Journal of Mathematical Inequalities
Volume 14, Number 4 (2020), 1269–1284 doi:10.7153/jmi-2020-14-82

HERMITE–HADAMARD TYPE INEQUALITIES FOR OPERATOR (p,h)–CONVEX FUNCTIONS

ZHIWEI HAO AND LIBO LI*

(*Communicated by J. Mićić Hot*)

Abstract. Motivated by the recent work on convex functions and operator convex functions, we investigate the Hermite-Hadamard inequalities for operator (p,h)-convex functions. We also present the estimates of both sides of the Hermite-Hadamard type inequality for operator (p,h)-convex functions, where h is a non-negative function with $h(t)+h(1-t)\leqslant\kappa$ (κ is a positive constant) for $t\in(0,1)$. The results are new even for the commutative case. Applications for particular cases of these inequalities are also provided.

1. Introduction

For every real convex function f on $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(t)dt\leqslant\frac{f(a)+f(b)}{2}, \tag{1}$$

which is well known in the literature as the Hermite-Hadamard inequality. It was first published by Hermite in 1883 in an elementary journal and independently proved in 1893 by Hadamard in [14]. The Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f:[a,b]\longrightarrow\mathbb{R}$. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from the Hermite-Hadamard inequality under the utility of peculiar convex function f.

The Hermite-Hadamard inequality plays a crucial role in analysis and in other areas of applied mathematics as well. For more related results, generalizations, improvements and refinements to the Hermite-Hadamard inequality, see [1–5, 7–8, 15–17, 19–31]. In particular, Dragomir [9, 10] studied the Hermite-Hadamard inequality for convex function of self-adjoint operators in Hilbert spaces. Since then, the Hermite-Hadamard inequality for *operator convex function* has increasingly becoming one of the hot topics, see [11, 13, 18, 32] and the references therein.

Mathematics subject classification (2010): 47A63, 46C05, 26D15.
Keywords and phrases: Hermite-Hadamard inequality, operator (p,h)-convex function, h-convex function, positive operator.

* Corresponding author

This project was supported by the National Natural Science Foundation of China (No. 11801001), Scientific Research Fund of Hunan Provincial Education Department (No. 20C0780, 18C0332) and Scientific Research Fund of Hunan University of Science and Technology (No. E51997, E51998).

In 2018, a new type operator convex function, *operator* (p,h)*-convex function* (see Section 2), was proposed by Dinh and Vo [6]. They showed the Jensen inequality, Hansen-Pedersen type inequality and Choi-Davis-Jensen type inequality for operator (p,h)-convex function. To the best of our knowledge, there seems to be relatively little work on the Hermite-Hadamard inequality as well as its generalizations, improvements and refinements for operator (p,h)-convex functions.

The purpose of this paper is to present the Hermite-Hadamard type inequalities for operator (p,h)-convex functions. More precisely, we firstly establish the Hermite-Hadamard inequality for operator (p,h)-convex functions. In the second place, the improvement and refinement of Hermite-Hadamard inequality for operator (p,h)-convex functions will be studied. Finally, applications for particular cases of these inequalities are provided.

2. Preliminaries

In this paper, $\mathbf{B}(\mathscr{H})$ stands for the $\mathscr{C}^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with inner product $\langle\cdot,\cdot\rangle$. An operator $A\in\mathbf{B}(\mathscr{H})$ is positive and write $A\geqslant 0$ if $\langle Ax,x\rangle\geqslant 0$ for all $x\in\mathscr{H}$. Denote by $\mathbf{B}(\mathscr{H})^+$ the set of all positive operators in $\mathbf{B}(\mathscr{H})$. Over $\mathbf{B}(\mathscr{H})$ there exists a partial order relation by means of

$$A\geqslant B \quad \text{if} \quad A-B\geqslant 0$$

for self-adjoint operators $A,B\in\mathbf{B}(\mathscr{H})$.

Let $A\in\mathbf{B}(\mathscr{H})$ be a self-adjoint operator. The Gelfand map established a $*$-isometrically isomorphism Φ between the set $\mathscr{C}(\sigma(A))$ of all continuous functions defined on the spectrum of A, denoted by $\sigma(A)$, and the $\mathscr{C}^*$-algebra $\mathscr{C}^*(A)$ generated by A and the identity operator $\mathbb{I}_{\mathscr{H}}$ on $\mathscr{H}$ (see [12]) as follows:

For any $f,g\in\mathscr{C}(\sigma(A))$ and $\alpha,\beta\in\mathbb{C}$ (Complex numbers) we have

(1) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(2) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\overline{f})=\Phi(f)^*$;

(3) $\|\Phi(f)\|=\|f\|:=\sup_{t\in\sigma(A)} f(t)$;

(4) $\Phi(f_0)=\mathbb{I}_{\mathscr{H}}$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in\sigma(A)$.

With this notations, we define

$$f(A):=\Phi(f) \quad \text{for all} \quad f\in\mathscr{C}(\sigma(A))$$

and we call it the continuous functional calculus for a self-adjoint operator A.

If $A\in\mathbf{B}(\mathscr{H})$ is a self-adjoint operator and f is a real valued continuous function on $\sigma(A)$, then $f(t)\geqslant 0$ for any $t\in\sigma(A)$ implies that $f(A)\geqslant 0$, i.e. $f(A)$ is a positive operator on $\mathscr{H}$. Moreover, if both f and g are real valued functions on $\sigma(A)$, then the following important property holds: $f(t)\geqslant g(t)$ for any $t\in\sigma(A)$ implies that $f(A)\geqslant g(A)$ in the partial order of $\mathbf{B}(\mathscr{H})$.

A real valued continuous function f on an interval I is said to be operator convex if

$$f\big((1-\alpha)A+\alpha B\big) \leqslant (1-\alpha)f(A)+\alpha f(B)$$

in the partial order, for all $\alpha \in [0,1]$ and all self-adjoint operators $A,B \in \mathbf{B}(\mathscr{H})$ whose spectra are contained in I. For some fundamental results on operator convex function, see [12] and the references therein.

Now we introduce the operator (p,h)-convex function which generalizes the operator convex function. Assume that p is a positive constant, J is an interval in $\mathbb{R}^+$ such that $[0,1] \subset J$. Let K be a subset of $\mathbb{R}^+$, we say that K is a *p-convex*, if $\big(\lambda x^p+(1-\lambda)y^p\big)^{1/p} \in K$ for all $x,y \in K$ and $\lambda \in [0,1]$.

DEFINITION 2.1. [6] Let $h: J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$ and K be a p-convex subset of $\mathbb{R}^+$. A non-negative continuous function $f: K \to \mathbb{R}$ is said to be operator (p,h)-convex if

$$f\Big(\big[\alpha A^p+(1-\alpha)B^p\big]^{1/p}\Big) \leqslant h(\alpha)f(A)+h(1-\alpha)f(B) \tag{2}$$

for all $A,B \in \mathbf{B}(\mathscr{H})^+$ whose spectra are in K, and $\alpha \in (0,1)$.

REMARK 2.2. Note that the notation of operator (p,h)-convex function unifies and generalizes the known classes of operator h-convex function, operator convex function, operator s-convex function, operator P-function and operator Q-class function. To be precise,

(1) if $p=1$, one gets the definition of operator h-convex function on $\mathbf{B}(\mathscr{H})^+$;

(2) if $p=1$ and $h(t)=t$, one gets the definition of operator convex function on $\mathbf{B}(\mathscr{H})^+$;

(3) if $p=1$ and $h(t)=t^s$, one gets the definition of operator s-convex function on $\mathbf{B}(\mathscr{H})^+$;

(4) if $p=1$ and $h(t)=1$, one gets the definition of operator P-function on $\mathbf{B}(\mathscr{H})^+$;

(5) if $p=1$ and $h(t)=1/t$, one gets the definition of operator Q-class function on $\mathbf{B}(\mathscr{H})^+$.

The following result shows the connection between operator (p,h)-convex function and h-convex function, which will be useful in the sequel. We refer to [28] for the definition of h-convex function.

LEMMA 2.3. *Let $h: J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$ and $f: K \to \mathbb{R}$ be an operator (p,h)-convex function, then $\varphi_{x,A,B}: [0,1] \to \mathbb{R}$ is a h-convex function for all $A,B \in \mathbf{B}(\mathscr{H})^+$ whose spectra are in K and any $x \in \mathscr{H}$ with $\|x\|=1$, where*

$$\varphi_{x,A,B}(\alpha) = \Big\langle f\big(\big[\alpha A^p+(1-\alpha)B^p\big]^{1/p}\big)x,x\Big\rangle, \qquad \forall\, \alpha \in [0,1].$$

Proof. Let f be an operator (p,h)-convex function, then for $u,v\in[0,1]$ we have

$$\begin{aligned}
&\varphi_{x,A,B}(\alpha u+(1-\alpha)v)\\
&=\Big\langle f\Big(\Big[(\alpha u+(1-\alpha)v)A^p+\big(1-[\alpha u+(1-\alpha)v]\big)B^p\Big]^{1/p}\Big)x,x\Big\rangle\\
&=\Big\langle f\Big(\Big[\alpha\big(uA^p+(1-u)B^p\big)+(1-\alpha)\big(vA^p+(1-v)B^p\big)\Big]^{1/p}\Big)x,x\Big\rangle\\
&=\Big\langle f\Big(\Big[\alpha\big[\big(uA^p+(1-u)B^p\big)^{1/p}\big]^p+(1-\alpha)\big[\big(vA^p+(1-v)B^p\big)^{1/p}\big]^p\Big]^{1/p}\Big)x,x\Big\rangle\\
&\leqslant h(\alpha)\Big\langle f\big(\big[uA^p+(1-u)B^p\big]^{1/p}\big)x,x\Big\rangle+h(1-\alpha)\Big\langle f\big(\big[vA^p+(1-v)B^p\big]^{1/p}\big)x,x\Big\rangle\\
&=h(\alpha)\varphi_{x,A,B}(u)+h(1-\alpha)\varphi_{x,A,B}(v).
\end{aligned}$$

This implies that $\varphi_{x,A,B}$ is a h-convex function on $[0,1]$. □

3. Main results

In this section, we firstly establish the Hermite-Hadamard inequality for the class of operator (p,h)-convex functions. More importantly, we study refinements of Hermite-Hadamard inequality for operator (p,h)-convex functions. Finally, applications for particular cases of these inequalities are also provided. To simplify the writing, we always assume that $[0,1]\subset J$ is an interval in $\mathbb{R}^+$ and K is a p-convex.

THEOREM 3.1. (Hermite-Hadamard inequality for operator (p,h)-convex function) *Let $h:J\to\mathbb{R}$ be a non-negative function, $h(1/2)\neq 0$ and $f:K\to\mathbb{R}$ be an operator (p,h)-convex function, then for every $A,B\in\mathbf{B}(\mathscr{H})^+$ with $\sigma(A),\sigma(B)\in K$ we have*

$$\frac{1}{2h(\frac{1}{2})}f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)\leqslant\int_0^1 f\Big(\big[tA^p+(1-t)B^p\big]^{1/p}\Big)dt\leqslant\big(f(A)+f(B)\big)\int_0^1 h(\alpha)d\alpha.$$

For the proof of Theorem 3.1, we need the following Lemma, proved in [28].

LEMMA 3.2. *Let $h:J\to\mathbb{R}$ be a non-negative function with $h(1/2)\neq 0$. If $g:[0,1]\longrightarrow\mathbb{R}$ is a h-convex function, then for $a,b\in[0,1]$ with $a<b$, we have*

$$\frac{1}{2h(\frac{1}{2})}g\Big(\frac{a+b}{2}\Big)\leqslant\frac{1}{b-a}\int_a^b g(t)dt\leqslant\big[g(a)+g(b)\big]\int_0^1 h(\alpha)d\alpha.$$

Let us proceed now to prove Theorem 3.1.

Proof. Let f be an operator (p,h)-convex function. For every $x\in\mathscr{H}$ with $\|x\|=1$ and $\alpha\in[0,1]$, we suppose that

$$\varphi_{x,A,B}(\alpha)=\Big\langle f\big(\big[\alpha A^p+(1-\alpha)B^p\big]^{1/p}\big)x,x\Big\rangle.$$

It follows from Lemma 2.3 that $\varphi_{x,A,B}(\cdot)$ is a h-convex function on $[0,1]$. According to Lemma 3.2, we have

$$\frac{1}{2h(\frac{1}{2})}\varphi_{x,A,B}\Big(\frac{0+1}{2}\Big) \leqslant \frac{1}{1-0}\int_0^1 \varphi_{x,A,B}(t)dt \leqslant \big[\varphi_{x,A,B}(0)+\varphi_{x,A,B}(1)\big]\int_0^1 h(\alpha)d\alpha,$$

which implies that

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}\Big\langle f\Big(\Big[\frac{A^p+B^p}{2}\Big]^{1/p}\Big)x,x\Big\rangle &\leqslant \int_0^1 \Big\langle f\big(\big[tA^p+(1-t)B^p\big]^{1/p}\big)x,x\Big\rangle dt\\ &\leqslant \big[\langle f(A)x,x\rangle+\langle f(B)x,x\rangle\big]\int_0^1 h(\alpha)d\alpha\\ &= \big\langle \big(f(A)+f(B)\big)x,x\big\rangle \int_0^1 h(\alpha)d\alpha.\end{aligned}$$

Now, the desired result follows by taking into account that

$$\int_0^1 \Big\langle f\big(\big[tA^p+(1-t)B^p\big]^{1/p}\big)x,x\Big\rangle dt = \Big\langle\Big\{\int_0^1 f\big(\big[tA^p+(1-t)B^p\big]^{1/p}\big)dt\Big\}x,x\Big\rangle. \quad \square$$

Now we establish the refinement of the Hermite-Hadamard inequality for operator (p,h)-convex functions by Theorem 3.1.

THEOREM 3.3. *Let $h: J\to\mathbb{R}$ be a non-negative function, $h(1/2)\neq 0$ and $f: K\to\mathbb{R}$ be an operator (p,h)-convex function. For every $A,B\in\mathbf{B}(\mathscr{H})^+$ with $\sigma(A)$, $\sigma(B)\in K$, then the following inequalities hold:*

(i) *If n is a positive even, we have*

$$\begin{aligned}&\frac{1}{4h(\frac{1}{2})^2}f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)\\ &\leqslant \frac{1}{2nh(\frac{1}{2})}\sum_{m=0}^{n-1} f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big)\\ &\leqslant \int_0^1 f\Big(\big[tA^p+(1-t)B^p\big]^{1/p}\Big)dt\\ &\leqslant \frac{2}{n}\Big[\sum_{m=1}^{n-1} f\Big(\Big(\frac{(n-m)A^p+mB^p}{n}\Big)^{1/p}\Big)+\frac{f(A)+f(B)}{2}\Big]\int_0^1 h(t)dt.\end{aligned}$$

(ii) *If n is a positive odd, we have*

$$\begin{aligned}&\frac{n+2h(\frac{1}{2})-1}{n}\frac{1}{4h(\frac{1}{2})^2}f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)\\ &\leqslant \frac{1}{2nh(\frac{1}{2})}\sum_{m=0}^{n-1} f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big)\end{aligned}$$

$$\leqslant \int_0^1 f\Big(\big[tA^p+(1-t)B^p\big]^{1/p}\Big)dt$$
$$\leqslant \frac{2}{n}\left[\sum_{m=1}^{n-1} f\Big(\Big(\frac{(n-m)A^p+mB^p}{n}\Big)^{1/p}\Big)+\frac{f(A)+f(B)}{2}\right]\int_0^1 h(t)dt.$$

Proof. According to the definition of operator (p,h)-convex function, it holds for every $0\leqslant\lambda\leqslant 1$ and $m\in\{0,1,\ldots,n-1\}$ that

$$f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big) = f\Big(\Big(\frac{C^p_{\lambda,A,B}+D^p_{\lambda,A,B}}{2}\Big)^{1/p}\Big)$$
$$\leqslant h\Big(\frac{1}{2}\Big)\Big\{f(C_{\lambda,A,B})+f(D_{\lambda,A,B})\Big\},$$

where

$$C_{\lambda,A,B}=\Big[\lambda\frac{(n-m)A^p+mB^p}{n}+(1-\lambda)\frac{(n-m-1)A^p+(m+1)B^p}{n}\Big]^{1/p}$$

and

$$D_{\lambda,A,B}=\Big[(1-\lambda)\frac{(n-m)A^p+mB^p}{n}+\lambda\frac{(n-m-1)A^p+(m+1)B^p}{n}\Big]^{1/p}.$$

Because of the monotonicity and linearity of the integral operator, we have

$$f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big) \tag{3}$$
$$\leqslant h\Big(\frac{1}{2}\Big)\Big\{\int_0^1 f(C_{\lambda,A,B})d\lambda+\int_0^1 f(D_{\lambda,A,B})d\lambda\Big\}.$$

Let $t=1-\lambda$, we have

$$\int_0^1 f(C_{\lambda,A,B})d\lambda=\int_0^1 f(D_{\lambda,A,B})d\lambda. \tag{4}$$

Let $t=\frac{m+\lambda}{n}$ for $\lambda\in[0,1]$, we have

$$\int_0^1 f(D_{\lambda,A,B})d\lambda = \int_0^1 f\Big(\Big(\Big[1-\frac{m+\lambda}{n}\Big]A^p+\frac{m+\lambda}{n}B^p\Big)^{1/p}\Big)d\lambda \tag{5}$$
$$= n\int_{\frac{m}{n}}^{\frac{m+1}{n}} f\Big(\big[(1-t)A^p+tB^p\big]^{1/p}\Big)dt.$$

Applying Theorem 3.1, one can get

$$\int_0^1 f(C_{\lambda,A,B})d\lambda = \int_0^1 f\Bigg(\Bigg[\lambda\Big(\Big[\frac{(n-m)A^p+mB^p}{n}\Big]^{1/p}\Big)^p \tag{6}$$
$$+(1-\lambda)\Big(\Big[\frac{(n-m-1)A^p+(m+1)B^p}{n}\Big]^{1/p}\Big)^p\Bigg]^{1/p}\Bigg)d\lambda$$

$$\leqslant \Big[f\Big(\Big[\frac{(n-m)A^p+mB^p}{n}\Big]^{1/p}\Big) + f\Big(\Big[\frac{(n-m-1)A^p+(m+1)B^p}{n}\Big]^{1/p}\Big)\Big]\int_0^1 h(\alpha)d\alpha.$$

Hence, it follows from formulas (3), (4), (5) and (6) that

$$\begin{aligned} &f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big) \\ &\leqslant 2nh\Big(\frac{1}{2}\Big)\int_{\frac{m}{n}}^{\frac{m+1}{n}} f\Big(\big[(1-t)A^p+tB^p\big]^{1/p}\Big)dt \\ &\leqslant 2h\Big(\frac{1}{2}\Big)\Big[f\Big(\Big[\frac{(n-m)A^p+mB^p}{n}\Big]^{1/p}\Big) \\ &\quad + f\Big(\Big[\frac{(n-m-1)A^p+(m+1)B^p}{n}\Big]^{1/p}\Big)\Big]\int_0^1 h(\alpha)d\alpha. \end{aligned}$$

Sum these inequalities above over m, we get

$$\begin{aligned} &\sum_{m=0}^{n-1} f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big) \\ &\leqslant 2nh\Big(\frac{1}{2}\Big)\int_0^1 f\Big(\big[(1-t)A^p+tB^p\big]^{1/p}\Big)dt \\ &\leqslant 4h\Big(\frac{1}{2}\Big)\Big[\frac{f(A)+f(B)}{2}+\sum_{m=1}^{n-1} f\Big(\Big(\frac{(n-m)A^p+m)B^p}{n}\Big)^{1/p}\Big)\Big]\int_0^1 h(\alpha)d\alpha. \end{aligned} \tag{7}$$

Now we prove the inequality (i). If $n=2k$ is even, we get

$$\begin{aligned} &\sum_{m=0}^{n-1} f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big) \\ &= \sum_{m=0}^{2k-1} f\Big(\Big(\frac{(4k-2m-1)A^p+(2m+1)B^p}{4k}\Big)^{1/p}\Big) \\ &= \sum_{m=0}^{k-1}\Big[f\Big(\Big(\frac{(4k-2m-1)A^p+(2m+1)B^p}{4k}\Big)^{1/p}\Big) \\ &\quad + f\Big(\Big(\frac{(2m+1)A^p+(4k-2m-1)B^p}{4k}\Big)^{1/p}\Big)\Big]. \end{aligned}$$

By the definition of operator (p,h)-convex function, it follows

$$\begin{aligned} f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right) &= f\left(\left(\frac{\frac{(4k-2m-1)A^p+(2m+1)B^p}{4k}+\frac{(2m+1)A^p+(4k-2m-1)B^p}{4k}}{2}\right)^{1/p}\right) \\ &\leqslant h\left(\frac{1}{2}\right)\left[f\left(\left(\frac{(4k-2m-1)A^p+(2m+1)B^p}{4k}\right)^{1/p}\right)\right. \\ &\quad \left.+f\left(\left(\frac{(2m+1)A^p+(4k-2m-1)B^p}{4k}\right)^{1/p}\right)\right]. \end{aligned}$$

This implies that

$$\begin{aligned} k\cdot f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right) &= \sum_{m=0}^{k-1} f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right) \\ &\leqslant h\left(\frac{1}{2}\right)\sum_{m=0}^{k-1}\left[f\left(\left(\frac{(4k-2m-1)A^p+(2m+1)B^p}{4k}\right)^{1/p}\right)\right. \\ &\quad \left.+f\left(\left(\frac{(2m+1)A^p+(4k-2m-1)B^p}{4k}\right)^{1/p}\right)\right] \\ &= h\left(\frac{1}{2}\right)\sum_{m=0}^{n-1} f\left(\left(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\right)^{1/p}\right). \end{aligned}$$

Combining (7) with the above inequality, (i) holds. Finally, we prove the inequality (ii). If $n=2k+1$ is odd, we have

$$\begin{aligned} &\sum_{m=0}^{n-1} f\left(\left(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\right)^{1/p}\right) \\ &= \sum_{m=0}^{2k} f\left(\left(\frac{(4k-2m+1)A^p+(2m+1)B^p}{4k+2}\right)^{1/p}\right) \\ &= \sum_{m=0}^{k-1}\left[f\left(\left(\frac{(4k-2m+1)A^p+(2m+1)B^p}{4k+2}\right)^{1/p}\right)\right. \\ &\quad \left.+f\left(\left(\frac{(2m+1)A^p+(4k-2m-1)B^p}{4k+2}\right)^{1/p}\right)+f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right)\right] \end{aligned}$$

which implies that

$$\begin{aligned} &\left[k+h\left(\frac{1}{2}\right)\right]\cdot f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right) \\ &= \sum_{m=0}^{k-1} f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right)+h\left(\frac{1}{2}\right)f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right) \end{aligned}$$

$$\leqslant h\Big(\frac{1}{2}\Big)\sum_{m=0}^{k-1}\Big[f\Big(\Big(\frac{(4k-2m+1)A^p+(2m+1)B^p}{4k+2}\Big)^{1/p}\Big)$$
$$+f\Big(\Big(\frac{(2m+1)A^p+(4k-2m-1)B^p}{4k+2}\Big)^{1/p}\Big)\Big]+h\Big(\frac{1}{2}\Big)f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)$$
$$=h\Big(\frac{1}{2}\Big)\sum_{m=0}^{n-1}f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big).$$

Thus it follows that

$$\frac{n+2h(\frac{1}{2})-1}{n}\frac{1}{4h(\frac{1}{2})^2}f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)$$
$$\leqslant\frac{1}{2nh(\frac{1}{2})}\sum_{m=0}^{n-1}f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big).$$

Combining (7) with the above inequality, we obtain the result. □

If we put $n=1$ in Theorem 3.3 (ii), then we can obtain Theorem 3.1. As an application of Theorem 3.3, we state the following result, which is a refinement of Theorem 3.1.

COROLLARY 3.4. *Let $h:J\to\mathbb{R}$ be a non-negative function, $h(1/2)\neq 0$ and $f:K\to\mathbb{R}$ be an operator (p,h)-convex function. For every $A,B\in\mathbf{B}(\mathscr{H})^+$ with $\sigma(A),\sigma(B)\in K$, then the following inequality holds:*

$$\frac{1}{4h(\frac{1}{2})^2}f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)\leqslant\frac{1}{4h(\frac{1}{2})}\Big[f\Big(\Big(\frac{3A^p+B^p}{4}\Big)^{1/p}\Big)+f\Big(\Big(\frac{A^p+3B^p}{4}\Big)^{1/p}\Big)\Big]$$
$$\leqslant\int_0^1 f\Big(\big[tA^p+(1-t)B^p\big]^{1/p}\Big)dt$$
$$\leqslant\Big[f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)+\frac{f(A)+f(B)}{2}\Big]\int_0^1 h(t)dt$$
$$\leqslant\Big(C_h+\frac{1}{2}\Big)\big[f(A)+f(B)\big]\int_0^1 h(t)dt,$$

where

$$C_h=\min\Big\{h\Big(\frac{1}{2}\Big),2h\Big(\frac{1}{2}\Big)\int_0^1 h(t)dt\Big\}.$$

Proof. Setting $n=2$ in Theorem 3.3, we have

$$\frac{1}{4h(\frac{1}{2})^2}f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)\leqslant\frac{1}{4h(\frac{1}{2})}\Big[f\Big(\Big(\frac{3A^p+B^p}{4}\Big)^{1/p}\Big)+f\Big(\Big(\frac{A^p+3B^p}{4}\Big)^{1/p}\Big)\Big]$$
$$\leqslant\int_0^1 f\Big(\big[tA^p+(1-t)B^p\big]^{1/p}\Big)dt$$
$$\leqslant\Big[f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)+\frac{f(A)+f(B)}{2}\Big]\int_0^1 h(t)dt.$$

Since f is an operator (p,h)-convex function, it follows from Theorem 3.1 that

$$f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big) \leqslant 2h\Big(\frac{1}{2}\Big)\big[f(A)+f(B)\big]\int_0^1 h(t)dt.$$

According to the definition of operator (p,h)-convex function, we have

$$f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big) \leqslant h\Big(\frac{1}{2}\Big)\big[f(A)+f(B)\big],$$

which implies that

$$\Big[f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)+\frac{f(A)+f(B)}{2}\Big] \leqslant \min\Big\{h\Big(\frac{1}{2}\Big),2h\Big(\frac{1}{2}\Big)\int_0^1 h(t)dt\Big\}\big[f(A)+f(B)\big].$$

Hence the result is proved. □

Let $p=1$ in Corollary 3.4, then we immediately get the following result:

COROLLARY 3.5. *Let $h: J \to \mathbb{R}$ be a non-negative function, $h(1/2)\neq 0$ and $f: K \to \mathbb{R}$ be an operator h-convex function. For every $A,B \in \mathbf{B}(\mathscr{H})^+$ with $\sigma(A),\sigma(B) \in K$, then the following inequality holds:*

$$\begin{aligned}\frac{1}{4h(\frac{1}{2})^2} f\Big(\frac{A+B}{2}\Big) &\leqslant \frac{1}{4h(\frac{1}{2})}\Big[f\Big(\frac{3A+B}{4}\Big)+f\Big(\frac{A+3B}{4}\Big)\Big]\\ &\leqslant \int_0^1 f\big(tA+(1-t)B\big)dt\\ &\leqslant \Big[f\Big(\frac{A+B}{2}\Big)+\frac{f(A)+f(B)}{2}\Big]\int_0^1 h(t)dt\\ &\leqslant \Big(C_h+\frac{1}{2}\Big)\big[f(A)+f(B)\big]\int_0^1 h(t)dt,\end{aligned}$$

where

$$C_h = \min\Big\{h\Big(\frac{1}{2}\Big),2h\Big(\frac{1}{2}\Big)\int_0^1 h(t)dt\Big\}.$$

Now we give the generalization of Corollary 3.4 and Corollary 3.5 as follows.

THEOREM 3.6. *Let $h: J \to \mathbb{R}$ be a non-negative function, $h(1/2)\neq 0$ and $f: K \to \mathbb{R}$ be an operator (p,h)-convex function. For every $A,B \in \mathbf{B}(\mathscr{H})^+$ with $\sigma(A)$, $\sigma(B) \in K$ and for each $\lambda \in [0,1]$, then the following inequality holds:*

$$\begin{aligned}&\frac{1}{2h(\frac{1}{2})}\Big[(1-\lambda)f\Big(\Big(\frac{(1-\lambda)A^p+(1+\lambda)B^p}{2}\Big)^{1/p}\Big)+\lambda f\Big(\Big(\frac{(2-\lambda)A^p+\lambda B^p}{2}\Big)^{1/p}\Big)\Big]\\ &\leqslant \int_0^1 f\Big(\big[tA^p+(1-t)B^p\big]^{1/p}\Big)dt\end{aligned}$$

$$\leqslant \left[(1-\lambda)f(B)+\lambda f(A)+f\left(\left((1-\lambda)A^p+\lambda B^p\right)^{1/p}\right)\right]\int_0^1 h(t)dt$$
$$\leqslant \left[\left(h(1-\lambda)+\lambda\right)f(A)+\left(h(\lambda)+1-\lambda\right)f(B)\right]\int_0^1 h(t)dt.$$

Moreover, if $h(t)\leqslant \kappa t$ *for* $t\in(0,1)$ *and* κ *is a positive constant, then we have*

$$\frac{1}{2\kappa h(\frac{1}{2})}f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right)$$
$$\leqslant \frac{1}{2h(\frac{1}{2})}\left[(1-\lambda)f\left(\left(\frac{(1-\lambda)A^p+(1+\lambda)B^p}{2}\right)^{1/p}\right)+\lambda f\left(\left(\frac{(2-\lambda)A^p+\lambda B^p}{2}\right)^{1/p}\right)\right]$$
$$\leqslant \int_0^1 f\left(\left[tA^p+(1-t)B^p\right]^{1/p}\right)dt$$
$$\leqslant \frac{\kappa}{2}\left[\left(h(1-\lambda)+\lambda\right)f(A)+\left(h(\lambda)+1-\lambda\right)f(B)\right].$$

Proof. Let $C=\left((1-\lambda)A^p+\lambda B^p\right)^{1/p}$. It follows from Theorem 3.1 that

$$\frac{1}{2h(\frac{1}{2})}f\left(\left(\frac{B^p+C^p}{2}\right)^{1/p}\right)\leqslant \int_0^1 f\left(\left[tB^p+(1-t)C^p\right]^{1/p}\right)dt \qquad (8)$$
$$\leqslant \left[f(B)+f(C)\right]\int_0^1 h(t)dt$$

and

$$\frac{1}{2h(\frac{1}{2})}f\left(\left(\frac{C^p+A^p}{2}\right)^{1/p}\right)\leqslant \int_0^1 f\left(\left[tC^p+(1-t)A^p\right]^{1/p}\right)dt \qquad (9)$$
$$\leqslant \left[f(C)+f(A)\right]\int_0^1 h(t)dt.$$

We use the change of variables $x=\lambda+t-t\lambda$, $x=t\lambda$ for (8) and (9) with $\lambda\neq 1$ and $\lambda\neq 0$, respectively, we have

$$\int_0^1 f\left(\left[tB^p+(1-t)C^p\right]^{1/p}\right)dt=\frac{1}{1-\lambda}\int_\lambda^1 f\left(\left[(1-t)A^p+tB^p\right]^{1/p}\right)dt$$

and

$$\int_0^1 f\left(\left[tC^p+(1-t)A^p\right]^{1/p}\right)dt=\frac{1}{\lambda}\int_0^\lambda f\left(\left[(1-t)A^p+tB^p\right]^{1/p}\right)dt,$$

which implies that

$$\int_0^1 f\left(\left[tA^p+(1-t)B^p\right]^{1/p}\right)dt=\int_0^1 f\left(\left[(1-t)A^p+tB^p\right]^{1/p}\right)dt$$
$$=(1-\lambda)\int_0^1 f\left(\left[tB^p+(1-t)C^p\right]^{1/p}\right)dt+\lambda\int_0^1 f\left(\left[tC^p+(1-t)A^p\right]^{1/p}\right)dt$$

$$\leqslant \left[\lambda f(A)+(1-\lambda)f(B)+f(C)\right]\int_0^1 h(t)dt$$
$$\leqslant \left[\lambda f(A)+(1-\lambda)f(B)+h(1-\lambda)f(A)+h(\lambda)f(B)\right]\int_0^1 h(t)dt$$
$$=\left[(h(1-\lambda)+\lambda)f(A)+(h(\lambda)+1-\lambda)f(B)\right]\int_0^1 h(t)dt.$$

That is

$$\frac{1}{2h(\frac{1}{2})}\left[(1-\lambda)f\Big(\Big(\frac{(1-\lambda)A^p+(1+\lambda)B^p}{2}\Big)^{1/p}\Big)+\lambda f\Big(\Big(\frac{(2-\lambda)A^p+\lambda B^p}{2}\Big)^{1/p}\Big)\right]$$
$$=\frac{1-\lambda}{2h(\frac{1}{2})}f\Big(\Big(\frac{B^p+C^p}{2}\Big)^{1/p}\Big)+\frac{\lambda}{2h(\frac{1}{2})}f\Big(\Big(\frac{C^p+A^p}{2}\Big)^{1/p}\Big)$$
$$\leqslant \int_0^1 f\Big(\left[tA^p+(1-t)B^p\right]^{1/p}\Big)dt$$
$$\leqslant \left[(h(1-\lambda)+\lambda)f(A)+(h(\lambda)+1-\lambda)f(B)\right]\int_0^1 h(t)dt.$$

Hence we obtain the first result. Now we turn to last part of the proof. Let f be an operator (p,h)-convex function, we have

$$f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)$$
$$=f\Big(\Big((1-\lambda)\frac{(1-\lambda)A^p+(1+\lambda)B^p}{2}+\lambda\frac{(2-\lambda)A^p+\lambda B^p}{2}\Big)^{1/p}\Big)$$
$$=f\Big(\Big((1-\lambda)\Big[\Big(\frac{(1-\lambda)A^p+(1+\lambda)B^p}{2}\Big)^{1/p}\Big]^p+\lambda\Big[\Big(\frac{(2-\lambda)A^p+\lambda B^p}{2}\Big)^{1/p}\Big]^p\Big)^{1/p}\Big)$$
$$\leqslant h(1-\lambda)f\Big(\Big(\frac{(1-\lambda)A^p+(1+\lambda)B^p}{2}\Big)^{1/p}\Big)+h(\lambda)f\Big(\Big(\frac{(2-\lambda)A^p+\lambda B^p}{2}\Big)^{1/p}\Big)$$
$$=h(1-\lambda)f\Big(\Big(\frac{B^p+C^p}{2}\Big)^{1/p}\Big)+h(\lambda)f\Big(\Big(\frac{C^p+A^p}{2}\Big)^{1/p}\Big).$$

Since $h(t)\leqslant \kappa t$ for $t\in(0,1)$, we can obtain that

$$f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)\leqslant \kappa\left[(1-\lambda)f\Big(\Big(\frac{B^p+C^p}{2}\Big)^{1/p}\Big)+\lambda f\Big(\Big(\frac{C^p+A^p}{2}\Big)^{1/p}\Big)\right].$$

This means that

$$\frac{1}{2\kappa h(\frac{1}{2})}f\Big(\Big(\frac{A^p+B^p}{2}\Big)^{1/p}\Big)$$
$$\leqslant \frac{1}{2h(\frac{1}{2})}\left[(1-\lambda)f\Big(\Big(\frac{(1-\lambda)A^p+(1+\lambda)B^p}{2}\Big)^{1/p}\Big)+\lambda f\Big(\Big(\frac{(2-\lambda)A^p+\lambda B^p}{2}\Big)^{1/p}\Big)\right]$$
$$\leqslant \int_0^1 f\Big(\left[tA^p+(1-t)B^p\right]^{1/p}\Big)dt$$
$$\leqslant \frac{\kappa}{2}\Big[\big(h(1-\lambda)+\lambda\big)f(A)+\big(h(\lambda)+1-\lambda\big)f(B)\Big]. \quad \square$$

If the non-negative function h satisfies with $h(t)+h(1-t)\leqslant\kappa$ (κ is a positive constant) for each $t\in(0,1)$, in Theorem 3.3, we have the following new result for operator (p,h)-convex functions.

THEOREM 3.7. *Let $h:J\to\mathbb{R}$ be a non-negative function, $h(1/2)\neq 0$ with $h(t)+h(1-t)\leqslant\kappa$ (κ is a positive constant) for each $t\in(0,1)$, and $f:K\to\mathbb{R}$ be an operator (p,h)-convex function. For every $A,B\in\mathbf{B}(\mathscr{H})^+$ with $\sigma(A),\sigma(B)\in K$, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{2nh(\frac{1}{2})}\sum_{m=0}^{n-1} f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big)\\
&\leqslant \int_0^1 f\Big(\big[tA^p+(1-t)B^p\big]^{1/p}\Big)dt\\
&\leqslant \frac{\kappa}{n}\Big[\sum_{m=1}^{n-1} f\Big(\Big(\frac{(n-m)A^p+(m)B^p}{n}\Big)^{1/p}\Big)+\frac{f(A)+f(B)}{2}\Big]\\
&\leqslant \frac{(n-1)\kappa^2+\kappa}{2n}\big[f(A)+f(B)\big].
\end{aligned}$$

Proof. Since $h:(0,1)\to\mathbb{R}$ is a non-negative function and $h(t)+h(1-t)\leqslant\kappa$ for any $t\in(0,1)$, we have

$$\begin{aligned}
\int_0^1 h(t)dt &= \frac{1}{2}\Big(\int_0^1 h(t)dt+\int_0^1 h(1-t)dt\Big)=\frac{1}{2}\int_0^1 \big(h(t)+h(1-t)\big)dt \qquad (10)\\
&\leqslant \frac{1}{2}\int_0^1 \kappa\, dt=\frac{\kappa}{2}.
\end{aligned}$$

From Theorem 3.3 and the inequality (10), we have

$$\begin{aligned}
&\frac{1}{2nh(\frac{1}{2})}\sum_{m=0}^{n-1} f\Big(\Big(\frac{(2n-2m-1)A^p+(2m+1)B^p}{2n}\Big)^{1/p}\Big)\\
&\leqslant \int_0^1 f\Big(\big[tA^p+(1-t)B^p\big]^{1/p}\Big)dt\\
&\leqslant \frac{2}{n}\Big[\sum_{m=1}^{n-1} f\Big(\Big(\frac{(n-m)A^p+mB^p}{n}\Big)^{1/p}\Big)+\frac{f(A)+f(B)}{2}\Big]\int_0^1 h(t)dt\\
&\leqslant \frac{\kappa}{n}\Big[\sum_{m=1}^{n-1} f\Big(\Big(\frac{(n-m)A^p+mB^p}{n}\Big)^{1/p}\Big)+\frac{f(A)+f(B)}{2}\Big].
\end{aligned}$$

According to the definition of operator (p,h)-convex function and $h(t)+h(1-t)\leqslant\kappa$ for $t\in(0,1)$, we have

$$\begin{aligned}
&\frac{\kappa}{n}\Big[\sum_{m=0}^{n-1} f\Big(\Big(\frac{(n-m)A^p+m^p}{n}\Big)^{1/p}\Big)+\frac{f(A)+f(B)}{2}\Big] \qquad (11)\\
&=\frac{\kappa}{2n}\sum_{m=1}^{n-1}\Big[f\Big(\Big(\frac{(n-m)A^p+mB^p}{n}\Big)^{1/p}\Big)+f\Big(\Big(\frac{mA^p+(n-m)B^p}{n}\Big)^{1/p}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa\left[f(A)+f(B)\right]}{2n}\\
&\leqslant \frac{\kappa}{2n}\sum_{m=1}^{n-1}\left[h\Big(\frac{n-m}{n}\Big)f(A)+h\Big(\frac{m}{n}\Big)f(B)+h\Big(\frac{m}{n}\Big)f(A)+h\Big(\frac{n-m}{n}\Big)f(B)\right]\\
&\quad+\frac{\kappa\left[f(A)+f(B)\right]}{2n}\\
&=\frac{\kappa\left[f(A)+f(B)\right]}{2n}\left[1+\sum_{m=1}^{n-1}\Big(h\Big(\frac{n-m}{n}\Big)+h\Big(\frac{m}{n}\Big)\Big)\right]\\
&\leqslant \frac{\kappa\left[f(A)+f(B)\right]}{2n}\left[1+\sum_{m=1}^{n-1}\kappa\right]=\frac{(n-1)\kappa^2+\kappa}{2n}\left[f(A)+f(B)\right].
\end{aligned}
$$

This completes the proof. □

Note that the condition $h(t)+h(1-t)\leqslant\kappa$ is weaker than $h(t)\leqslant\kappa t$ (in Theorem 3.6) for $t\in(0,1)$. Indeed, if $h(t)\leqslant\kappa t$ holds for $t\in(0,1)$, then, for any $t\in(0,1)$ we have $h(t)+h(1-t)\leqslant\kappa t+\kappa(1-t)=\kappa$. But the inverse is not true. As applications, we give the following two corollaries.

COROLLARY 3.8. *Suppose that $s\in(0,1]$. Let $f:K\to\mathbb{R}$ be an operator s-convex function. For every $A,B\in\mathbf{B}(\mathscr{H})^+$ with $\sigma(A),\sigma(B)\in K$, then the following inequality holds:*

$$
\begin{aligned}
4^{s-1}f\Big(\frac{A+B}{2}\Big)&\leqslant\frac{2^{s-1}}{n}\sum_{m=0}^{n-1}f\Big(\frac{(2n-2m-1)A+(2m+1)B}{2n}\Big)\\
&\leqslant\int_0^1 f\big(tA+(1-t)B\big)dt\\
&\leqslant\frac{2}{n(s+1)}\Big[\sum_{m=1}^{n-1}f\Big(\frac{(n-m)A+mB}{n}\Big)+\frac{f(A)+f(B)}{2}\Big]\\
&\leqslant\frac{2n-1}{n(s+1)}\big[f(A)+f(B)\big].
\end{aligned}
$$

Proof. Let $h(t)=t^s$ for $0\leqslant t\leqslant 1$, it is easy to calculate that

$$n+2h(1/2)-1=n+2^{1-s}-1\geqslant n,\quad 4h(1/2)^2=4^{1-s}$$

which means

$$\frac{n+2h(\frac{1}{2})-1}{n}\geqslant 1,\quad \frac{1}{4h(\frac{1}{2})^2}=4^{s-1},\quad \int_0^1 h(t)dt=\frac{1}{1+s}$$

and, $h(t)+h(1-t)=t^s+(1-t)^s\leqslant 2$ for each $t\in[0,1]$. Combining Theorem 3.3 and the inequality (11), we can obtain the result. □

From results above, one can easily get the refinement of the Hermite-Hadamard inequality for operator convex function.

COROLLARY 3.9. *Let* $f : K \to \mathbb{R}$ *be an operator convex function. For every* $A, B \in \mathbf{B}(\mathscr{H})^+$ *with* $\sigma(A), \sigma(B) \in K$*, the following inequality holds*

$$f\Big(\frac{A+B}{2}\Big) \leqslant \frac{1}{n}\sum_{m=0}^{n-1} f\Big(\frac{(2n-2m-1)A+(2m+1)B}{2n}\Big) \leqslant \int_0^1 f\big(tA+(1-t)B\big)dt \leqslant \frac{1}{n}\Big[\sum_{m=1}^{n-1} f\Big(\frac{(n-m)A+mB}{n}\Big)+\frac{f(A)+f(B)}{2}\Big] \leqslant \frac{f(A)+f(B)}{2}. \tag{12}$$

Proof. Applying Theorem 3.7 and Corollary 3.8 by setting $h(t) = t$, we can get inequality (12). □

Acknowledgements. The authors are thankful to the referee for giving valuable comments and suggestions which helped to improve the final version of this paper.

REFERENCES

[1] B. AHMAD, A. ALSAEDI, M. KIRANE AND B. TOREBEK, *Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals*, J. Comput. Appl. Math. 353 (2019), 120–129.

[2] M. AWAN, M. NOOR, T. DU AND K. NOOR, *New refinements of fractional Hermite-Hadamard inequality*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. 113 (1) (2019), 21–29.

[3] M. BACHAR, A. GUESSAB, O. MOHAMMED AND Y. ZAIM, *New cubature formulas and Hermite-Hadamard type inequalities using integrals over some hyperplanes in the d-dimensional hyper-rectangle*, Appl. Math. Comput. 315 (2017), 347–362.

[4] M. BESSENYEI, *The Hermite-Hadamard inequality in Beckenbach's setting*, J. Math. Anal. Appl. 364 (2010), 366–383.

[5] H. CHEN AND U. KATUGAMPOLA, *Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals*, J. Math. Anal. Appl. 446 (2017), 1274–1291.

[6] T. DINH AND B. VO, *Some inequalities for operator (p,h)-convex functions*, Linear Multilinear A. 66 (2018), 580–592.

[7] S. DRAGOMIR, *Two mappings in connection to Hadamard's inequality*, J. Math. Anal. Appl. 167 (1992), 49–56.

[8] S. DRAGOMIR AND C. PEARCE, *Selected topics on Hermite-Hadamard inequalities and applications*, Victoria University, 2000.

[9] S. DRAGOMIR, *Hermite-Hadamard's type inequalities for operator convex functions*, Appl. Math. Comput. 218 (2011), 766–772.

[10] S. DRAGOMIR, *Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Linear Algebra Appl. 436 (2012), 1503–1515.

[11] Y. ERDAŞ, E. ÜNLÜYOL AND S. SALAŞ, *Some New Inequalities of Operator m-Convex Functions and Applications for Synchronous-Asynchronous Functions*, Complex Anal. Oper. Th. 13 (8) (2019), 3871–3881.

[12] T. FURUTA, J. MIĆIĆ, J. PEČARIĆ AND Y. SEO, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[13] A. GHAZANFARI AND A. BARANI, *Some Hermite-Hadamard type inequalities for the product of two operator preinvex functions*, Banach J. Math. Anal. 9 (2015), 9–20.

[14] J. HADAMARD, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl. 58 (1893), 171–216.
[15] S. HWANG, S. YEH AND K. TSENG, *Refinements and similar extensions of Hermite-Hadamard inequality for fractional integrals and their applications*, Appl. Math. Comput. 249 (2014), 103–113.
[16] İ. İŞCAN, *Generalization of different type integral inequalities for s-convex functions via fractional integrals*, Appl. Anal. 93 (2014), 1846–1862.
[17] İ. İŞCAN, *On generalization of different type inequalities for harmonically quasi-convex functions via fractional integrals*, Appl. Math. Comput. 275 (2016), 287–298.
[18] R. KAUR, M. MOSLEHIAN, M. SINGH AND C. CONDE, *Further refinements of the Heinz inequality*, Linear Algebra Appl. 447 (2014), 26–37.
[19] U. KIRMACI, M. BAKULA, M. ÖZDEMIR AND J. PEČARIĆ, *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput. 193 (2007), 26–35.
[20] D. KOTRYS, *Hermite-Hadamard inequality for convex stochastic processes*, Aequ. Math. 83 (2012), 143–151.
[21] L. LI AND Z. HAO, *On Hermite-Hadamard inequality for h-convex stochastic processes*, Aequ. Math. 91 (2017), 909–920.
[22] J. MAKÓ AND Z. PÁLES, *Approximate Hermite-Hadamard type inequalities for approximately convex functions*, Math. Inequal. Appl. 16 (2013), 507–526.
[23] K. MEHREZ AND P. AGARWAL, *New Hermite-Hadamard type integral inequalities for convex functions and their applications*, J. Comput. Appl. Math. 350 (2019), 274–285.
[24] M. NIEZGODA, *Fejér and Hermite-Hadamard type results for H-invex functions with applications*, Positivity 23 (2019), 531–543.
[25] M. NOWICKA AND A. WITKOWSKI, *A refinement of the right-hand side of the Hermite-Hadamard inequality for simplices*, Aequ. Math. 91 (2017), 121–128.
[26] A. OLBRYŚ, *On the $\mathbb{K}$-Riemann integral and Hermite-Hadamard inequalities for $\mathbb{K}$-convex functions*, Aequ. Math. 91 (2017), 429–444.
[27] M. ÖZDEMIR, M. AVCI AND H. KAVURMACI, *Hermite-Hadamard-type inequalities via (α,m)-convexity*, Comput. Math. Appl. 61 (2011), 2614–2620.
[28] M. SARIKAYA, A. SAGLAM AND H. YILDIRIM, *On some Hadamard-type inequalities for h-convex functions*, J. Math. Inequal. 2 (2008), 335–341.
[29] E. SET, İ. İŞCAN, M. SARIKAYA AND M. ÖZDEMIR, *On new inequalities of Hermite-Hadamard-Fejér type for convex functions via fractional integrals*, Appl. Math. Comput. 259 (2015), 875–881.
[30] S. STEINERBERGER, *The Hermite-Hadamard Inequality in Higher Dimensions*, J. Geom. Anal. 30 (1) (2020), 466–483.
[31] T. SZOSTOK, *Functional inequalities involving numerical differentiation formulas of order two*, B. Malays. Math. Sci. So. 41 (2018), 2053–2066.
[32] A. TAGHAVI, V. DARVISH, H. NAZARI AND S. DRAGOMIR, *Hermite-Hadamard type inequalities for operator geometrically convex functions*, Monatsh. Math. 181 (2016), 187–203.

(Received October 2, 2019)

Zhiwei Hao
College of Mathematics and Computing Science
Hunan University of Science and Technology
Xiangtan, Hunan 411201, China
e-mail: zw_hao8@163.com

Libo Li
College of Mathematics and Computing Science
Hunan University of Science and Technology
Xiangtan, Hunan 411201, China
e-mail: lilibohnu@163.com