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WEIGHTED ESTIMATES FOR BILINEAR FRACTIONAL INTEGRAL OPERATOR OF ITERATED PRODUCT COMMUTATORS ON MORREY SPACES

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Abstract. In this paper we prove several weighted estimates for iterated product commutators generated by BMO-functions and the bilinear fractional integral operators on Morrey spaces. As a corollary we obtain new weighted estimates for Adams type inequality.

1. Introduction

In this paper, we study the bilinear fractional integral form as follows

$$I_{\alpha,2}(f,g)(x)=\int_{\mathbb{R}^{2n}}\frac{f(y_1)g(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}dy_1dy_2,\quad 0<\alpha<2n. \tag{1.1}$$

Given a linear operator T and a function b, the commutator $[b,T]$ is defined to be

$$[b,T](f)=bT(f)-T(bf).$$

Coifman, Rochberg and Weiss [2] considered commutators of singular integral operators as a tool to extend the classical factorization theory of H^p spaces. They proved that if $b\in\mathrm{BMO}$ and T is a singular integral operator, then $[b,T]$ in bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. Here BMO denote the space of functions of bounded mean oscillation, i.e. functions b such that

$$\|b\|_{\mathrm{BMO}}=\sup_Q \fint_Q |b(x)-b_Q|dx<\infty,$$

where $b_Q:=\fint_Q b(x)dx$ and $\fint_Q b(x)dx$ denotes the usual integral average of b over Q.

Weighted estimates for the linear fractional integral operator I_α acting on Lebesgue spaces with one weight were obtained by Muckenhoupt and Wheedn [7]. Pérez [8] found a sufficient condition on weights w and v which ensures the boundedness of the

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fractional integral I_α from $L^p(v,\mathbb{R}^n)$ to $L^p(w,\mathbb{R}^n)$ with $1<p\leqslant q<\infty$, where I_α is defined as

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy,\quad 0<\alpha<n.$$

The commutator of I_α was first considered by Chanillo [1], who showed that if $b\in$ BMO, then $[b,I_\alpha]$ maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ with $1/p-1/q=\alpha/n$. Iida, Sato, Sawano and Tanaka [5] introduced the condition of two weights for the boundedness of I_α on Morrey space. Weighted estimates for $[b,I_\alpha]$ were studied by Iida in [4] where it was shown that if $b\in \mathrm{BMO}$, $0<\alpha<n$, $1<p\leqslant p_0<\infty$, $1<q\leqslant q_0<r_0<\infty$, $1/p_0>\alpha/n\geqslant 1/r_0$, $1/q_0=1/p_0+1/r_0-\alpha/n$, $q/q_0=p/p_0$, $1<a<r_0/q_0$ and (v,w) is a pair of weights satisfying

$$\sup_{Q\in\mathscr{D}(\mathbb{R}^n)}\left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{aq_0}}|Q'|^{\frac{1}{r_0}}\left(⨍_Q v(x)^{aq}dx\right)^{\frac{1}{aq}}\left(⨍_{Q'} w(x)^{-(p/a)'}dx\right)^{\frac{1}{(p/a)'}}<\infty,$$

we have

$$\|[b,I_\alpha](f)v\|_{\mathscr{M}_q^{q_0}}\leqslant C\|b\|_{\mathrm{BMO}}\|fw\|_{\mathscr{M}_p^{p_0}},$$

where $\mathscr{D}(\mathbb{R}^n)$ denotes the family of all dyadic cubes (see below) and $\mathscr{M}_p^{p_0}(\mathbb{R}^n)=\mathscr{M}_p^{p_0}$ denotes Morrey space which is defined by the norm

$$\|f\|_{\mathscr{M}_p^{p_0}}:=\sup_{Q\in\mathscr{D}(\mathbb{R}^n)}|Q|^{\frac{1}{p_0}}\left(⨍_Q|f(x)|^p dx\right)^{\frac{1}{p}}<\infty.$$

When considering a bilinear operator $I_{\alpha,2}$, we define the commuators $[b,I_{\alpha,2}]_1$ and $[b,I_{\alpha,2}]_2$ on the first and the second components to be

$$[b,I_{\alpha,2}]_1(f,g)=bI_{\alpha,2}(f,g)-I_{\alpha,2}(bf,g)$$

and

$$[b,I_{\alpha,2}]_2(f,g)=bI_{\alpha,2}(f,g)-I_{\alpha,2}(f,bg).$$

Let $\vec{b}=(b_1,\ldots,b_N)$, where b_i's are given functions, and $\vec{\beta}=(\beta_1,\ldots,\beta_N)\in\{1,2\}^N$, the iterated product commutators of $I_{\alpha,2}$ is given by

$$[\vec{b},I_{\alpha,2}]_{\vec{\beta}}=[b_N,[b_{N-1},\cdots,[b_2,[b_1,I_{\alpha,2}]_{\beta_1}]_{\beta_2}\cdots]_{\beta_{N-1}}]_{\beta_N}. \tag{1.2}$$

Now we consider bilinear fractional integral operators having rough kernel which were studied by Iida [3]. Let $0<\alpha<2n$. For a measurable function Ω on $\mathbb{R}^{2n}\setminus\{0,0\}$ and a measurable function b, we define

$$I_{\Omega,\alpha}(f,g)(x):=\int_{\mathbb{R}^{2n}}\frac{\Omega(x-y_1,x-y_2)f(y_1)g(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}dy_1dy_2$$

and

$$[\vec{b},I_{\Omega,\alpha}]_{\vec{\beta}}=[b_N,[b_{N-1},\cdots,[b_2,[b_1,I_{\Omega,\alpha}]_{\beta_1}]_{\beta_2}\cdots]_{\beta_{N-1}}]_{\beta_N}.$$

In [11], Pérez and Rivera-Rios studied these types of commutators in the linear case. Given a bilinear operator $I_{\alpha,2}$, we may rearrange the commutators in any order as the following proposition states.

PROPOSITION 1.1. *For any permutation σ on $\{1,\dots,N\}$,*

$$[\sigma(\vec{b}),I_{\alpha,2}]_{\sigma(\vec{b})}=[\vec{b},I_{\alpha,2}]_{\vec{\beta}}, \tag{1.3}$$

where $\sigma(\vec{b})=(b_{\sigma(1)},\dots,b_{\sigma(N)})$ and $\sigma(\vec{\beta})=(\beta_{\sigma(1)},\dots,\beta_{\sigma(N)})$. In particular, equality (1.3) is valid for any permutation σ_0 be such that $\sigma_0(\vec{\beta})=(1,\dots,1,2,\dots,2)$.

For simplicity in the notation and proof, from now on we will always assume that $\vec{\beta}=(1,\dots,1,2,\dots,2)$, and reserve the notation $m=m(\vec{\beta})$ to denote the number of 1's in $\vec{\beta}$.

In this paper, a symbol C is a positive constant. Whenever we evaluate the operator, the constant C may be change from one constant to another. Let $|E|$ denote the Lebesgue measure of E. All cubes are assumed to have their sides parallel to the coordinate axes. For a cube $Q\subset\mathbb{R}^n$, we use cQ to denote the cube with the same center as Q but with side-length c times. For $1<p,p'<\infty$, p and p' are conjugate indices, i.e., $1/p+1/p'=1$.

2. Main results and their corollaries

Our departure is the following result of Iida, Sato, Sawano and Tanaka [5].

THEOREM 2.1. *Let $\vec{w}=(w_1,w_2)$ be a collection of two weights on $\mathbb{R}^n$, $0<\alpha<2n$, $\vec{P}=(p_1,p_2)$, $1<p_1,p_2<\infty$, $0<p\leqslant p_0<\infty$, $1<q\leqslant q_0<r_0\leqslant\infty$ and $1<a<\min(r_0/q_0,p_1,p_2)$. Here, p is given by $1/p=1/p_1+1/p_2$. Assume that*

$$\frac{1}{p_0}>\frac{\alpha}{n}\geqslant\frac{1}{r_0},\quad \frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n}\quad\text{and}\quad\frac{q}{q_0}=\frac{p}{p_0}.$$

Suppose that the weights v and $\vec{w}$ satisfy condition:

$$\begin{aligned}[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}:=&\sup_{\substack{Q,Q'\in\mathscr{D}(\mathbb{R}^n)\\ Q\subset Q'}}\left(\frac{|Q|}{|Q'|}\right)^{1/aq_0}|Q'|^{1/r_0}\left(\fint_Q v(x)^{aq}dx\right)^{1/aq}\\ &\times\prod_{i=1}^{2}\left(\fint_{Q'}w_i(x)^{-(p_i/a)'}dx\right)^{1/(p_i/a)'}<\infty.\end{aligned} \tag{2.4}$$

Then we have

$$\begin{aligned}\|I_{\alpha,2}(f,g)v\|_{\mathscr{M}^{q_0}_q}\leqslant C[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}&\sup_{Q\in\mathscr{D}(\mathbb{R}^n)}|Q|^{1/p_0}\left(\fint_Q(|f(x)|w_1(x))^{p_1}dx\right)^{1/p_1}\\ &\times\left(\fint_Q(|g(x)|w_2(x))^{p_2}dx\right)^{1/p_2}.\end{aligned}$$

For the general commutators defined as in (1.2), we investigate the boundedness of $[\vec{b}, I_{\alpha,2}]_{\vec{\beta}}$ on Morrey spaces corresponding to Theorem 2.1. In this paper, we obtain two main theorems.

Now, we formulate our main results as follows.

THEOREM 2.2. *Let $\vec{w}$, p, p_0 q, q_0, r_0, α, a be same as in Theorem 2.1. Suppose that the weights v and $\vec{w}$ satisfy condition (2.4). Then, for $\vec{b} \in \mathrm{BMO}^N(\mathbb{R}^n)$, we have*

$$\|[\vec{b}, I_{\alpha,2}]_{\vec{\beta}}(f,g)v\|_{\mathscr{M}_q^{q_0}} \leqslant C\|\vec{b}\|_{\mathrm{BMO}^N}[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a} \sup_{Q\in\mathscr{D}(\mathbb{R}^n)} |Q|^{1/p_0} \times \left(\fint_Q (|f(x)|w_1(x))^{p_1}dx\right)^{1/p_1}\left(\fint_Q (|g(x)|w_2(x))^{p_2}dx\right)^{1/p_2}.$$

REMARK 2.3. The condition of Theorem 2.2 corresponds with condition of Theorem 2.1. This implies that Theorem 2.2 gives us the same type of corollaries as in Theorem 2.1.

Taking $w_1(x) = w_2(x) \equiv 1$, we obtain the following corollary.

COROLLARY 2.4. *Let p, p_0 q, q_0, r_0, α, a be same as in Theorem 2.1. Suppose that $v \in \mathscr{M}^{r_0}_{aq}(\mathbb{R}^n)$. Then, for $\vec{b} \in \mathrm{BMO}^N(\mathbb{R}^n)$, we have*

$$\|[\vec{b}, I_{\alpha,2}]_{\vec{\beta}}(f,g)v\|_{\mathscr{M}_q^{q_0}} \leqslant C\|\vec{b}\|_{\mathrm{BMO}^N}\|v\|_{\mathscr{M}^{r_0}_{aq}} \sup_{Q\in\mathscr{D}(\mathbb{R}^n)} |Q|^{1/p_0}\left(\fint_Q |f(x)|^{p_1}dx\right)^{1/p_1} \times \left(\fint_Q |g(x)|^{p_2}dx\right)^{1/p_2}.$$

On the other hand, letting $r_0 \to \infty$, $1/p_0 = 1/p_0^1 + 1/p_0^2$ and $p/p_0 = p_1/p_0^1 = p_2/p_0^2$, we obtain the weighted inequality Adams type for the general commutator $[\vec{b}, I_{\alpha,2}]_{\vec{\beta}}$.

COROLLARY 2.5. *Let $\vec{w} = (w_1, w_2)$ be a collection of two weights on $\mathbb{R}^n$, $0 < \alpha < 2n$, $\vec{P} = (p_1, p_2)$, $1 < p_1, p_2 < \infty$, $0 < p \leqslant p_0 < \infty$, $1 < q \leqslant q_0 < \infty$ and $1 < a < \min(p_1, p_2)$. Here, p is given by $1/p = 1/p_1 + 1/p_2$ and $1/p_0 = 1/p_0^1 + 1/p_0^2$. Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \textit{and} \quad \frac{q}{q_0} = \frac{p_1}{p_0^1} = \frac{p_2}{p_0^2}.$$

Suppose that the weights v and $\vec{w}$ satisfy condition:

$$[v,\vec{w}]^{aq_0}_{aq,\vec{P}/a} := \sup_{\substack{Q,Q'\in\mathscr{D}(\mathbb{R}^n)\\ Q\subset Q'}} \left(\frac{|Q|}{|Q'|}\right)^{1/aq_0}\left(\fint_Q v(x)^{aq}dx\right)^{1/aq} \times \prod_{i=1}^{2}\left(\fint_{Q'} w_i(x)^{-(p_i/a)'}dx\right)^{1/(p_i/a)'} < \infty. \tag{2.5}$$

Then, for $\vec{b} \in \mathrm{BMO}^N(\mathbb{R}^n)$*, we have*

$$\|[\vec{b}, I_{\alpha,2}]_{\vec{\beta}}(f,g)v\|_{\mathscr{M}^{q_0}_q} \leqslant C\|\vec{b}\|_{\mathrm{BMO}^N}[v,\vec{w}]^{aq_0}_{aq,\vec{P}/a}\|fw_1\|_{\mathscr{M}^{p^1_0}_{p_1}}\|gw_2\|_{\mathscr{M}^{p^2_0}_{p_2}}.$$

Corollary 2.5 gives us the following inequality in letting $q = q_0$, $p_1 = p^1_0$, $p_2 = p^2_0$ and $v = w_1w_2$.

COROLLARY 2.6. *Let* $\vec{w} = (w_1, w_2)$ *be a collection of two weights on* $\mathbb{R}^n$, $0 < \alpha < 2n$, $\vec{P} = (p_1, p_2)$, $1 < p_1, p_2 < \infty$, $1 < q < \infty$ *and* $1/p = 1/p_1 + 1/p_2$*. Assume that* $1/q = 1/p - \alpha/n$*. Suppose that a vector of the weights* $\vec{w}$ *satisfies* $A_{\vec{P},q}(\mathbb{R}^n)$*, i.e.*

$$[\vec{w}]_{A_{\vec{P},q}(\mathbb{R}^n)} := \sup_{Q\in\mathscr{D}(\mathbb{R}^n)} \left(\fint_Q (w_1(x)w_2(x))^q dx\right)^{1/q} \prod_{i=1}^{2} \left(\fint_{Q'} w_i(x)^{-p'_i} dx\right)^{1/p'_i} < \infty. \quad (2.6)$$

Then, for $\vec{b} \in \mathrm{BMO}^N(\mathbb{R}^n)$*, we have*

$$\|[\vec{b}, I_{\alpha,2}]_{\vec{\beta}}(f,g)\|_{L^q((w_1w_2)^q)} \leqslant C\|\vec{b}\|_{\mathrm{BMO}^N}[\vec{w}]_{A_{\vec{P},q}(\mathbb{R}^n)}\|f\|_{L^{p_1}(w_1^{p_1})}\|g\|_{L^{p_2}(w_2^{p_2})}.$$

By a similar argument to the proof Theorem 2.2, we have the following theorem.

THEOREM 2.7. *Let* $\vec{w} = (w_1, w_2)$ *be a collection of two weights on* $\mathbb{R}^n$, $1 < s \leqslant \infty$, $0 < \alpha < 2n$, $\vec{P} = (p_1, p_2)$, $1 \leqslant s' < p_1, p_2 < \infty$, $0 < p \leqslant p_0 < \infty$, $1 < q \leqslant q_0 < r_0 \leqslant \infty$ *and* $1 < a < \min(r_0/q_0, p_1, p_2)$*. Here,* p *is given by* $1/p = 1/p_1 + 1/p_2$*. Assume that* (2.1) *holds and the weights* v *and* $\vec{w}$ *satisfy* $[v^{s'}, \vec{w}^{s'}]^{\frac{r_0}{s'}, \frac{aq_0}{s'}}_{\frac{aq}{s'}, \frac{\vec{P}}{s'a}} < \infty$*. Moreover, assume that* $\Omega \in L^s(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ *satisfies the following homogeneity: For any* $\lambda_1, \lambda_2 > 0$, $\Omega(\lambda_1x_1, \lambda_2x_2) = \Omega(x_1, x_2)$*. Then, for* $\vec{b} \in \mathrm{BMO}^N(\mathbb{R}^n)$*, we have*

$$\|[\vec{b}, I_{\Omega,\alpha}]_{\vec{\beta}}(f,g)v\|_{\mathscr{M}^{q_0}_q} \leqslant C\|\vec{b}\|_{\mathrm{BMO}^N}\|\Omega\|_{L^s(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}\left([v^{s'}, \vec{w}^{s'}]^{\frac{r_0}{s'}, \frac{aq_0}{s'}}_{\frac{aq}{s'}, \frac{\vec{P}}{s'a}}\right)^{1/s'}$$
$$\times \sup_{Q\in\mathscr{D}(\mathbb{R}^n)} |Q|^{1/p_0}\prod_{i=1}^{2}\left(\fint_Q (|f_i(x)|w_i(x))^{p_i}dx\right)^{1/p_i}.$$

Theorem 2.7 recovers the following result.

COROLLARY 2.8. *Let* $\vec{w} = (w_1, w_2)$ *be a collection of two weights on* $\mathbb{R}^n$, $1 < s \leqslant \infty$, $0 < \alpha < 2n$, $\vec{P} = (p_1, p_2)$, $1 \leqslant s' < p_1, p_2 < \infty$, $1 < q < \infty$ *and* $1/p = 1/p_1 + 1/p_2$*. Assume that* $1/q = 1/p - \alpha/n$*. Suppose that a vector of the weights* $\vec{w}^{s'}$ *satisfies* $A_{\frac{\vec{P}}{s'}, \frac{q}{s'}}(\mathbb{R}^n)$ *and* $\Omega \in L^s(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ *satisfies the following homogeneity: for any* $\lambda_1, \lambda_2 > 0$, $\Omega(\lambda_1x_1, \lambda_2x_2) = \Omega(x_1, x_2)$*. Then, for* $\vec{b} \in \mathrm{BMO}^N(\mathbb{R}^n)$*, we have*

$$\|[\vec{b}, I_{\Omega,\alpha}]_{\vec{\beta}}(f,g)\|_{L^q((w_1w_2)^q)} \leqslant C\|\vec{b}\|_{\mathrm{BMO}^N}\|\Omega\|_{L^s(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}[\vec{w}^{s'}]^{\frac{1}{s'}}_{A_{\frac{\vec{P}}{s'}, \frac{q}{s'}}(\mathbb{R}^n)}$$
$$\times\|f\|_{L^{p_1}(w_1^{p_1})}\|g\|_{L^{p_2}(w_2^{p_2})}.$$

In this paper, we only give proof of Theorem 2.2. Theorem 2.7 can be proved using similar techniques as in the proof of Theorem 2.2.

3. Preliminaries

In this section, we prepare some lemmas for proving main results.

The following inequality about BMO function which is given by John and Nirenberg [6].

LEMMA 3.1. *Let* $1 \leqslant p < \infty$ *and let* Q *be a cube. Then there exists a constant* $C > 0$ *such that*

$$\left(\fint_Q |b(x) - b_Q|^p dx\right)^{\frac{1}{p}} \leqslant C\|b\|_{\mathrm{BMO}}$$

for all $b \in \mathrm{BMO}(\mathbb{R}^n)$.

We invoke the following decomposition which is derived in [8, 9, 10].

A dyadic grid $\mathscr{D}(\mathbb{R}^n)$ is a countable collection of cubes that satisfies the following properties:

(a) $Q \in \mathscr{D}(\mathbb{R}^n) \Rightarrow \ell(Q) = 2^k$ for some $k \in \mathbb{Z}$.

(b) For each $k \in \mathbb{Z}$, the set $\{Q \in \mathscr{D}(\mathbb{R}^n) : \ell(Q) = 2^k\}$ forms a partition of $\mathbb{R}^n$.

(c) $Q, P \in \mathscr{D}(\mathbb{R}^n) \Rightarrow Q \cap P \in \{\emptyset, P, Q\}$.

One very clear example for this concept is the dyadic grid that is formed by translating and then dilating the unit cube $[0,1)^n$ all over $\mathbb{R}^n$. More precisely, it is formulated as

$$\mathscr{D}(\mathbb{R}^n) = \{2^{-k}([0,1)^n + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

Let $\mathscr{D}(Q_0)$ be the collection of all dyadic subcubes of Q_0, that is, all those cubes obtained by dividing Q_0 into 2^n congruent cubes of half its length, dividing each of those into 2^n congruent cubes. By convention Q_0 itself to $\mathscr{D}(Q_0)$, and so on.

The following lemma resembles the results in [12, 13].

LEMMA 3.2. *For* $\theta_1, \theta_2 > 1$, *let* $A = (4 \cdot 18^n)^{\frac{1}{\theta_1} + \frac{1}{\theta_2}}$ *and*

$$\gamma := \left(\fint_{3Q_0} |f(y_1)|^{\theta_1} dy_1\right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_0} |g(y_2)|^{\theta_2} dy_2\right)^{\frac{1}{\theta_2}}.$$

For $k = 1, 2, \cdots$ *we take*

$$D_k := \bigcup \left\{ Q \in \mathscr{D}(Q_0) : \left(\fint_{3Q} |f(y_1)|^{\theta_1} dy_1\right)^{\frac{1}{\theta_1}} \left(\fint_{3Q} |g(y_2)|^{\theta_2} dy_2\right)^{\frac{1}{\theta_2}} > \gamma A^k \right\}.$$

Considering the maximality cube, we have

$$D_k = \bigcup_j Q_j^k$$

Then we have

$$\gamma A^k < \left(\fint_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \leqslant 2^{n(\frac{1}{\theta_1}+\frac{1}{\theta_2})} \gamma A^k.$$

Let $E_0 := Q_0 \setminus D_1$ *and* $E_j^k := Q_j^k \setminus D_{k+1}$. *Moreover, we obtain*

$$|Q_0| \leqslant 2|E_0| \quad \textit{and} \quad |Q_j^k| \leqslant 2|E_j^k|$$

Proof. By the maximality of Q_j^k, we obtain the following:

$$\gamma A^k < \left(\fint_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \leqslant 2^{n(\frac{1}{\theta_1}+\frac{1}{\theta_2})} \gamma A^k. \tag{3.7}$$

Let $E_0 = Q_0 \setminus D_1$ and $E_j^k = Q_j^k \setminus D_{k+1}$. Then $\{E_0\}$ and $\{E_j^k\}$ are disjoint and satisfied

$$E_0 \bigcup \Big(\bigcup_{k,j} E_j^k \Big) = Q_0.$$

Fix fixed Q_j^k we set

$$A_1 := \left[\left(\int_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\int_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \right]^{-\frac{\theta_2}{\theta_1+\theta_2}} (\gamma A^{k+1})^{\frac{\theta_2}{\theta_1+\theta_2}} \times \left(\int_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}}$$

and

$$A_2 := \left[\left(\int_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\int_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \right]^{-\frac{\theta_1}{\theta_1+\theta_2}} (\gamma A^{k+1})^{\frac{\theta_1}{\theta_1+\theta_2}} \times \left(\int_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}}.$$

Observe that $A_1 A_2 = \gamma A^{k+1}$. Define

$$M_{\theta_1}(f)(x) := \sup_{\mathscr{D}(\mathbb{R}^n) \ni Q \ni x} \left(\fint_Q |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}}$$

and

$$M_{\theta_2}(g)(x) := \sup_{\mathscr{D}(\mathbb{R}^n) \ni Q \ni x} \left(\fint_Q |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}}$$

and

$$M_{\theta_1,\theta_2}(f,g)(x) := \sup_{\mathscr{D}(\mathbb{R}^n)\ni Q\ni x} \left(\fint_Q |f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}} \left(\fint_Q |g(y_2)|^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}$$

By (3.7), we see that

$$\begin{aligned} Q_j^k \cap D_{k+1} &\subset \left\{x\in Q_j^k : M_{\theta_1,\theta_2}(\chi_{Q_j^k}f,\chi_{Q_j^k}g)(x) > \gamma A^{k+1}\right\} \\ &\subset \left\{x\in Q_j^k : M_{\theta_1}(\chi_{Q_j^k}f)(x)M_{\theta_2}(\chi_{Q_j^k}g)(x) > \gamma A^{k+1}\right\} \\ &\subset \left\{x\in Q_j^k : M_{\theta_1}(\chi_{Q_j^k}f)(x) > A_1\right\}\bigcup\left\{x\in Q_j^k : M_{\theta_2}(\chi_{Q_j^k}g)(x) > A_2\right\} \\ &\subset \left\{x\in \mathbb{R}^n : M(\chi_{Q_j^k}f^{\theta_1})(x) > A_1^{\theta_1}\right\}\bigcup\left\{x\in \mathbb{R}^n : M(\chi_{Q_j^k}g^{\theta_2})(x) > A_2^{\theta_2}\right\}. \end{aligned}$$

Using the weak-$(1,1)$ boundedness of M, we have

$$\begin{aligned} \left|Q_j^k\cap D_{k+1}\right| &\leqslant \left|\left\{x\in\mathbb{R}^n : M(\chi_{Q_j^k}f^{\theta_1})(x) > A_1^{\theta_1}\right\}\right| + \left|\left\{x\in\mathbb{R}^n : M(\chi_{Q_j^k}g^{\theta_2})(x) > A_2^{\theta_2}\right\}\right| \\ &\leqslant \frac{3^n}{A_1^{\theta_1}}\int_{3Q_j^k}|f(y)|^{\theta_1}dy + \frac{3^n}{A_2^{\theta_2}}\int_{3Q_j^k}|f(z)|^{\theta_2}dz \\ &= 2\cdot 3^n\left[\frac{1}{\gamma A^{k+1}}\left(\int_{3Q_j^k}|f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}}\left(\int_{3Q_j^k}|g(y_2)|^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}\right]^{\frac{\theta_1\theta_2}{\theta_1+\theta_2}}, \end{aligned}$$

where we have used the defintions of A_1 and A_2. From (3.7) we further have

$$\begin{aligned} \left|Q_j^k\cap D_{k+1}\right| \leqslant 2\cdot 3^n &\left[\frac{1}{\gamma A^{k+1}}\left(\fint_{3Q_j^k}|f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}}\left(\fint_{3Q_j^k}|g(y_2)|^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}\right]^{\frac{\theta_1\theta_2}{\theta_1+\theta_2}} \\ &\times\left|3Q_j^k\right| \leqslant \frac{1}{2}\left|3Q_j^k\right|. \end{aligned}$$

Similarly, we see that $|Q_0| \leqslant 2|E_0|$. This finishes the proof of Lemma 3.2. □

The following key lemma essentially due to Iida [4].

LEMMA 3.3. *Let* p_1, p_2 *and* a *satisfy the conditions of Theorem* 2.2. *Let* θ_i $(1\leqslant i\leqslant 6)$ *satisfy the following conditions :*

1. $\theta_1,\theta_4 \in (1,p_1)$, $\theta_2,\theta_5\in(1,p_2)$ *and* $\theta_3,\theta_6 > 1$.
2. *For* $L>1$ *and* $s\in(q,r_0)$ *such that* $s\theta_3 < Lq$ *and* $s'\theta_3 < q'$.
3. *For the indices* $\theta_1\in(1,p_1)$ *and* $\theta_2\in(1,p_2)$, *we can choose* $a_*>1$ *such that*

$$a_*\theta_1 < p_1 \quad \text{and} \quad a_*\theta_2 < p_2.$$

Assume in addition that, for these indices,

$$a \geqslant \max\left\{\frac{p_1}{(\theta_1(\frac{p_1}{a_*\theta_1})')'}, \frac{p_2}{(\theta_2(\frac{p_2}{a_*\theta_2})')'}, \frac{p_1}{(\theta_4(\frac{p_1}{\theta_4})')'}, \frac{p_2}{(\theta_5(\frac{p_2}{\theta_5})')'}, \theta_6\right\} > 1.$$

Then we obtain

$$\max\left\{\theta_1\left(\frac{p_1}{a_*\theta_1}\right)', \theta_4\left(\frac{p_1}{\theta_4}\right)'\right\} \leqslant \left(\frac{p_1}{a}\right)'$$

and

$$\max\left\{\theta_2\left(\frac{p_2}{a_*\theta_2}\right)', \theta_5\left(\frac{p_2}{\theta_5}\right)'\right\} \leqslant \left(\frac{p_2}{a}\right)'.$$

Let $0 \leqslant \alpha < 2n$. For f,g be locally integrable functions and let $\vec{R} = (r_1, r_2)$. Define a bilinear maximal operator as follows

$$M_{\alpha,\vec{R}}(f,g)(x) = \sup_{\mathscr{D}(\mathbb{R}^n)\ni Q\ni x} |Q|^{\frac{\alpha}{n}} \left(\fint_Q |f(y_1)|^{r_1} dy_1\right)^{\frac{1}{r_1}} \left(\fint_Q |g(y_2)|^{r_2} dy_2\right)^{\frac{1}{r_2}}.$$

For bilinear maximal operator $M_{\alpha,\vec{R}}$, Iida, Sato, Sawano and Tanaka [5] obtained the following result.

LEMMA 3.4. *Let* $0 \leqslant \alpha < 2n$. *Set* $\vec{P} = (p_1, p_2)$ *and* $\vec{R} = (r_1, r_2)$. *Assume in addition that* $0 < r_i < p_i < \infty$ *with* $1 = 1, 2$. *If* $0 < q \leqslant q_0 < \infty$ *and* $0 < p \leqslant p_0 < \infty$ *satisfy*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \textit{and} \quad \frac{q}{q_0} = \frac{p}{p_0},$$

where p *is given by* $1/p = 1/p_1 + 1/p_2$, *then*

$$\|M_{\alpha,\vec{R}}(f,g)\|_{\mathscr{M}^{q_0}_q} \leqslant C \sup_{Q\in\mathscr{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p_0}} \left(\fint_Q |f(y_1)|^{p_1} dy_1\right)^{\frac{1}{p_1}} \left(\fint_Q |g(y_2)|^{p_2} dy_2\right)^{\frac{1}{p_2}}.$$

4. Proof of Theorem 2.2

Proof of Theorem 2.2. Without loss of generality, we may assume that f and g are non-negative, bounded and compactly supported. By induction, we can prove that

$$\begin{aligned}[\vec{b}, I_{\alpha,2}]_{\vec{\beta}}(f,g)(x) = & \int_{\mathbb{R}^{2n}} \prod_{i=1}^{m} (b_i(x) - b_i(y_1)) \\ & \times \prod_{i=m+1}^{N} (b_i(x) - b_i(y_2)) \frac{f(y_1)g(y_2)}{(|x-y_1| + |x-y_2|)^{2n-\alpha}} dy_1 dy_2. \quad (4.8)\end{aligned}$$

Fix a dyadic cube $Q_0 \in \mathscr{D}(\mathbb{R}^n)$. For each $Q \in \mathscr{D}(Q_0)$, letting $\lambda_i = \lambda_i(Q) = \fint_Q b_i(x)dx$ with $i = 1, \dots, N$, we have

$$\prod_{i=1}^{m}(b_i(x)-b_i(y_1)) = \prod_{i=1}^{m}[(b_i(x)-\lambda_i)+(\lambda_i-b_i(y_1))] = \sum_{A\subseteq\{1,\dots,m\}}\prod_{i\in A}(b_i(x)-\lambda_i)\prod_{i\in\bar{A}}(\lambda_i-b_i(y_1))$$

and similarly,

$$\prod_{i=m+1}^{N}(b_i(x)-b_i(y_2)) = \prod_{i=m+1}^{N}[(b_i(x)-\lambda_i)+(\lambda_i-b_i(y_2))] = \sum_{B\subseteq\{m+1,\dots,N\}}\prod_{i\in B}(b_i(x)-\lambda_i)\prod_{i\in\bar{B}}(\lambda_i-b_i(y_2)).$$

Hence, we have that

$$\prod_{i=1}^{m}(b_i(x)-b_i(y_1))\prod_{i=m+1}^{N}(b_i(x)-b_i(y_2)) = \sum_{A\subseteq\{1,\dots,m\}}\sum_{B\subseteq\{m+1,\dots,N\}}\prod_{i\in A\cup B}(b_i(x)-\lambda_i)\prod_{i\in\bar{A}}(\lambda_i-b_i(y_1))\prod_{i\in\bar{B}}(\lambda_i-b_i(y_2)).$$

The volume of the elements in $\mathscr{D}(\mathbb{R}^n)$ is 2^{nv} for some $v \in \mathbb{Z}$. For $x \in Q_0$, we have

$$\begin{aligned}
&|[\vec{b}, I_\alpha]_{\vec{\beta}}(f,g)(x)| \\
&\leqslant \sum_{A\subseteq\{1,\dots,m\}}\sum_{B\subseteq\{m+1,\dots,N\}}\int_{\mathbb{R}^{2n}}\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\prod_{i\in\bar{A}}|\lambda_i-b_i(y_1)| \\
&\quad\times\prod_{i\in\bar{B}}|\lambda_i-b_i(y_2)|\frac{|f(y_1)g(y_2)|}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}dy_1dy_2 \\
&\leqslant C\sum_{A\subseteq\{1,\dots,m\}}\sum_{B\subseteq\{m+1,\dots,N\}}\sum_{v\in\mathbb{Z}}\sum_{\substack{Q\in\mathscr{D}(\mathbb{R}^n)\\|Q|=2^{vn}}}2^{-v(2n-\alpha)}\chi_Q(x) \\
&\quad\times\int_{3Q}\int_{3Q}\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\prod_{i\in\bar{A}}|\lambda_i-b_i(y_1)| \\
&\quad\times\prod_{i\in\bar{B}}|\lambda_i-b_i(y_2)|f(y_1)g(y_2)dy_1dy_2 \\
&= C\sum_{A\subseteq\{1,\dots,m\}}\sum_{B\subseteq\{m+1,\dots,N\}}\sum_{v\in\mathbb{Z}}\Bigg(\sum_{\substack{Q\in\mathscr{D}(\mathbb{R}^n)\\Q\subseteq Q_0}}+\sum_{\substack{Q\in\mathscr{D}(\mathbb{R}^n)\\Q\supsetneq Q_0}}\Bigg)2^{-v(2n-\alpha)}\chi_Q(x) \\
&\quad\times\int_{3Q}\int_{3Q}\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\prod_{i\in\bar{A}}|\lambda_i-b_i(y_1)| \\
&\quad\times\prod_{i\in\bar{B}}|\lambda_i-b_i(y_2)||f(y_1)g(y_2)|dy_1dy_2 \\
&=: C(X_1+X_2).
\end{aligned}$$

Estimate for X_1. Taking $\theta_1,\theta_2>1$, and using Hölder's inequality, we have

$$\begin{aligned}
X_1 =& \sum_{A\subset\{1,\dots,m\}}\sum_{B\subset\{m+1,\dots,N\}}\sum_{Q\in\mathscr{D}(Q_0)}|Q|^{\frac{\alpha}{n}}⨍_{3Q}⨍_{3Q}\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\prod_{i\in\bar{A}}|\lambda_i-b_i(y_1)|\\
&\times\prod_{i\in\bar{B}}|\lambda_i-b_i(y_2)||f(y_1)g(y_2)|dy_1dy_2\\
\leqslant& \sum_{A\subset\{1,\dots,m\}}\sum_{B\subset\{m+1,\dots,N\}}\sum_{Q\in\mathscr{D}(Q_0)}|Q|^{\frac{\alpha}{n}}\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\\
&\times\left(⨍_{3Q}\prod_{i\in\bar{A}}|\lambda_i-b_i(y_1)|^{\theta_1'}dy_1\right)^{\frac{1}{\theta_1'}}\left(⨍_{3Q}\prod_{i\in\bar{B}}|\lambda_i-b_i(y_2)|^{\theta_2'}dy_2\right)^{\frac{1}{\theta_2'}}\\
&\times\left(⨍_{3Q}|f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}}\left(⨍_{3Q}|g(y_2|)^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}. \qquad (4.9)
\end{aligned}$$

By Hölder's inequality and Lemma 3.1, we obtain that

$$\left(⨍_{3Q}\prod_{i\in\bar{A}}|\lambda_i-b_i(y_1)|^{\theta_1'}dy_1\right)^{\frac{1}{\theta_1'}}\leqslant C\prod_{i\in\bar{A}}\|b_i\|_{\mathrm{BMO}} \qquad (4.10)$$

and

$$\left(⨍_{3Q}\prod_{i\in\bar{B}}|\lambda_i-b_i(y_2)|^{\theta_2'}dy_2\right)^{\frac{1}{\theta_2'}}\leqslant C\prod_{i\in\bar{B}}\|b_i\|_{\mathrm{BMO}}.$$

Combining (4.9) and (4.10), it implies that

$$\begin{aligned}
X_1\leqslant& C\sum_{A\subset\{1,\dots,m\}}\sum_{B\subset\{m+1,\dots,N\}}\prod_{i\in\bar{A}\cup\bar{B}}\|b_i\|_{\mathrm{BMO}}\\
&\times\sum_{Q\in\mathscr{D}(Q_0)}|Q|^{\frac{\alpha}{n}}\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\left(⨍_{3Q}|f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}}\left(⨍_{3Q}|g(y_2)|^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}\\
=:& C\sum_{A\subset\{1,\dots,m\}}\sum_{B\subset\{m+1,\dots,N\}}\prod_{i\in\bar{A}\cup\bar{B}}\|b_i\|_{\mathrm{BMO}}II \qquad (4.11)
\end{aligned}$$

Now we will estimate II. Let

$$\mathscr{D}_0(Q_0):=\left\{Q\in\mathscr{D}_0(Q_0):\left(⨍_{3Q_0}|f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}}\left(⨍_{3Q_0}|g(y_2)|^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}\leqslant\gamma A\right\} \qquad (4.12)$$

and for $k\geqslant 1$

$$\begin{aligned}
\mathscr{D}_j^k(Q_0):=\Big\{&Q\in\mathscr{D}_0(Q_0):Q\subset Q_j^k,\gamma A^k\\
&<\left(⨍_{3Q}|f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}}\left(⨍_{3Q}|g(y_2)|^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}\leqslant\gamma A^{k+1}\Big\}, \qquad (4.13)
\end{aligned}$$

where Q_j^k is defined as in Lemma 3.2. Using the properties of (4.12) and (4.13), we show that

$$\mathscr{D}(Q_0) = \mathscr{D}_0(Q_0) \cup \left(\bigcup_{k,j} \mathscr{D}_j^k(Q_0) \right).$$

By the duality argument, we obtain

$$\left(\int_{Q_0} II^q \cdot v(x)^q dx \right)^{\frac{1}{q}} = \sup_{\|h\|_{L^{q'}(Q_0)}=1} \left(\int_{Q_0} I \cdot v(x)|h(x)|dx \right).$$

Let $\operatorname{supp}(h) \subset Q_0$, $\|h\|_{L^{q'}(Q_0)} = 1$. Then we get

$$\begin{aligned}
&\int_{Q_0} II \cdot v(x)|h(x)|dx \\
&\leqslant C \sum_{Q \in \mathscr{D}(Q_0)} |Q|^{\frac{\alpha}{n}} \left(\fint_{3Q} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \\
&\quad \times \int_Q \prod_{i \in A \cup B} |b_i(x) - \lambda_i| v(x) h(x) dx \\
&= C \left(\sum_{Q \in \mathscr{D}(Q_0)} + \sum_{k,j} \sum_{Q \in \mathscr{D}_j^k(Q_0)} \right) |Q|^{\frac{\alpha}{n}} \left(\fint_{3Q} |f(y_1)| dy_1 \right) \int_Q \prod_{i \in A \cup B} |b_i(x) - \lambda_i| v(x) h(x) dx \\
&= II_0 + \sum_{k,j} II_j^k.
\end{aligned}$$

We need to estimate II_0 and II_j^k. For II_j^k, if $Q \in \mathscr{D}_j^k(Q_0)$, we have

$$\left(\fint_{3Q} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \leqslant \gamma A^{k+1}.$$

Hence, we obtain

$$\begin{aligned}
II_j^k &\leqslant \sum_{Q \in \mathscr{D}_j^k(Q_0)} |Q|^{\frac{\alpha}{n}} \gamma A^{k+1} \int_Q \prod_{i \in A \cup B} |b(x) - \lambda_i| v(x) h(x) dx \\
&\leqslant A \sum_{Q \in \mathscr{D}_j^k(Q_0)} |Q|^{\frac{\alpha}{n}} \gamma A^k \int_Q \prod_{i \in A \cup B} |b(x) - \lambda_i| v(x) h(x) dx.
\end{aligned}$$

Since

$$\gamma A^k \leqslant \left(\fint_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}},$$

we obtain

$$II_j^k \leqslant A \sum_{Q \in \mathscr{D}_j^k(Q_0)} |Q_j^k|^{\frac{\alpha}{n}} \left(\fint_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \times \int_Q \prod_{i \in A \cup B} |b(x) - \lambda_i| v(x) h(x) dx.$$

By Hölder's inequality for $\theta_3 > 1$ as in Lemma 3.3, we obtain

$$\begin{aligned} II_j^k &\leqslant A|Q_j^k|^{\frac{\alpha}{n}} \left(\fint_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \sum_{Q \in \mathscr{D}_j^k(Q_0)} \\ &\quad \times \int_Q \prod_{i \in A \cup B} |b(x) - \lambda_i| v(x) h(x) dx \\ &\leqslant A|Q_j^k|^{\frac{\alpha}{n}} \left(\fint_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \\ &\quad \times \sum_{Q \in \mathscr{D}_j^k(Q_0)} |Q| \left(\fint_Q \prod_{i \in A \cup B} |b(x) - \lambda_i|^{\theta_3'} dx \right)^{\frac{1}{\theta_3'}} \left(\fint_Q v(x)^{\theta_3} h(x)^{\theta_3} dx \right)^{\frac{1}{\theta_3}}. \end{aligned}$$

Using Lemma 3.1, we obtain

$$\begin{aligned} II_j^k &\leqslant A \prod_{i \in A \cup B} \|b_i\|_{\mathrm{BMO}} |Q_j^k|^{\frac{\alpha}{n}} \left(\fint_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \\ &\quad \times \sum_{Q \in \mathscr{D}_j^k(Q_0)} \int_Q \left(\fint_Q (v(y)h(y))^{\theta_3} dy \right)^{\frac{1}{\theta_3}} dx \\ &\leqslant A \prod_{i \in A \cup B} \|b_i\|_{\mathrm{BMO}} |Q_j^k|^{\frac{\alpha}{n}} \left(\fint_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \\ &\quad \times \sum_{Q \in \mathscr{D}_j^k(Q_0)} \int_Q M[(v_j^k h)^{\theta_3}](x)^{\frac{1}{\theta_3}} dx, \end{aligned}$$

where $v_j^k = v\chi_{Q_j^k}$ and the symbol M is the ordinary Hardy-Littlewood maximal operator. Using Lemma 3.2, we have

$$II_j^k \leqslant A \prod_{i \in A \cup B} \|b_i\|_{\mathrm{BMO}} |Q_j^k| |Q_j^k|^{\frac{\alpha}{n}} \left(\fint_{3Q_j^k} |f(y_1)|^{\theta_1} dy_1 \right)^{\frac{1}{\theta_1}} \left(\fint_{3Q_j^k} |g(y_2)|^{\theta_2} dy_2 \right)^{\frac{1}{\theta_2}} \times \fint_{Q_j^k} M[(v_j^k h)^{\theta_3}](x)^{\frac{1}{\theta_3}} dx$$

$$\leqslant 2A\prod_{i\in A\cup B}\|b_i\|_{\mathrm{BMO}}|E_j^k||Q_j^k|^{\frac{\alpha}{n}}\left(\fint_{3Q_j^k}|f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}}\left(\fint_{3Q_j^k}|g(y_2)|^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}$$
$$\times\fint_{Q_j^k}M[(v_j^kh)^{\theta_3}](x)^{\frac{1}{\theta_3}}dx$$
$$= 2A\prod_{i\in A\cup B}\|b_i\|_{\mathrm{BMO}}\int_{E_j^k}|Q_j^k|^{\frac{\alpha}{n}}\left(\fint_{3Q_j^k}|f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}}\left(\fint_{3Q_j^k}|g(y_2)|^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}$$
$$\times\left(\fint_{Q_j^k}M[(v_j^kh)^{\theta_3}](x)^{\frac{1}{\theta_3}}dx\right)dy.$$

Taking $s\in(q,r_0)$ and $L>1$ as in Lemma 3.3. Using Hölder's inequality for $s>1$, we have

$$M[(v_j^kh)^{\theta_3}](x)^{\frac{1}{\theta_3}}\leqslant M[(v_j^k)^{s\theta_3}](x)^{\frac{1}{s\theta_3}}M[h^{s'\theta_3}](x)^{\frac{1}{s'\theta_3}}.$$

Using Hölder's inequality for $Lq>1$, we obtain the following inequality:

$$\left(\fint_{Q_j^k}M[(v_j^kh)^{\theta_3}](x)^{\frac{1}{\theta_3}}dx\right)\leqslant\left(\fint_{Q_j^k}M[(v_j^k)^{s\theta_3}](x)^{\frac{Lq}{s\theta_3}}dx\right)^{\frac{1}{Lq}}\left(\fint_{Q_j^k}M[h^{s'\theta_3}](x)^{\frac{(Lq)'}{s'\theta_3}}dx\right)^{\frac{1}{(Lq)'}}.$$

Since $s\theta_3<Lq$, the boundedness of $M:L^{\frac{Lq}{s\theta_3}}(\mathbb{R}^n)\to L^{\frac{Lq}{s\theta_3}}(\mathbb{R}^n)$ gives us the following inequality:

$$\left(\fint_{Q_j^k}M[(v_j^kh)^{\theta_3}](x)^{\frac{1}{\theta_3}}dx\right)\leqslant C\left(\frac{1}{|Q_j^k|}\int_{\mathbb{R}^n}v_j^k(x)^{Lq}dx\right)^{\frac{1}{Lq}}\left(\fint_{Q_j^k}M[h^{s'\theta_3}](x)^{\frac{(Lq)'}{s'\theta_3}}dx\right).$$

Since $a\geqslant L>1$, using Hölder's inequality for $\frac{a}{L}\geqslant 1$, we obtain

$$\left(\fint_{Q_j^k}M[(v_j^kh)^{\theta_3}](x)^{\frac{1}{\theta_3}}dx\right)\leqslant C\left(\fint_{Q_j^k}v(x)^{aq}dx\right)^{\frac{1}{aq}}\left(\fint_{Q_j^k}M[h^{s'\theta_3}](x)^{\frac{(Lq)'}{s'\theta_3}}dx\right).$$

Using Lemma 3.2, this implies that

$$II_j^k\leqslant 2A\prod_{i\in A\cup B}\|b_i\|_{\mathrm{BMO}}\int_{E_j^k}M_{\alpha,aq}^{\theta_1,\theta_2}(f,g,v)(x)\cdot M[M[h^{s'\theta_3}]^{\frac{(Lq)'}{s'\theta_3}}](x)^{\frac{1}{(Lq)'}}dx,$$

where

$$M_{\alpha,aq}^{\theta_1,\theta_2}(f,g,v)(x):=\sup_{\mathscr{D}(\mathbb{R}^n)\ni Q\ni x}|Q|^{\frac{\alpha}{n}}\left(\fint_{3Q}|f(y_1)|^{\theta_1}dy_1\right)^{\frac{1}{\theta_1}}\left(\fint_{3Q}|g(y_2)|^{\theta_2}dy_2\right)^{\frac{1}{\theta_2}}$$
$$\times\left(\fint_Q v(x)^{aq}dx\right)^{\frac{1}{aq}}.$$

A similar argument gives us the following estimate:

$$II_0 \leqslant 2A \prod_{i\in A\cup B} \|b_i\|_{\mathrm{BMO}} \int_{E_0} M^{\theta_1,\theta_2}_{\alpha,aq}(f,g,v)(x)\cdot M[M[h^{s'\theta_3}]^{\frac{(Lq)'}{s'\theta_3}}](x)^{\frac{1}{(Lq)'}}\,dx.$$

By summing up II_0 and II^k_j, we obtain

$$II_0 + \sum_{k,j} II^k_j \leqslant 2A \prod_{i\in A\cup B} \|b_i\|_{\mathrm{BMO}} \int_{Q_0} M^{\theta_1,\theta_2}_{\alpha,aq}(f,g,v)(x)\cdot M[M[h^{s'\theta_3}]^{\frac{(Lq)'}{s'\theta_3}}](x)^{\frac{1}{(Lq)'}}\,dx.$$

Using Hölder's inequality for $q>1$, we have

$$\begin{aligned}
&\int_{Q_0} M^{\theta_1,\theta_2}_{\alpha,aq}(f,g,v)(x)\cdot M[M[h^{s'\theta_3}]^{\frac{(Lq)'}{s'\theta_3}}](x)^{\frac{1}{(Lq)'}}\,dx\\
&\leqslant \left(\int_{Q_0} M^{\theta_1,\theta_2}_{\alpha,aq}(f,g,v)(x)^q dx\right)^{\frac{1}{q}} \left(\int_{Q_0} M[M[h^{s'\theta_3}]^{\frac{(Lq)'}{s'\theta_3}}](x)^{\frac{q'}{(Lq)'}}\,dx\right)^{\frac{1}{q'}}
\end{aligned}$$

Since $(Lq)'<q'$, the boundedness of $M : L^{\frac{(q)'}{(Lq)'}}(\mathbb{R}^n) \to L^{\frac{q'}{(Lq)'}}(\mathbb{R}^n)$ gives us the following inequality:

$$\begin{aligned}
\left(\int_{Q_0} M[M[h^{s'\theta_3}]^{\frac{(Lq)'}{s'\theta_3}}](x)^{\frac{q'}{(Lq)'}}\,dx\right)^{\frac{1}{q'}} &\leqslant C\left(\int_{\mathbb{R}^n} M[h^{s'\theta_3}](x)^{\frac{(Lq)'}{s'\theta_3}\cdot\frac{q'}{(Lq)'}}\,dx\right)^{\frac{1}{q'}}\\
&= C\left(\int_{\mathbb{R}^n} M[h^{s'\theta_3}](x)^{\frac{q'}{s'\theta_3}}\,dx\right)^{\frac{1}{q'}}.
\end{aligned}$$

Since $s'\theta_3<q'$, the boundedness of $M : L^{\frac{q'}{s'\theta_3}}(\mathbb{R}^n) \to L^{\frac{q'}{s'\theta_3}}(\mathbb{R}^n)$ gives us the following inequality:

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} M[h^{s'\theta_3}](x)^{\frac{q'}{s'\theta_3}}\,dx\right)^{\frac{1}{q'}} &\leqslant C\left(\int_{Q_0} |h(x)|^{s'\theta_3\frac{q'}{s'\theta_3}}(x)dx\right)^{\frac{1}{q'}}\\
&= C\left(\int_{Q_0} |h(x)|^{q'}(x)dx\right)^{\frac{1}{q'}} = C.
\end{aligned}$$

Using Hölder's inequality for $\frac{p_1}{a_*\theta_1}>1$ and $\frac{p_2}{a_*\theta_2}>1$ as in Lemma 3.3, we obtain

$$\begin{aligned}
&M^{\theta_1,\theta_2}_{\alpha,aq}(f,g,v)(x)\\
&\leqslant \sup_{\mathscr{D}(\mathbb{R}^n)\ni Q\ni x} |Q|^{\frac{\alpha}{n}} \left(\fint_{3Q} |f(y_1)w_1(y_1)|^{\frac{p_1}{a_*}}dy_1\right)^{\frac{a_*}{p_1}} \left(\fint_{3Q} |g(y_2)w_2(y_2)|^{\frac{p_2}{a_*}}dy_2\right)^{\frac{a_*}{p_2}}\\
&\times \left(\fint_{Q} v(x)^{aq}dx\right)^{\frac{1}{aq}} \left(\int_{3Q} w_1(y_1)^{-\theta_1(\frac{p_1}{a_*\theta_1})'}dy_1\right)^{\frac{1}{\theta_1(\frac{p_1}{a_*\theta_1})'}}\\
&\times \left(\int_{3Q} w_2(y_2)^{-\theta_1(\frac{p_2}{a_*\theta_2})'}dy_2\right)^{\frac{1}{\theta_2(\frac{p_2}{a_*\theta_2})'}}.
\end{aligned}$$

Using the results $\theta_1\left(\frac{p_1}{a_*\theta_1}\right)' \leqslant \left(\frac{p_1}{a}\right)'$ and $\theta_2\left(\frac{p_2}{a_*\theta_2}\right)' \leqslant \left(\frac{p_2}{a}\right)'$ as in Lemma 3.3 and Hölder's inequality, we have

$$\begin{aligned}
&M^{\theta_1,\theta_2}_{\alpha,aq}(f,g,v)(x)\\
&\leqslant \sup_{\mathscr{D}(\mathbb{R}^n)\ni Q\ni x} |Q|^{\frac{\alpha}{n}}\left(\fint_{3Q}|f(y_1)w_1(y_1)|^{\frac{p_1}{a_*}}dy_1\right)^{\frac{a_*}{p_1}}\left(\fint_{3Q}|g(y_2)w_2(y_2)|^{\frac{p_2}{a_*}}dy_2\right)^{\frac{a_*}{p_2}}\\
&\quad\times\left(\frac{|3Q|}{|Q|}\right)^{\frac{1}{aq_0}}|3Q|^{-\frac{1}{r_0}}\left(\frac{|Q|}{|3Q|}\right)^{\frac{1}{aq_0}}|3Q|^{\frac{1}{r_0}}\left(\fint_Q v(x)^{aq}dx\right)^{\frac{1}{aq}}\\
&\quad\times\left(\int_{3Q}w_1(y_1)^{-(\frac{p_1}{a})'}dy_1\right)^{\frac{1}{(\frac{p_1}{a})'}}\left(\int_{3Q}w_2(y_2)^{-(\frac{p_2}{a})'}dy_2\right)^{\frac{1}{(\frac{p_2}{a})'}}.
\end{aligned}$$

Using condition (2.4), we obtain

$$\begin{aligned}
&M^{\theta_1,\theta_2}_{\alpha,aq}(f,g,v)(x)\\
&\leqslant C[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}\sup_{\mathscr{D}(\mathbb{R}^n)\ni Q\ni x}|Q|^{\frac{\alpha}{n}-\frac{1}{r_0}}\left(\fint_{3Q}|f(y_1)w_1(y_1)|^{\frac{p_1}{a_*}}dy_1\right)^{\frac{a_*}{p_1}}\\
&\quad\times\left(\fint_{3Q}|g(y_2)w_2(y_2)|^{\frac{p_2}{a_*}}dy_2\right)^{\frac{a_*}{p_2}}\\
&\leqslant C[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}M_{\alpha-\frac{n}{r_0},\vec{P}/a_*}(fw_1,gw_2)(x).
\end{aligned}$$

This implies that

$$|Q_0|^{\frac{1}{q_0}}\left(\fint_{Q_0}M^{\theta_1,\theta_2}_{\alpha,aq}(f,g,v)(x)^q dx\right)^{\frac{1}{q}} \leqslant C[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}\|M_{\alpha-\frac{n}{r_0},\vec{P}/a_*}(fw_1,gw_2)\|_{\mathscr{M}^{q_0}_q}.$$

Since

$$\frac{1}{q_0}=\frac{1}{p_0}-\frac{(\alpha-\frac{n}{r_0})}{n}\quad\text{and}\quad\frac{q}{q_0}=\frac{p}{p_0},$$

using Lemma 3.4, we obtain

$$\begin{aligned}
&\|M_{\alpha-\frac{n}{r_0},\vec{P}/a_*}(fw_1,gw_2)\|_{\mathscr{M}^{q_0}_q}\\
&\leqslant C\sup_{Q\in\mathscr{D}(\mathbb{R}^n)}|Q|^{1/p_0}\left(\fint_Q(|f(x)|w_1(x))^{p_1}dx\right)^{1/p_1}\left(\fint_Q(|g(x)|w_2(x))^{p_2}dx\right)^{1/p_2}.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
&\|X_1\cdot v\|\vec{b}\|_{\mathrm{BMO}^N}\|_{\mathscr{M}^{q_0}_q}\\
&\leqslant C[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}\sup_{Q\in\mathscr{Q}}|Q|^{1/p_0}\left(\fint_Q(|f(x)|w_1(x))^{p_1}dx\right)^{1/p_1}\left(\fint_Q(|g(x)|w_2(x))^{p_2}dx\right)^{1/p_2}.
\end{aligned}$$

Estimate for X_2. A normalization allows us to assume that

$$\sup_{Q\in\mathscr{D}(\mathbb{R}^n)} |Q|^{1/p_0}\left(\fint_Q (|f(x)|w_1(x))^{p_1}dx\right)^{1/p_1}\left(\fint_Q (|g(x)|w_2(x))^{p_2}dx\right)^{1/p_2}=1. \quad (4.14)$$

Using Hölder's ineqeulity for $\theta_4\in(1,p_1)$ and $\theta_5\in(1,p_2)$ as in Lemma 3.3, we obtain

$$\begin{aligned}X_2 \leqslant & \sum_{A\subset\{1,\dots,m\}}\sum_{B\subset\{m+1,\dots,N\}}\sum_{\substack{Q\supsetneq Q_0\\ Q\in\mathscr{D}(\mathbb{R}^n)}} |Q|^{\frac{\alpha}{n}}\chi_Q(x)\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\\ &\times\left(\fint_{3Q} f(y_1)^{\theta_4}dy_1\right)^{\frac{1}{\theta_4}}\left(\fint_{3Q} g(y_2)^{\theta_5}dy_2\right)^{\frac{1}{\theta_5}}\\ &\times\left(\fint_{3Q}\prod_{i\in\bar{A}}|\lambda_i-b_i(y_1)|^{\theta_4'}dy_1\right)^{\frac{1}{\theta_4'}}\left(\fint_{3Q}\prod_{i\in\bar{B}}|\lambda_i-b_i(y_2)|^{\theta_5'}dy_2\right)^{\frac{1}{\theta_5'}}. \quad (4.15)\end{aligned}$$

Applying Lemma 3.1 and (4.15), we have

$$\begin{aligned}X_2\leqslant C &\sum_{A\subset\{1,\dots,m\}}\sum_{B\subset\{m+1,\dots,N\}}\prod_{i\in\bar{A}\cup\bar{B}}\|b_i\|_{\mathrm{BMO}}\\ &\times\sum_{\substack{Q\supsetneq Q_0\\ Q\in\mathscr{D}(\mathbb{R}^n)}}|Q|^{\frac{\alpha}{n}}\chi_Q(x)\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\left(\fint_{3Q} f(y_1)^{\theta_4}dy_1\right)^{\frac{1}{\theta_4}}\left(\fint_{3Q} g(y_2)^{\theta_5}dy_2\right)^{\frac{1}{\theta_5}}.\end{aligned}$$

Using Hölder's inequality for $\frac{p_1}{\theta_4}>1$ and $\frac{p_2}{\theta_5}>1$, we obtain

$$\begin{aligned}X_2\leqslant C&\sum_{A\subset\{1,\dots,m\}}\sum_{B\subset\{m+1,\dots,N\}}\prod_{i\in\bar{A}\cup\bar{B}}\|b_i\|_{\mathrm{BMO}}\sum_{\substack{Q\supsetneq Q_0\\ Q\in\mathscr{D}(\mathbb{R}^n)}}|Q|^{\frac{\alpha}{n}-\frac{1}{p_0}}\chi_Q(x)\\ &\times\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\left(\fint_{3Q} w_1(y_1)^{-\theta_4(\frac{p_1}{\theta_4})'}dy_1\right)^{\frac{1}{\theta_4(\frac{p_1}{\theta_4})'}}\\ &\times\left(\fint_{3Q} w_2(y_2)^{-\theta_5(\frac{p_2}{\theta_5})'}dy_2\right)^{\frac{1}{\theta_5(\frac{p_2}{\theta_5})'}}|3Q|^{\frac{1}{p_0}}\left(\fint_{3Q}(f(y_1)w_1(y_1))^{p_1}dy_1\right)^{\frac{1}{p_1}}\\ &\times\left(\fint_{3Q}(g(y_2)w_2(y_2))^{p_2}dy_2\right)^{\frac{1}{p_2}}. \quad (4.16)\end{aligned}$$

Since the assumption conditions $a\geqslant\frac{p_1}{(\theta_4(\frac{p_1}{\theta_4})')'}>1$ and $a\geqslant\frac{p_2}{(\theta_5(\frac{p_2}{\theta_5})')'}>1$ as in Lemma 3.3, using Hölder's inequality and (4.14), we show that

$$\begin{aligned}X_2\leqslant C&\sum_{A\subset\{1,\dots,m\}}\sum_{B\subset\{m+1,\dots,N\}}\prod_{i\in\bar{A}\cup\bar{B}}\|b_i\|_{\mathrm{BMO}}\sum_{\substack{Q\supsetneq Q_0\\ Q\in\mathscr{D}(\mathbb{R}^n)}}|Q|^{\frac{\alpha}{n}-\frac{1}{p_0}}\chi_Q(x)\prod_{i\in A\cup B}|b_i(x)-\lambda_i|\\ &\times\left(\fint_{3Q} w_1(y_1)^{-(\frac{p_1}{a})'}dy_1\right)^{\frac{1}{(\frac{p_1}{a})'}}\left(\fint_{3Q} w_2(y_2)^{-(\frac{p_2}{a})'}dy_2\right)^{\frac{1}{(\frac{p_2}{a})'}}.\end{aligned}$$

The integral of $X_2 \cdot v(x)$ on Q_0 is evaluated as follows:

$$\begin{aligned}
&|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} |X_2 \cdot v(x)|^q dx \right)^{\frac{1}{q}} \\
&\leqslant C \sum_{A \subset \{1,\dots,m\}} \sum_{B \subset \{m+1,\dots,N\}} \prod_{i \in \bar{A} \cup \bar{B}} \|b_i\|_{\mathrm{BMO}} \\
&\quad \times \sum_{\substack{Q \supsetneq Q_0 \\ Q \in \mathscr{D}(\mathbb{R}^n)}} |Q|^{\frac{\alpha}{n} - \frac{1}{p_0}} |Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} \prod_{i \in A \cup B} |b_i(x) - \lambda_i|^q v(x)^q dx \right)^{\frac{1}{q}} \\
&\quad \times \left(\fint_{3Q} w_1(y_1)^{-(\frac{p_1}{a})'} dy_1 \right)^{\frac{1}{(\frac{p_1}{a})'}} \left(\fint_{3Q} w_2(y_2)^{-(\frac{p_2}{a})'} dy_2 \right)^{\frac{1}{(\frac{p_2}{a})'}}.
\end{aligned}$$

Using Hölder's inequality for $a \geqslant \theta_6 > 1$ and Lemma 3.1, we have

$$\begin{aligned}
&|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} |X_2 \cdot v(x)|^q dx \right)^{\frac{1}{q}} \\
&\leqslant C \|\vec{b}\|_{\mathrm{BMO}^N} \sum_{\substack{Q \supsetneq Q_0 \\ Q \in \mathscr{D}(\mathbb{R}^n)}} |Q|^{\frac{\alpha}{n} - \frac{1}{p_0}} |Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} v(x)^{aq} dx \right)^{\frac{1}{aq}} \left(\fint_{3Q} w_1(y_1)^{-(\frac{p_1}{a})'} dy_1 \right)^{\frac{1}{(\frac{p_1}{a})'}} \\
&\quad \times \left(\fint_{3Q} w_2(y_2)^{-(\frac{p_2}{a})'} dy_2 \right)^{\frac{1}{(\frac{p_2}{a})'}} \\
&\leqslant C \|\vec{b}\|_{\mathrm{BMO}^N} \sum_{\substack{Q \supsetneq Q_0 \\ Q \in \mathscr{D}(\mathbb{R}^n)}} \left(\frac{|Q_0|}{|3Q|} \right)^{\frac{1}{q_0}(1 - \frac{1}{a})} \left(\frac{|Q_0|}{|3Q|} \right)^{\frac{1}{aq}} |3Q|^{\frac{1}{r_0}} \\
&\quad \times \left(\fint_{Q_0} v(x)^{aq} dx \right)^{\frac{1}{aq}} \left(\fint_{3Q} w_1(y_1)^{-(\frac{p_1}{a})'} dy_1 \right)^{\frac{1}{(\frac{p_1}{a})'}} \left(\fint_{3Q} w_2(y_2)^{-(\frac{p_2}{a})'} dy_2 \right)^{\frac{1}{(\frac{p_2}{a})'}}
\end{aligned}$$

Using condition (2.4), we obtain

$$\begin{aligned}
|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} |X_2 \cdot v(x)|^q dx \right)^{\frac{1}{q}} &\leqslant C \|\vec{b}\|_{\mathrm{BMO}^N} [v, \vec{w}]^{r_0, aq_0}_{aq, \vec{P}/a} \sum_{\substack{Q \supsetneq Q_0 \\ Q \in \mathscr{D}(\mathbb{R}^n)}} \left(\frac{|Q_0|}{|3Q|} \right)^{\frac{1}{q_0}(1 - \frac{1}{a})} \\
&\leqslant C \|\vec{b}\|_{\mathrm{BMO}^N} [v, \vec{w}]^{r_0, aq_0}_{aq, \vec{P}/a}.
\end{aligned}$$

This finishes proof of Theorem 2.2. $\square$

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