

LEMMA 2.1. *Let* $z(t)$ *be defined as in* (1.3). *Then the principal branch of Lambert* W *function satisfies the inequality*

$$W_0(z(t))+1 < \frac{e^t-1}{e^t+1}, \qquad \textit{for all} \quad t > 0. \tag{2.1}$$

Proof. Let the function

$$Q(t) = (e^t + 1)W_0(z(t)) + 2$$

be defined for all $t > 0$. It is easy from the asymptotic expansion (3.1) in [1] to find that

$$Q(t) = (e^t + 1)W_0(z(t)) + 2 < (e^t + 1)z(t) + 2 = q_1(t)$$

Differentiation gives

$$\begin{aligned} q_1'(t) &= -\frac{e^{\frac{1}{2}t-1}}{2(e^t-1)^2}\left[e^{2t}(t+2) - 4te^t - t - 2\right] \\ &= -\frac{e^{\frac{1}{2}t-1}}{2(e^t-1)^2}\sum_{n=1}^{\infty}\frac{t^n}{n!}\left[2^{n-1}(n+4) - 4n\right] < 0 \end{aligned}$$

and $q_1(0) = 2(1 - e^{-1}) > 0$ and $\lim_{t\to\infty} q_1(t) = -\infty$ which lead to the function $q_1(t)$ has a unique zero at $q_1(t) = 0$ which can be easily computed by software of Mathematica as $t = 1.61948$. These mean that $q_1(t) < 0$ for all $t > 1.62$ and so is the function $Q(t)$.

Also, from (1.6), we find that

$$Q(t) = (e^t + 1)W_0(z(t)) + 2 < (e^t + 1)\left(\frac{\sqrt{3}}{6}t - 1\right) + 2 = q_2(t)$$

Differentiation gives

$$\begin{aligned} q_2'(t) &= \frac{\sqrt{3}}{6}e^t(t+1) - e^t + \frac{\sqrt{3}}{6} \\ q_2''(t) &= \frac{\sqrt{3}}{6}e^t(t + 2 - 2\sqrt{3}). \end{aligned}$$

It is obvious that $q_2''(t) > 0$ for all $t > 2(\sqrt{3} - 1)$ and $q_2''(t) < 0$ for all $t < 2(\sqrt{3} - 1)$ which lead to the function $q_2'(t)$ is increasing on $(2(\sqrt{3} - 1), \infty)$ and decreasing on $(0, 2(\sqrt{3} - 1))$. Since $q'(0) = (\sqrt{3} - 3)/3 < 0$ and $q_2'(2) = e^2(\sqrt{3} - 2)/2 + \sqrt{3}/6 \simeq -0.701271$, then $q_2'(t) < 0$ for all $t < 2$ which yields that $q_2(t)$ is decreasing on $(0, 2)$. By virtue of $q_2(0) = 0$, we get $q_2(t) < 0$ for all $t \in (0, 2)$ and so is the function $Q(t)$. Conclusion, the function $Q(t) < 0$ for all $t > 0$ which gives the desired result. □

LEMMA 2.2. *Let $z(t)$ be defined as in (1.3) and $g(t) = z'(t)/z(t)$. Then $g(t)$ is decreasing on $(0, \infty)$ and $g'(t)$ is increasing on $(0, \infty)$ and*

$$\frac{g(t)g'(t)}{g''(t)} > \frac{e^t - 1}{e^t + 1}. \tag{2.2}$$

Proof. Salem [9] proved that $-1/2 < g(t) < 0$. To prove the monotonicity of $g(t)$, we have $g'(t) = -\lambda(t)/[t^2(e^t - 1)^2]$ where

$$\lambda(t) = e^{2t} - e^t(t^2 + 2) + 2.$$

Whence

$$\lambda'(t) = e^t[2e^t - 2 - 2t - t^2] > 0 \qquad \text{and} \qquad \lambda(0) = 0$$

which leads to $\lambda(t) > 0$ for all $t > 0$. This proves the decreasing of $g(t)$. Also, we have $g''(t) = \mu(t)/[t^3(e^t-1)^3]$ where

$$\mu(t) = 2e^{3t} - e^{2t}(t^3+6) - e^t(t^3-6) - 2.$$

The series of exponential function and derivative give

$$\begin{aligned}\mu'(t) &= e^t[6e^{2t} - e^t(t^3+3t^2+6) - t^3 - 3t^2 + 6] \\ &> e^t[6te^t - t^3 - 3t^2 + 6] \\ &> e^t[2t^3 + 3t^2 + 6t + 6] > 0 \qquad \text{and} \qquad \mu(0) = 0\end{aligned}$$

which leads to $\mu(t) > 0$ for all $t > 0$. This proves the increasing of $g'(t)$. Finally, we have

$$\frac{g(t)g'(t)}{g''(t)} - \frac{e^t-1}{e^t+1} = \frac{u(t)}{2(e^t+1)v(t)}$$

where

$$\begin{aligned}u(t) &= e^{4t}(t-6) + e^{3t}(t^3+2t^2+20) - 2e^{2t}(t^3+t+12) - e^t(3t^3+2t^2-12) + t - 2 \\ v(t) &= 2e^{3t} - e^{2t}(t^3+6) - e^t(t^3-6) - 2\end{aligned}$$

The proof of positivity of the two functions $u(t)$ and $v(t)$ is similar, so we suffice to prove $v(t) > 0$. Differentiation gives $v'(t) = e^t v_1(t)$ where

$$v_1(t) = 6e^{2t} - e^t(2t^3+3t^2+12) - t^3 - 3t^2 + 6.$$

Differentiations again, give

$$\begin{aligned}v_1'(t) =& 12e^{2t} - e^t(2t^3+9t^2+6t+12) - 3t^2 - 3t^2 - 6t \\ v_1''(t) =& 24e^{2t} - e^t(2t^3+15t^2+24t+18) - 6t - 6 \\ v_1'''(t) =& 48e^{2t} - e^t(2t^3+21t^2+54t+42) - 6 \\ v_1^{(4)}(t) =& e^t(96e^t - 2t^3 - 27t^2 - 96t - 96) \\ >& e^t\left[96\left(1+t+\frac{1}{2}t^2+\frac{1}{6}t^3\right) - 2t^3 - 27t^2 - 96t - 96\right] \\ =& 7t^2e^t(2t+3] > 0.\end{aligned}$$

It is easy to see that $v_1^{(i)}(0) = 0, i = 0,1,2,3$ which lead to $v_1(t) > 0$ for all $t > 0$ and so does $v'(t)$. Since $v(0) = 0$, then $v(t) > 0$ for all $t > 0$. This completes the proof. □

LEMMA 2.3. *Let $z(t)$ be defined as in (1.3). Then the principal branch of Lambert W function satisfies the functional inequality*

$$(W_0(z(t))+1)^2 > \frac{g^2(t)}{g'(t)} W_0(z(t)) \qquad t>0. \tag{2.3}$$

Proof. Let the function

$$y(t) = (W_0(z(t))+1)^2 - \frac{g^2(t)}{g'(t)} W_0(z(t)). \tag{2.4}$$

be defined for all $t>0$. Differentiating with using results in Lemma 2.1 and Lemma 2.2 give

$$\begin{aligned} y'(t) &= \frac{g^2(t)g''(t)}{g'^2(t)} \frac{W_0(z(t))}{W_0(z(t))+1}\left[W_0(z(t))+1-\frac{g(t)g'(t)}{g''(t)}\right] \\ &> \frac{g^2(t)g''(t)}{g'^2(t)} \frac{W_0(z(t))}{W_0(z(t))+1}\left[\frac{e^t-1}{e^t+1}-\frac{g(t)g'(t)}{g''(t)}\right] > 0. \end{aligned}$$

L'Hospital rule gives $\lim_{t\to 0} y(t) = 0$ which means that $y(t)>0$ for all $t>0$. This ends the proof. □

THEOREM 2.4. *Let $z(t)$ be defined as in (1.3). Then the principal branch of Lambert W function satisfies the inequality*

$$-\alpha g(t) < W_0(z(t))+1 < -\beta g(t) \qquad t>0. \tag{2.5}$$

for all $\alpha \leqslant 2$ and $\beta \geqslant 2\sqrt{3}$ with the best possible constant $\alpha = 2$ and $\beta = 2\sqrt{3}$.

Proof. Let the function

$$f(a) = W_0(z(t))+1+ag(t)$$

be defined for all $a\in\mathbb{R}$, then $f'(a) = g(t) < 0$. Thus $f(a)$ is decreasing on $\mathbb{R}$. It is clear that $f(0)>0$ and $\lim_{a\to\infty} f(a) = -\infty$ which mean that the function $f(a)$ has a unique root at $a = a(t)$ where

$$a(t) = -\frac{W_0(z(t))+1}{g(t)} \qquad t>0.$$

Differentiation gives

$$a'(t) = \frac{g'(t)y(t)}{g^2(t)(W_0(z(t))+1)} < 0$$

where $y(t)$ is defined in (2.4), which concludes that the function $a(t)$ is decreasing on $(0,\infty)$ and

$$2 = \lim_{t\to\infty} a(t) < a(t) < \lim_{t\to 0} a(t) = 2\sqrt{3}.$$

Here, the lower bound comes from $\lim_{t\to\infty} W_0(z(t)) = W_0(0) = 0$ and $\lim_{t\to\infty}(-1/g(t)) = 2$ and the upper bound comes from $\lim_{t\to 0}(W_0(z(t))+1)/t = \sqrt{3}/6$ (see the relation (3.4) in [9]) and $\lim_{t\to 0}(-t)/g(t) = 12$ (using L'Hospital rule). In view of the above, we obtain $f(a)\geqslant 0$ for all $a\leqslant 2$ and $f(a)\leqslant 0$ for all $a\geqslant 2\sqrt{3}$ which yield the desired results. □

LEMMA 2.5. *Let t be non-negative real. Then, the function*

$$p(t) = t(t-1) + 2\sqrt{3}g(t)(t-2\sqrt{3}) \tag{2.6}$$

is non-negative.

Proof. In view of the definition of $g(t)$, the function $p(t)$ can be read as $p(t)t(e^t - 1) = p_1(t)$ where

$$p_1(t) = e^t(t^3 - (1+\sqrt{3})t^2 + 2(3+\sqrt{3})t - 12) - t^3 + (1-\sqrt{3})t^2 + 2(3-\sqrt{3})t + 12.$$

On differentiating and the fact that $e^t > 1+t$, we find that

$$\begin{aligned} p_1'(t) &= e^t(t^3 + (2-\sqrt{3})t^2 + 4t - 2(3-\sqrt{3})) - 3t^2 + 2(1-\sqrt{3})t + 2(3-\sqrt{3}) \\ p_1''(t) &= e^t(t^3 + (5-\sqrt{3})t^2 + 2(4-\sqrt{3})t - 2(1-\sqrt{3})) - 6t + 2(1-\sqrt{3}) \\ &> t^2(t^2 + (6-\sqrt{3})t + 13 - 2\sqrt{3}) > 0 \end{aligned}$$

Since $p_1(0) = p_1'(0) = p_1''(0) = 0$, then $p_1(t) > 0$ for all $t > 0$ and so does the function $p(t)$. □

THEOREM 2.6. *Let $z(t)$ be defined as in (1.3). Then the principal branch of Lambert W function satisfies the inequality*

$$W_0(z(t)) + 1 > \frac{\sqrt{3}}{6}te^{-t}, \qquad t > 0 \tag{2.7}$$

with the best possible constant $\sqrt{3}/6$.

Proof. Let the function

$$h(t) = (W_0(z(t)) + 1)\frac{e^t}{t}$$

be defined for all $t > 0$. Differentiation gives

$$h'(t) = \frac{e^t}{t^2(W_0(z(t)) + 1)}[(t-1)(W_0(z(t)) + 1)^2 + tg(t)W_0(z(t))].$$

It is clear that $h'(t) > 0$ for all $t \geqslant 1$. Now, let $0 < t < 1$. Then, from (1.6), we get

$$h'(t) > \frac{e^t}{12t(W_0(z(t)) + 1)}p(t) > 0$$

where $p(t)$ is defined in (2.6), which yields $h(t)$ is increasing on $(0, \infty)$. By reference to the relation (3.4) in [9], we get $\lim_{t \to 0} h(t) = \sqrt{3}/6$. This ends the proof. □

LEMMA 2.7. *Let $z(t)$ be defined as in (1.3) and the functions*

$$\ell_1(t) = e^t(t^2-6t+12)-t^2-6t-12$$
$$\ell_2(t) = 6e^t(t-2)+\sqrt{3}t^2e^{-t}-\sqrt{3}t^2+6t+12$$

be defined for all $t \geqslant 0$. The function $\ell_1(t) \geqslant 0$ for all $t \geqslant 0$ and $\ell_2(t)$ is less than zero for all $t < t_0 \sim 0.554405$ and greater than zero for all $t > t_0$.

Proof. It is easy to see that $\ell_1'''(t) = t^2e^t \geqslant 0$, $t \geqslant 0$ and $\ell_1(0) = \ell_1'(0) = \ell_1''(0) = 0$ which reveal that $\ell_1(t) \geqslant 0$, $t \geqslant 0$. On differentiating $\ell_2(t)$ gives

$$\ell_2'(t) = 6e^t(t-1)-\sqrt{3}te^{-t}(t-2)-2\sqrt{3}t+6$$
$$\ell_2''(t) = 6te^t+\sqrt{3}e^{-t}(t^2-4t+2)-2\sqrt{3}$$
$$\ell_2'''(t) = 6e^t(t+1)-\sqrt{3}e^{-t}(t^2-6t+6)$$
$$\ell_2^{(4)}(t) = e^{-t}[6e^{2t}(t+2)+\sqrt{3}(t^2-8t+12)].$$

It is well known that $e^{2t} > 1+2t$ which leads to $\ell_2^{(4}(t) > 0$ for all $t > 0$. Since $\ell_2'''(0) = 6(1-\sqrt{3}) < 0$ and $\ell_2'''(1) \sim 32 > 0$, then there exists a unique number $0 < t_3 < 1$ such that $\ell_2'''(t_3) = 0$. Thus $\ell_2'''(t) < 0$ for all $t < t_3$ and $\ell_2'''(t) > 0$ for all $t > t_3$ which reveals that $\ell_2''(t)$ is decreasing on $(0, t_3)$ and decreasing on (t_3, ∞). Since $\ell_2''(0) = 0$ and $\ell_2''(1) \sim 12.2$, then there exists a unique number $t_3 < t_2 < 1$ such that $0 < \ell_2''(t_2) = 0$. Thus $\ell_2''(t) \leqslant 0$ for all $0 \leqslant t < t_2$ and $\ell_2''(t) > 0$ for all $t > t_2$ which reveals that $\ell_2'(t)$ is decreasing on $(0, t_2)$ and decreasing on (t_2, ∞). Similarly, we can deduce that there exists a unique number $0 < t_0 < 1$ such that $\ell_0(t) \leqslant 0$ for all $0 \leqslant t < t_0$ and $\ell_0(t) > 0$ for all $t > t_0$. By carrying out Mathematica 9 software, we find that $t_0 \sim 0.554405$. This ends the proof. □

REMARK 2.8. Although some bounds have simpler form than others but may not be the sharpest. Therefore, we will do a comparison to determine the best upper and lower bounds among all. By subtracting the upper bounds of (1.6) and (2.5), we find that

$$-2\sqrt{3}g(t)-\frac{\sqrt{3}}{6}t = -\frac{\ell_1(t)}{2\sqrt{3}(e^t-1)} < 0, \qquad t > 0$$

which means that the upper bound of (2.5) is better than the upper bound of (1.6) for all $t > 0$. Also, to compare the lower bounds of (2.5) and (2.7), we find that

$$\frac{\sqrt{3}}{6}te^{-t}+2g(t) = -\frac{\ell_2(t)}{6t(e^t-1)} \begin{cases} > 0 & \text{if} \quad 0 < t < t_0 \sim 0.554405 \\ < 0 & \text{if} \quad t > t_0 \end{cases}$$

which means that the lower bound of (2.7) is better than the lower bound of (2.5) if $t < t_0$ (near zero) and the lower bound of (2.5) is better than the lower bound of (2.7) if $t > t_0$ (large t).

3. The branch $W_{-1}(z)$

In this section, we provide tractable closed-form upper and lower bounds for the branch $W_{-1}(z(t))$ for all positive real variable t where $z(t)$ is defined in (1.3). To do these, we use functional analysis methods and the monotonicity properties for some certain functions involving the branch $W_{-1}(z(t))$.

LEMMA 3.1. *Let $z(t)$ be defined as in (1.3). Then the branch W_{-1} satisfies the functional inequality*

$$\frac{tg(t)}{t+1}W_{-1}(z(t)) < (W_{-1}(z(t))+1)^2 < \frac{g^2(t)}{g'(t)}W_{-1}(z(t)) \qquad t>0. \tag{3.1}$$

Proof. Using (1.7) gives

$$\begin{aligned}(W_{-1}(z(t))+1)^2 - \frac{tg(t)}{t+1}W_{-1}(z(t)) &> \frac{1}{12}t^2 - \frac{tg(t)}{t+1}\left(-\frac{1}{2}t-1\right)\\ &> \frac{\eta(t)}{12(t+1)(e^t-1)}.\end{aligned}$$

where

$$\eta(t) = e^t(t^3-2t^2+12) - t^3 - 4t^2 - 12t - 12$$

It is easy as above to prove that $\eta(t) > 0$ which reveals the lower bound. In order to prove the upper bound, let the function

$$\delta(t) = (W_{-1}(z(t))+1)^2 - \frac{g^2(t)}{g'(t)}W_{-1}(z(t))$$

be defined for all $t>0$. Differentiation gives

$$\delta'(t) = \frac{g^2(t)W_{-1}(z(t))}{g'^2(t)(W_{-1}(z(t))+1)}[g''(t)(W_{-1}(z(t))+1) - g(t)g'(t)]$$

According to Lemma 2.2, it turns out that $\delta(t)$ is decreasing on $(0,\infty)$ which yields, with noting $\delta(0)=0$, that $\delta(t)<0$ for all $t>0$. □

THEOREM 3.2. *Let $z(t)$ be defined as in (1.3). Then the branch W_{-1} satisfies the inequality*

$$W_{-1}(z(t))+1 > -\frac{\sqrt{3}}{6}te^t, \qquad t>0 \tag{3.2}$$

with the best possible constant $-\sqrt{3}/6$.

Proof. Let the function

$$x(t) = (W_{-1}(z(t))+1)\frac{e^{-t}}{t}$$

be defined for all $t>0$. Differentiation gives

$$x'(t) = -\frac{e^{-t}(t+1)}{t^2(W_{-1}(z(t))+1)}[(W_{-1}(z(t))+1)^2 - \frac{tg(t)}{t+1}W_{-1}(z(t))]$$

It is clear, from Lemma 3.1, that $x'(t)>0$ for all $t>0$ which yields that $x(t)$ is increasing on $(0,\infty)$. Invoking relation (3.5) in [9] to show that $\lim_{t\to 0} x(t) = -\sqrt{3}/6$. This ends the proof. □

REMARK 3.3. In order to compare the lower bound of (1.7) and the bound of (3.2), we see that

$$-\frac{\sqrt{3}}{6}te^t - \left(-\frac{1}{2}t\right) = -\frac{\sqrt{3}}{6}t(e^t-\sqrt{3})\begin{cases} >0 & \text{if} \quad 0<t<\ln\sqrt{3} \\ <0 & \text{if} \quad t>\ln\sqrt{3}. \end{cases}$$

Therefore, the bound of (3.2) is greater (better) than the lower bound of (1.7) for all $0<t<\ln\sqrt{3}$ and vice versa for all $t>\ln\sqrt{3}$.

THEOREM 3.4. *Let $z(t)$ be defined as in (1.3). Then the branch W_{-1} satisfies the inequality*

$$W_{-1}(z(t))+1 < 2\sqrt{3}g(t), \tag{3.3}$$

$$W_{-1}(z(t))+1 < 2\sqrt{3}tg'(t) \tag{3.4}$$

with the best possible constant $2\sqrt{3}$.

Proof. Let the function

$$r(t) = \frac{W_{-1}(z(t))+1}{g(t)}$$

be defined for all $t>0$. Differentiation gives

$$r'(t) = -\frac{g'(t)}{g^2(t)(W_{-1}(z(t))+1)}[(W_{-1}(z(t))+1)^2 - \frac{g^2(t)}{g'(t)}W_{-1}(z(t))]$$

It is clear, from Lemma 3.1, that $r'(t)>0$ for all $t>0$ which yields that $r(t)$ is increasing on $(0,\infty)$. Invoking relation (3.5) in [9] and $\lim_{t\to 0} t/g(t) = -12$ to show that $\lim_{t\to 0} r(t) = 2\sqrt{3}$. Lemma 1.1 tells that $(W_{-1}(z)+1)/t$ is decreasing and negative on $(0,\infty)$ and since $1/g'(t)$ is also decreasing and negative on $(0,\infty)$, then the function $(W_{-1}(z(t))+1)/(tg'(t))$ is increasing on $(0,\infty)$. Therefore $(W_{-1}(z(t))+1)/(tg'(t)) > 2\sqrt{3}$. This ends the proof. □

THEOREM 3.5. *Let $z(t)$ be defined as in (1.3). Then the branch W_{-1} satisfies the inequality*

$$-\alpha t(1+e^{-t}) < W_{-1}(z)+1 < -\beta t(1+e^{-t}), \qquad \textit{for all} \quad t>0. \tag{3.5}$$

for all possible constants $\alpha \geqslant 1/2$ *and* $\beta \leqslant \sqrt{3}/12$ *with the best possible constants* $\alpha = 1/2$ *and* $\beta = \sqrt{3}/12$.

Proof. It is easy to see that the function $1/(1+e^{-t})$ is increasing on $(0,\infty)$ and satisfying $1/2 < 1/(1+e^{-t}) < 1$. In view of the above and Lemma 1.1, the function $-(W_{-1}(z)+1)/[t(1+e^{-t})]$ is increasing on $(0,\infty)$ and satisfying $\sqrt{3}/12 < -(W_{-1}(z)+1)/[t(1+e^{-t})] < 1/2$. This ends the proof. □

THEOREM 3.6. *Let* $z(t)$ *be defined as in* (1.3). *Then the branch* W_{-1} *satisfies the inequality*

$$\frac{\alpha t}{g(t)-1} < W_{-1}(z)+1 < \frac{\beta t}{g(t)-1}, \qquad t > 0. \tag{3.6}$$

for all possible constants $\alpha \geqslant 3/4$ *and* $\beta \leqslant \sqrt{3}/6$ *with the best possible constants* $\alpha = 3/4$ *and* $\beta = \sqrt{3}/6$.

Proof. Let the function

$$\psi(d) = W_{-1}(z)+1-\frac{bt}{g(t)-1}, \qquad t > 0$$

be defined for all $d \in \mathbb{R}$. Differentiation gives $\psi'(d) = -t/(g(t)-1) > 0$ which means that $\psi(d)$ is increasing on R. It is obvious that $\psi(0) < 0$ and $\psi(\infty) = \infty$, hence there is a unique root for the function $\psi(d)$ at $d = d(t)$ where

$$d(t) = \frac{W_{-1}(z)+1}{t}(g(t)-1), \qquad t > 0.$$

Lemma 1.1 tells that $(W_{-1}(z)+1)/t$ is decreasing and negative on $(0,\infty)$ and since $g(t)-1$ is also decreasing and negative on $(0,\infty)$, then $d(t)$ is increasing on $(0,\infty)$. Therefore $\sqrt{3}/6 < d(t) < 3/4$ which concludes that $\psi(d) < 0$ for all $d < \sqrt{3}/6$ and $\psi(d) > 0$ for all $d > 3/4$. This ends the proof. □

REMARK 3.7. Several upper and lower bounds to the standard branch W_{-1} are provided in the last three theorems which are best with their form and may be useful. However, we have to declare that they are not sharper than the bounds obtained in (1.7) and (3.2).

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