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ON THE ERDÖS–MORDELL INEQUALITY FOR TRIANGLES IN TAXICAB GEOMETRY

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Abstract. In this paper the Erdös-Mordell's inequality is examined for the case of a triangle ABC in the taxicab plane geometry. It is shown that the Erdös-Mordell's inequality $R_A+R_B+R_C \geqslant w(r_a+r_b+r_c)$ holds for triangles with appropriate positions for its points A, B and C, if $w=3/2$.

1. Introduction

Let the distance between two points, as well as the distance between a line and a point be defined in the Euclidean plane. Then, for a triangle ABC in such a plane the Erdös-Mordell's inequality holds [4], [19]:

$$R_A+R_B+R_C \geqslant 2\left(r_a+r_b+r_c\right) \tag{1}$$

where R_A, R_B and R_C are distances from the interior point M of $\triangle ABC$ to vertices A, B and C respectively and r_a, r_b and r_c are distances from the point M of the triangle to the corresponding edges which contain the vertices of $\triangle ABC$ (Fig. 1).

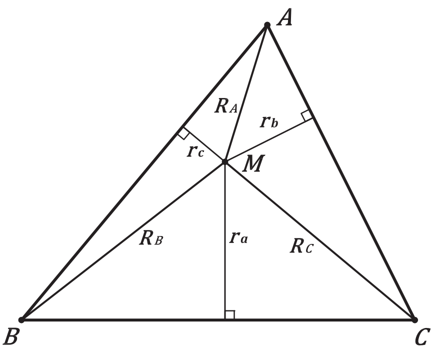

Figure 1: A geometric illustration of the Erdös-Mordell inequality in $\triangle ABC$

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Let there be two points, $A(x_A, y_A)$ and $B(x_B, y_B)$, then the distance between them in taxicab geometry is defined as:

$$d_1(A,B) = |x_A - x_B| + |y_A - y_B|. \tag{2}$$

This distance is also called the Manhattan or city block distance. This metric is a special case of the Minkowski metric of order k (where $k \geqslant 1$) which is defined by the following formula:

$$d_k(A,B) = \left(|x_A - x_B|^k + |y_A - y_B|^k\right)^{\frac{1}{k}} \tag{3}$$

The Minkowski metric contains in itself the taxicab metric for the value $k = 1$ and the Euclidean metric for $k = 2$ [9]. The term "taxicab" was first introduced by K. Menger [17]. A graphical representation of distances between points A and B is given in Fig. 2, in taxicab metric with d_1 (dashed/long dashed lines) and in Euclidean metric with d_2 (continuous line).

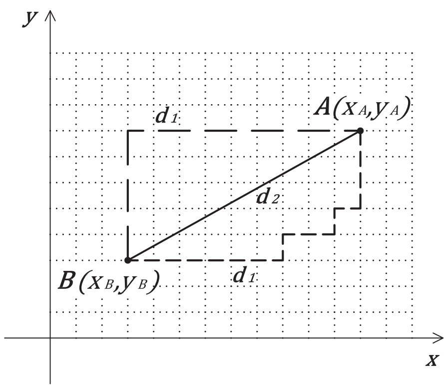

Figure 2: A geometric illustration of the Minkowski and the Euclidean distances between two points

In the rest of this paper, only taxicab distances are considered.

Let the $\triangle ABC$ be a triangle with vertices $A(0,r)$, $B(p,0)$, $C(q,0)$, $p \neq q$, $r \neq 0$. Without diminishing generality, let $p < q$. We denote by $M(x,y)$ an arbitrary point in the plane of the triangle $\triangle ABC$ (Fig. 1). The Taxicab distance from the point M to the points A, B and C, are given by functions:

$$\begin{aligned} R_A &= d_1(M,A) = |x| + |y - r|, \\ R_B &= d_1(M,B) = |x - p| + |y|, \\ R_C &= d_1(M,C) = |x - q| + |y|. \end{aligned} \tag{4}$$

Recently, general formulae for distance in taxicab geometry were analyzed in the paper [2]. Authors Kaya et al. [7] define the distance of a point to a line in taxicab plane geometry with the following statement:

LEMMA 1. *Distance of point $M(x_M, y_M)$ to the line $\ell: ax+by+c=0$ in the Taxicab plane is*:

$$d_1(M,\ell) = \frac{|ax_M + by_M + c|}{\max\{|a|,|b|\}}. \tag{5}$$

Let us notice that

$$r_a = d_1(M,\ell_{BC}),\ r_b = d_1(M,\ell_{AC}),\ r_c = d_1(M,\ell_{AB}). \tag{6}$$

Based on (4) and (6), the Erdös-Mordell's inequality (1) for $\triangle ABC$ in taxicab metric is defined by the following relation:

$$|x|+|y-r|+|x-p|+|x-q|+2|y| \geqslant 2\left(|y| + \frac{|qr-rx-qy|}{\max\{|r|,|q|\}} + \frac{|pr-rx-py|}{\max\{|r|,|p|\}}\right). \tag{7}$$

Inequalities in the taxicab geometry are the topic of recent research, see e.g [8]. Let us emphasize that the topic of the Erdös-Mordell inequality is current, as it has been shown in the papers [3], [5], [10]–[15], [23] and books [1] and [18]. V. Pambuccian proved that, in the plane of absolute geometry, the Erdös-Mordell inequality is an equivalent to the non-positive curvature [21]. In the paper [16] is given an extension of the Erdös-Mordell inequality on the interior of the Erdös-Mordell curve. In relation to the Erdös-Mordell inequality N. Dergiades in the paper [3] proved one extension of the Erdös-Mordell type inequality. Most notably, the Erdös-Mordell inequality has been considered in the taxicab plane geometry by N. Sönmez who has shown that (1) is a strict inequality: $R_A + R_B + R_C > 2(r_a + r_b + r_c)$, [22]. In this paper we prove that the conclusion reached by N. Sönmez is incorrect. That shall be shown through the following example.

EXAMPLE 1. (counterexample) Let the vertices of $\triangle ABC$ be given with $p = -20, q = 40, r = 30$ and let point $M(0,m)$ be defined with $m = 2$ (Fig. 3). The taxicab distance from the point M to the vertices of $\triangle ABC$ is given by (4) and the distance from point M to the lines $\ell_{BC}: y = 0$, $\ell_{AC}: -rx-qy+qr = 0$ and $\ell_{AB}: -rx-py+pr = 0$ is given by (5):

$$\begin{aligned} R_A &= d_1(M,A) = 28, \quad R_B = d_1(M,B) = 22, \quad R_C = d_1(M,C) = 42, \\ r_a &= d_1(M,\ell_{BC}) = 2, \quad r_b = d_1(M,\ell_{AC}) = 28, \quad r_c = d_1(M,\ell_{AB}) = \frac{56}{3}. \end{aligned} \tag{8}$$

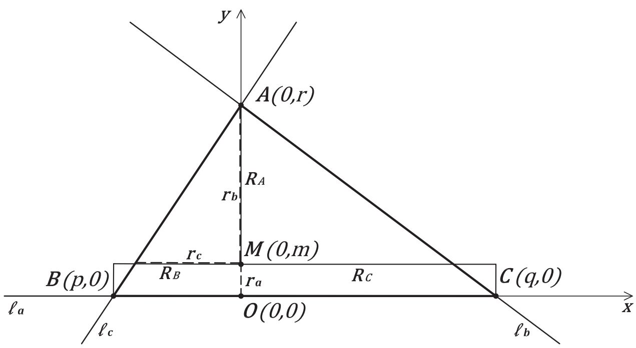

Figure 3: A geometric illustration of the counterexample

From (8) we obtain $L = R_A + R_B + R_C = 92$ and $R = r_a + r_b + r_c = \frac{146}{3}$. In the case of the Erdös-Mordell inequality, it holds that $L \geqslant 2R$ i.e $92 \geqslant 97.\overline{3}$. From this follows that the Erdös-Mordell inequality does not hold for all interior points of $\triangle ABC$. □

In the rest of this paper, the Erdös-Mordell inequality in taxicab geometry is considered in the form:

$$R_A + R_B + R_C \geqslant w\,(r_a + r_b + r_c)\,, \tag{9}$$

where the positive real number w is defined as such that the previous inequality holds for all interior points of $\triangle ABC$. The main goal of the paper is to, for all positive values of the weight coefficient w, determine a upper bound $\mathfrak{M}$ such that the Erdös-Mordell inequality holds for $0 < w \leqslant \mathfrak{M}$.

2. The main results

The Erdös-Mordell inequality in taxicab plane geometry has the following form:

$$|x| + |y-r| + |x-p| + |x-q| + 2|y| \geqslant w\left(|y| + \frac{|qr-rx-qy|}{\max\{|r|,|q|\}} + \frac{|pr-rx-py|}{\max\{|r|,|p|\}}\right). \tag{10}$$

It should be noted that the Erdös-Mordell inequality in the taxicab plane geometry defined by (10) refers to triangles ABC with the appropriate positions of points $A(0,r)$, $B(p,0)$ and $C(q,0)$ in two cases. The first case is when coordinates p, q and r are positive and the second case is when the p coordinate is negative, with positive q and r coordinates. Furthermore, we do not consider the general position of the triangle in the taxicab plane nor the rotation of such a triangle to $\triangle ABC$.

1° We analyze $\triangle ABC$ with $p,q,r > 0$ (see Fig. 4), then, for all interior points of the triangle holds:

$$\begin{aligned} |x| = x, \qquad |x-p| = \begin{cases} p-x : x < p \\ x-p : x \geqslant p \end{cases}, \qquad |x-q| = q-x, \\ |y| = y, \qquad |y-r| = r-y, \\ |qr-rx-qy| = qr-rx-qy, \qquad |pr-rx-py| = -pr+rx+py. \end{aligned} \tag{11}$$

Then, the form of the Erdös-Mordell inequality (10) becomes:

$$\begin{cases} q+r+y+p-x \geqslant w\left(y + \dfrac{qr-rx-qy}{\max\{r,q\}} + \dfrac{-pr+rx+py}{\max\{r,p\}}\right) : x < p \\[2ex] q+r+y+x-p \geqslant w\left(y + \dfrac{qr-rx-qy}{\max\{r,q\}} + \dfrac{-pr+rx+py}{\max\{r,p\}}\right) : x \geqslant p \end{cases} \tag{12}$$

Symmetric positions of $\triangle ABC$ relative to the coordinate axes can be analogously considered.

2° We analyze $\triangle ABC$ with $p<0$ and $q,r>0$ (see Fig. 4), then, for all interior points of the triangle holds:

$$|x|=\begin{cases}-x: x<0\\ x \;: x\geqslant 0\end{cases}, \qquad |x-p|=x-p, \qquad |x-q|=q-x, \qquad |y|=y, \qquad |y-r|=r-y,$$
$$|qr-rx-qy|=qr-rx-qy, \qquad |pr-rx-py|=-pr+rx+py. \tag{13}$$

Then, the form of the Erdös-Mordell inequality (10) becomes:

$$\begin{cases}-p+q+r+y-x\geqslant w\left(y+\dfrac{qr-rx-qy}{\max\{r,q\}}+\dfrac{-pr+rx+py}{\max\{r,-p\}}\right): x<0\\ -p+q+r+y+x\geqslant w\left(y+\dfrac{qr-rx-qy}{\max\{r,q\}}+\dfrac{-pr+rx+py}{\max\{r,-p\}}\right): x\geqslant 0\end{cases} \tag{14}$$

As in case 1°, symmetric positions of $\triangle ABC$ relative to the coordinate axes can be analogously considered.

Let us notice that for point $A(0,r)$, there exist the following subcases:

1°	$\langle \boldsymbol{a}\rangle$ $0<r\leqslant p<q$,	$\langle \boldsymbol{b}\rangle$ $0\leqslant p<r\leqslant q$,	$\langle \boldsymbol{c}\rangle$ $0\leqslant p<q<r$;
For this subcase, see Fig. $4/1^\circ$ with representations:	$\langle \boldsymbol{a}\rangle$ long and double-short dashed line,	$\langle \boldsymbol{b}\rangle$ dashed line,	$\langle \boldsymbol{c}\rangle$ continuous line;
2°	$\langle \boldsymbol{a}\rangle$ $0<r\leqslant -p\leqslant q$,	$\langle \boldsymbol{b}\rangle$ $0<-p\leqslant r\leqslant q$,	$\langle \boldsymbol{c}\rangle$ $0<-p\leqslant q<r$;
For this subcase, see Fig. $4/2^\circ$ with representations:	$\langle \boldsymbol{a}\rangle$ long and double-short dashed line,	$\langle \boldsymbol{b}\rangle$ dashed line,	$\langle \boldsymbol{c}\rangle$ continuous line.

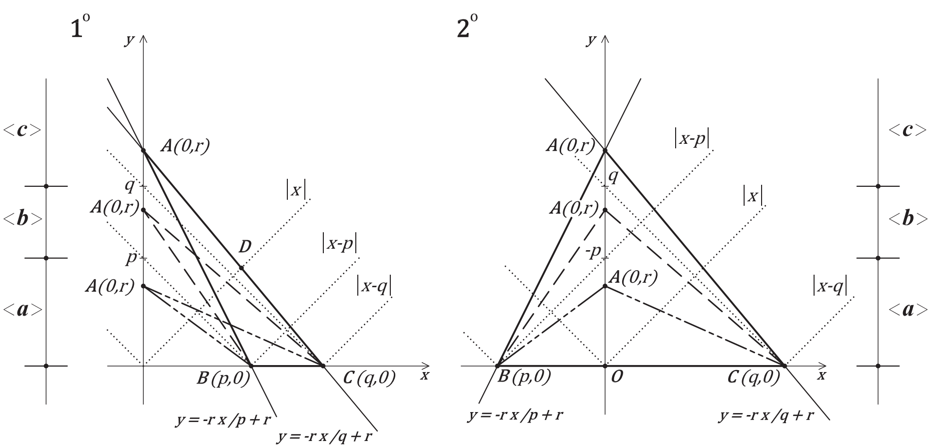

Figure 4: The two types of triangles ABC with subcasses

In formula (11), for the first triangle type $(i=1)$, branching is achieved for $x=p$, where p will then be denoted with x_1. In formula (13), for the second triangle type $(i=2)$, branching is achieved for $x=0$, where 0 will then be denoted with x_2. Then, the Erdös-Mordell inequality (10), with weight coefficient $w>0$, is considered with the following theorem:

THEOREM 1. *It holds:*

$$R_A+R_B+R_C \geqslant w(r_a+r_b+r_c) \iff \begin{cases} \alpha_{i1}x+\beta_{i1}y+\gamma_{i1} \geqslant 0 : x < x_i \ [\Pi_{i1}] \\ \alpha_{i2}x+\beta_{i2}y+\gamma_{i2} \geqslant 0 : x \geqslant x_i \ [\Pi_{i2}] \end{cases} \tag{15}$$

where coefficients $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ $(j=1,2)$, are given by Tab. 1 for $i=1$ and Tab. 2 for $i=2$.

1°		$\Pi_{1j} : \alpha_{1j}x+\beta_{1j}y+\gamma_{1j} \geqslant 0$		
		$\langle \boldsymbol{a} \rangle$ $0<r\leqslant p<q$	$\langle \boldsymbol{b} \rangle$ $0\leqslant p<r\leqslant q$	$\langle \boldsymbol{c} \rangle$ $0\leqslant p<q<r$
Π_{11}	α_{11}	$(p-q)wr-pq$	$r(w(r-q)-q)$	$-r$
	β_{11}	$-pq(w-1)$	$q(r-pw)$	$w(q-p-r)+r$
$x<p$	γ_{11}	$pq(p+q+r)$	$qr(w(p-r)+p+q+r)$	$r(w(p-q)+p+q+r)$
Π_{12}	α_{12}	$(p-q)wr+pq$	$r(w(r-q)+q)$	r
	β_{12}	$-pq(w-1)$	$q(r-pw)$	$w(q-p-r)+r$
$x\geqslant p$	γ_{12}	$pq(-p+q+r)$	$qr(w(p-r)-p+q+r)$	$r(w(p-q)-p+q+r)$

Table 1: The Erdös-Mordell inequality in the taxicab plane geometry for case 1°

2°		$\Pi_{2j} : \alpha_{2j}x+\beta_{2j}y+\gamma_{2j} \geqslant 0$		
		$\langle \boldsymbol{a} \rangle$ $0<r\leqslant -p\leqslant q$	$\langle \boldsymbol{b} \rangle$ $0<-p\leqslant r\leqslant q$	$\langle \boldsymbol{c} \rangle$ $0<-p\leqslant q<r$
Π_{21}	α_{21}	$pq-(p+q)wr$	$r(w(r-q)-q)$	$-r$
	β_{21}	$-pq(w+1)$	$q(r-pw)$	$w(q-p-r)+r$
$x<0$	γ_{21}	$pq(2rw+p-q-r)$	$qr(w(p-r)-p+q+r)$	$r(w(p-q)-p+q+r)$
Π_{22}	α_{22}	$-pq-(p+q)wr$	$r(w(r-q)+q)$	r
	β_{22}	$-pq(w+1)$	$q(r-pw)$	$w(q-p-r)+r$
$x\geqslant 0$	γ_{22}	$pq(2rw+p-q-r)$	$qr(w(p-r)-p+q+r)$	$r(w(p-q)-p+q+r)$

Table 2: The Erdös-Mordell inequality in the taxicab plane geometry for case 2°

Let us notice that the Erdös-Mordell inequality reduces to a problem of the positivity of the linear function

$$f_{ij}(x,y)=\alpha_{ij}x+\beta_{ij}y+\gamma_{ij} \geqslant 0,$$

for some choice of interior points (x,y) of a triangle, for concretely defined values of parameters α_{ij}, β_{ij} and γ_{ij} given by the above tables. The problem of determining the minimum and maximum of linear functions $f_{ij}(x,y)$ reduces down to the determining of the minimum and maximum in the vertices of the considered triangles, according to [6]. Given that, it is enough to consider the cases of the minima and maxima of linear functions $f_{ij}(x,y)$ in vertices of $\triangle ABD$ and $\triangle BCD$ for $A(0,r)$, $B(p,0)$, $C(q,0)$ and $D(p,\frac{r}{q}(q-p))$ when $i=1$ and in vertices of $\triangle ABO$ and $\triangle ACO$ for $A(0,r)$, $B(p,0)$, $C(q,0)$ and $O(0,0)$ when $i=2$.

The following statements hold:

STATEMENT 1. *Let $A(0,r)\in[\Pi_{11}]$. If the inequality (10) holds for $A(0,r)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{a}\rangle\, 0<r\leqslant p<q \,\vee\, \langle \boldsymbol{b}\rangle\, 0\leqslant p<r\leqslant q \,\vee\, \langle \boldsymbol{c}\rangle\, 0\leqslant p<q<r \implies w\leqslant 2+\frac{p+q}{r}. \tag{16}$$

Proof. From Table 1:

$\langle \boldsymbol{a}\rangle$ By substituting coordinates $x=0$ and $y=r$ into $f_{11}(x,y)=\alpha_{11}x+\beta_{11}y+\gamma_{11}$ the following is obtained:

$$\begin{aligned}
f_{11}(0,r)\geqslant 0 &\iff ((p-q)wr-pq)\cdot 0-pq(w-1)\cdot r+pq(p+q+r)\geqslant 0\\
&\iff -pq(w-1)\cdot r+pq(p+q+r)\geqslant 0\\
&\underset{pq>0}{\implies} -wr+p+q+2r\geqslant 0\\
&\underset{r>0}{\implies} w\leqslant 2+\frac{p+q}{r};
\end{aligned}$$

$\langle \boldsymbol{b}\rangle$ $q(r-pw)\cdot r+qr(w(p-r)+p+q+r)\geqslant 0$, from which follows $w\leqslant 2+\dfrac{p+q}{r}$;

$\langle \boldsymbol{c}\rangle$ $w(q-p-r)\cdot r+r(w(p-q)+p+q+r)+r\geqslant 0$, from which follows $w\leqslant 2+\dfrac{p+q}{r}$. □

STATEMENT 2. *Let $A(0,r)\in[\Pi_{22}]$. If the inequality (10) holds for $A(0,r)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{a}\rangle\, 0<r\leqslant -p\leqslant q \,\vee\, \langle \boldsymbol{b}\rangle\, 0<-p\leqslant r\leqslant q \,\vee\, \langle \boldsymbol{c}\rangle\, 0<-p\leqslant q<r \implies w\leqslant 2+\frac{q-p}{r}. \tag{17}$$

Proof. By Table 2:

$\langle \boldsymbol{a}\rangle$ $-pq(w+1)r+pq(2rw+p-q-r)\geqslant 0$, from which follows $w\leqslant 2+\dfrac{q-p}{r}$;

$\langle \boldsymbol{b}\rangle$ $q(r-pw)r+qr(w(p-r)-p+q+r)\geqslant 0$, from which follows $w\leqslant 2+\dfrac{q-p}{r}$;

$\langle \boldsymbol{c}\rangle$ $(w(q-p-r)+r)r+r(w(p-q)-p+q+r)\geqslant 0$, from which follows $w\leqslant 2+\dfrac{q-p}{r}$. □

STATEMENT 3. *Let $B(p,0) \in [\Pi_{12}]$. If the inequality (10) holds for $B(p,0)$, then the following conclusions hold for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\, 0<r\leqslant p<q \,\vee\, \langle \boldsymbol{b} \rangle\, 0\leqslant p<r\leqslant q \implies w \leqslant 1+\frac{q^2+pr}{r(q-p)}; \tag{18}$$

$$\langle \boldsymbol{c} \rangle\, 0\leqslant p<q<r \implies w \leqslant 1+\frac{p+r}{q-p}. \tag{19}$$

Proof. By Table 1:

$\langle \boldsymbol{a} \rangle$ $((p-q)wr+pq)p+pq(-p+q+r) \geqslant 0$, from which follows $w \leqslant 1+\dfrac{q^2+pr}{r(q-p)}$;

$\langle \boldsymbol{b} \rangle$ $r(w(r-q)+q)p+qr(w(p-r)-p+q+r) \geqslant 0$, from which follows $w \leqslant 1+\dfrac{q^2+pr}{r(q-p)}$;

$\langle \boldsymbol{c} \rangle$ $rp+r(w(p-q)-p+q+r) \geqslant 0$, from which follows $w \leqslant 1+\dfrac{p+r}{q-p}$. □

STATEMENT 4. *Let $B(p,0) \in [\Pi_{21}]$. If the inequality (10) holds for $B(p,0)$, then the following conclusions hold for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\, 0<r\leqslant -p\leqslant q \,\vee\, \langle \boldsymbol{b} \rangle\, 0<-p\leqslant r\leqslant q \implies w \leqslant \frac{q}{r}\left(1+\frac{r-p}{q-p}\right); \tag{20}$$

$$\langle \boldsymbol{c} \rangle\, 0<-p\leqslant q<r \implies w \leqslant 1+\frac{r-p}{q-p}. \tag{21}$$

Proof. By Table 2:

$\langle \boldsymbol{a} \rangle$ $(pq-(p+q)wr)p+pq(2rw+p-q-r) \geqslant 0$, from which follows $w \leqslant \dfrac{q}{r}\left(1+\dfrac{r-p}{q-p}\right)$;

$\langle \boldsymbol{b} \rangle$ $r(w(r-q)-q)p+qr(w(p-r)-p+q+r) \geqslant 0$, from which follows $w \leqslant \dfrac{q}{r}\left(1+\dfrac{r-p}{q-p}\right)$;

$\langle \boldsymbol{c} \rangle$ $-rp+r(w(p-q)-p+q+r) \geqslant 0$, from which follows $w \leqslant 1+\dfrac{r-p}{q-p}$. □

STATEMENT 5. *Let $C(q,0) \in [\Pi_{12}]$. If the inequality (10) holds for $C(q,0)$, then the following conclusions hold for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\, 0<r\leqslant p<q \implies w \leqslant \frac{p}{r}\left(1+\frac{q+r}{q-p}\right); \tag{22}$$

$$\langle \boldsymbol{b} \rangle\, 0\leqslant p<r\leqslant q \,\vee\, \langle \boldsymbol{c} \rangle\, 0\leqslant p<q<r \implies w \leqslant 1+\frac{q+r}{q-p}. \tag{23}$$

Proof. By Table 1:

$\langle \boldsymbol{a} \rangle$ $((p-q)wr+pq)q+pq(-p+q+r) \geqslant 0$, from which follows $w \leqslant \dfrac{p}{r}\left(1+\dfrac{q+r}{q-p}\right)$;

$\langle \boldsymbol{b} \rangle$ $r(w(r-q)+q)q+qr(w(p-r)-p+q+r) \geqslant 0$, from which follows $w \leqslant 1+\dfrac{q+r}{q-p}$;

$\langle \boldsymbol{c} \rangle$ $rq+r(w(p-q)-p+q+r) \geqslant 0$, from which follows $w \leqslant 1+\dfrac{q+r}{q-p}$. $\square$

STATEMENT 6. *Let $C(q,0) \in [\Pi_{22}]$. If the inequality (10) holds for $C(q,0)$, then the following conclusions hold for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\, 0<r\leqslant -p\leqslant q \implies w \leqslant \frac{-p}{r}\left(1+\frac{q+r}{q-p}\right); \tag{24}$$

$$\langle \boldsymbol{b} \rangle\, 0<-p\leqslant r\leqslant q \vee \langle \boldsymbol{c} \rangle\, 0<-p\leqslant q<r \implies w \leqslant 1+\frac{q+r}{q-p}. \tag{25}$$

Proof. By Table 2:

$\langle \boldsymbol{a} \rangle$ $(-pq-(p+q)wr)q+pq(2rw+p-q-r) \geqslant 0$, from which follows $w \leqslant \dfrac{-p}{r}\left(1+\dfrac{q+r}{q-p}\right)$;

$\langle \boldsymbol{b} \rangle$ $r(w(r-q)+q)q+qr(w(p-r)-p+q+r) \geqslant 0$, from which follows $w \leqslant 1+\dfrac{q+r}{q-p}$;

$\langle \boldsymbol{c} \rangle$ $rq+r(w(p-q)-p+q+r) \geqslant 0$, from which follows $w \leqslant 1+\dfrac{q+r}{q-p}$. $\square$

STATEMENT 7. *Let $D(p,\frac{r}{q}(q-p)) \in [\Pi_{12}]$. If the inequality (10) holds for $D(p,\frac{r}{q}(q-p))$, then the following conclusions hold for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\, 0<r\leqslant p<q \implies w \leqslant 1+\frac{q^2+pr}{2r(q-p)}; \tag{26}$$

$$\langle \boldsymbol{b} \rangle\, 0\leqslant p<r\leqslant q \vee \langle \boldsymbol{c} \rangle\, 0\leqslant p<q<r \implies w \leqslant 1+\frac{q-p}{r+p}+\frac{q}{q-p}. \tag{27}$$

Proof. By Table 1:

$\langle \boldsymbol{a} \rangle$ $((p-q)wr+pq)p-pq(w-1)\dfrac{r}{q}(q-p)+pq(-p+q+r) \geqslant 0$,

from which follows $w \leqslant 1+\dfrac{q^2+pr}{2r(q-p)}$;

$\langle \boldsymbol{b} \rangle$ $r(w(r-q)+q)p+q(r-pw)\dfrac{r}{q}(q-p)+qr(w(p-r)-p+q+r) \geqslant 0$,

from which follows $w \leqslant 1+\dfrac{q-p}{r+p}+\dfrac{q}{q-p}$;

$\langle \boldsymbol{c} \rangle$ $rp+(w(q-p-r)+r)\dfrac{r}{q}(q-p)+r(w(p-q)-p+q+r) \geqslant 0$,

from which follows $w \leqslant 1+\dfrac{q-p}{r+p}+\dfrac{q}{q-p}$. $\square$

STATEMENT 8. *Let $O(0,0) \in [\Pi_{22}]$. If the inequality* (10) *holds for $O(0,0)$, then the following conclusions hold for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\, 0<r\leqslant -p\leqslant q \implies w\leqslant \frac{1}{2}+\frac{q-p}{2r}; \tag{28}$$

$$\langle \boldsymbol{b} \rangle\, 0<-p\leqslant r\leqslant q \implies w\leqslant 1+\frac{q}{r-p}; \tag{29}$$

$$\langle \boldsymbol{c} \rangle\, 0<-p\leqslant q<r \implies w\leqslant 1+\frac{r}{q-p}. \tag{30}$$

Proof. By Table 2:

$\langle \boldsymbol{a} \rangle$ $pq(2rw+p-q-r)\geqslant 0$, from which follows $w\leqslant \frac{1}{2}+\frac{q-p}{2r}$;

$\langle \boldsymbol{b} \rangle$ $qr(w(p-r)-p+q+r)\geqslant 0$, from which follows $w\leqslant 1+\frac{q}{r-p}$;

$\langle \boldsymbol{c} \rangle$ $r(w(p-q)-p+q+r)\geqslant 0$, from which follows $w\leqslant 1+\frac{r}{q-p}$. □

Let the positions of points B and C be given. Then, let us consider the positions of point $A(0,r)$ in the concrete cases $\langle \boldsymbol{a} \rangle$, $\langle \boldsymbol{b} \rangle$, $\langle \boldsymbol{c} \rangle$ which were considered in Statements 1–8. Through the aforementioned Statements the functions of upper bounds ω for the weight coefficient w were obtained:

$$w\leqslant \omega(p,q,r).$$

Our goal is to, for the functions $\omega(p,q,r)$, dependent on concrete subcases $\langle\, \theta \,\rangle$, where $\theta \in \{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$, find the values:

$$\mathfrak{M}=\inf\{\omega(p,q,r)\,|\,\langle\, \theta\, \rangle\}. \tag{31}$$

In this way, the Erdös-Mordell inequality (9) holds for $w=\mathfrak{M}$ for all interior points of $\triangle ABC$. If $\mathfrak{M}$ is a minimum in this area, then an equality is also possible in (9).

2.1 Determining value of $\mathfrak{M}$ by areas

In this section of the paper, the values of $\mathfrak{M}$ by areas of $\triangle ABC$ are determined in dependence on cases $\langle\, \theta \,\rangle$, where $\theta \in \{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$.

The following three propositions are obtained on the basis of Statement 1.

PROPOSITION 1. *Let $A(0,r) \in [\Pi_{11}]$. If the inequality* (10) *holds for $A(0,r)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\; 0<r\leqslant p<q \implies w\leqslant \omega(p,q,r)=2+\frac{p+q}{r} \tag{32}$$

and in that case

$$\omega(p,q,r)\in(\mathfrak{M},\infty) \text{ and } \mathfrak{M}=4. \tag{33}$$

Proof. Let us consider $\langle \boldsymbol{a} \rangle\ 0<r\leqslant p<q$. Then, we notice the following expression holds:

$$\omega(p,q,r)=2+\frac{p+q}{r}\geqslant 2+\frac{p+q}{p}=3+\frac{q}{p}>4 \implies \mathfrak{M}=4.$$

The above conclusion is correct because the real number $\frac{q}{p}$ fulfills $\frac{q}{p}>1$ and it is possible to choose a number $\frac{q}{p}$ such that it is arbitrarily close to 1. □

PROPOSITION 2. *Let $A(0,r)\in[\Pi_{11}]$. If the inequality (10) holds for $A(0,r)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{b} \rangle\ 0\leqslant p<r\leqslant q \implies w\leqslant \omega(p,q,r)=2+\frac{p+q}{r} \tag{34}$$

and in that case

$$\omega(p,q,r)\in[\mathfrak{M},\infty) \text{ and } \mathfrak{M}=3. \tag{35}$$

Proof. Let us consider $\langle \boldsymbol{b} \rangle\ 0\leqslant p<r\leqslant q$. Then, we notice the following expression holds:

$$\omega(p,q,r)=2+\frac{p+q}{r}\geqslant 2+\frac{p+q}{q}=3+\frac{p}{q}\geqslant 3 \implies \mathfrak{M}=3. \quad \square$$

PROPOSITION 3. *Let $A(0,r)\in[\Pi_{11}]$. If the inequality (10) holds for $A(0,r)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{c} \rangle\ 0\leqslant p<q<r \implies w\leqslant \omega(p,q,r)=2+\frac{p+q}{r} \tag{36}$$

and in that case

$$\omega(p,q,r)\in(\mathfrak{M},\infty) \text{ and } \mathfrak{M}=2. \tag{37}$$

Proof. Let us consider $\langle \boldsymbol{c} \rangle\ 0\leqslant p<q<r$. Then, we notice the following expression holds:

$$\omega(p,q,r)=2+\frac{p+q}{r}>2 \implies \mathfrak{M}=2. \quad \square$$

The following three propositions are obtained on the basis of Statement 2.

PROPOSITION 4. *Let $A(0,r)\in[\Pi_{22}]$. If the inequality (10) holds for $A(0,r)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\ 0<r\leqslant -p\leqslant q \implies w\leqslant \omega(p,q,r)=2+\frac{q-p}{r} \tag{38}$$

and in that case

$$\omega(p,q,r)\in[\mathfrak{M},\infty) \text{ and } \mathfrak{M}=4. \tag{39}$$

Proof. Let us consider $\langle \boldsymbol{a} \rangle\ 0 < r \leqslant -p \leqslant q$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 2 + \frac{q-p}{r} \geqslant 2 + \frac{q-p}{-p} = 3 + \frac{q}{-p} \geqslant 4 \implies \mathfrak{M} = 4. \quad \square$$

PROPOSITION 5. *Let $A(0,r) \in [\Pi_{22}]$. If the inequality (10) holds for $A(0,r)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{b} \rangle\ 0 < -p \leqslant r \leqslant q \implies w \leqslant \omega(p,q,r) = 2 + \frac{q-p}{r} \tag{40}$$

and in that case

$$\omega(p,q,r) \in (\mathfrak{M}, \infty) \ \text{and}\ \mathfrak{M} = 3. \tag{41}$$

Proof. Let us consider $\langle \boldsymbol{b} \rangle\ 0 < -p \leqslant r \leqslant q$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 2 + \frac{q-p}{r} \geqslant 2 + \frac{q-p}{q} = 3 + \frac{-p}{q} > 3 \implies \mathfrak{M} = 3. \quad \square$$

PROPOSITION 6. *Let $A(0,r) \in [\Pi_{22}]$. If the inequality (10) holds for $A(0,r)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{c} \rangle\ 0 < -p \leqslant q < r \implies w \leqslant \omega(p,q,r) = 2 + \frac{q-p}{r} \tag{42}$$

and in that case

$$\omega(p,q,r) \in (\mathfrak{M}, \infty) \ \text{and}\ \mathfrak{M} = 2. \tag{43}$$

Proof. Let us consider $\langle \boldsymbol{c} \rangle\ 0 < -p \leqslant q < r$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 2 + \frac{q-p}{r} > 2 \implies \mathfrak{M} = 2. \quad \square$$

Similar to previous propositions, the following three propositions are obtained from Statement 3.

PROPOSITION 7. *Let $B(p,0) \in [\Pi_{12}]$. If the inequality (10) holds for $B(p,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\ 0 < r \leqslant p < q \implies w \leqslant \omega(p,q,r) = 1 + \frac{q^2}{r(q-p)} + \frac{p}{q-p} \tag{44}$$

and in that case

$$\omega(p,q,r) \in [\mathfrak{M}, \infty) \ \text{and}\ \mathfrak{M} = 3 + 2\sqrt{2}. \tag{45}$$

Proof. Let us consider $\langle \boldsymbol{a} \rangle$ $0<r\leqslant p<q$. Then, we notice the following expression holds:

$$\begin{aligned}\omega(p,q,r) &= 1+\frac{q^2}{r(q-p)}+\frac{p}{q-p}\\ &\geqslant 1+\frac{q^2}{p(q-p)}+\frac{p}{q-p}\\ &= 3+\frac{2p}{q-p}+\frac{q-p}{p}\geqslant 3+2\sqrt{2} \implies \mathfrak{M}=3+2\sqrt{2},\end{aligned}$$

because $t=\dfrac{p}{q-p}>0$ holds $2t+\dfrac{1}{t}\geqslant 2\sqrt{2}$. □

PROPOSITION 8. *Let $B(p,0)\in[\Pi_{12}]$. If inequality (10) holds for $B(p,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{b} \rangle\ 0\leqslant p<r\leqslant q \implies w\leqslant \omega(p,q,r)=1+\frac{q^2}{r(q-p)}+\frac{p}{q-p} \tag{46}$$

and in that case

$$\omega(p,q,r)\in[\mathfrak{M},\infty) \text{ and } \mathfrak{M}=2. \tag{47}$$

Proof. Let us consider $\langle \boldsymbol{b} \rangle$ $0\leqslant p<r\leqslant q$. Then, we notice the following expression holds:

$$\begin{aligned}\omega(p,q,r) &= 1+\frac{q^2}{r(q-p)}+\frac{p}{q-p}\\ &\geqslant 1+\frac{q^2}{q(q-p)}+\frac{p}{q-p}\\ &= 2+\frac{2p}{q-p}\geqslant 2 \implies \mathfrak{M}=2. \quad \square\end{aligned}$$

PROPOSITION 9. *Let $B(p,0)\in[\Pi_{12}]$. If the inequality (10) holds for $B(p,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{c} \rangle\ 0\leqslant p<q<r \implies w\leqslant \omega(p,q,r)=1+\frac{p+r}{q-p} \tag{48}$$

and in that case

$$\omega(p,q,r)\in(\mathfrak{M},\infty) \text{ and } \mathfrak{M}=2. \tag{49}$$

Proof. Let us consider $\langle \boldsymbol{c} \rangle$ $0\leqslant p<q<r$. Then, we notice the following expression holds:

$$\begin{aligned}\omega(p,q,r) &= 1+\frac{p+r}{q-p}\\ &> 1+\frac{p+q}{q-p}\\ &= 2+\frac{2p}{q-p}\geqslant 2 \implies \mathfrak{M}=2. \quad \square\end{aligned}$$

The following three propositions are obtained on the basis of Statement 4.

PROPOSITION 10. *Let $B(p,0) \in [\Pi_{21}]$. If the inequality (10) holds for $B(p,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\ 0 < r \leqslant -p \leqslant q \implies w \leqslant \omega(p,q,r) = \frac{q}{r}\left(1 + \frac{r-p}{q-p}\right) \tag{50}$$

and in that case

$$\omega(p,q,r) \in [\mathfrak{M}, \infty) \text{ and } \mathfrak{M} = 2. \tag{51}$$

Proof. Let us consider $\langle \boldsymbol{a} \rangle\ 0 < r \leqslant -p \leqslant q$. Then, we notice the following expression holds:

$$\begin{aligned} \omega(p,q,r) &= \frac{q}{r}\left(1 + \frac{r-p}{q-p}\right) \\ &\geqslant \frac{q}{r}\left(1 + \frac{r-p}{2q}\right) \\ &\geqslant \frac{q}{r}\left(1 + \frac{2r}{2q}\right) = \frac{q}{r} + 1 \geqslant 2 \implies \mathfrak{M} = 2. \quad \square \end{aligned}$$

PROPOSITION 11. *Let $B(p,0) \in [\Pi_{21}]$. If the inequality (10) holds for $B(p,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{b} \rangle\ 0 < -p \leqslant r \leqslant q \implies w \leqslant \omega(p,q,r) = \frac{q}{r}\left(1 + \frac{r-p}{q-p}\right) \tag{52}$$

and in that case

$$\omega(p,q,r) \in [\mathfrak{M}, \infty) \text{ and } \mathfrak{M} = 2. \tag{53}$$

Proof. Let us consider $\langle \boldsymbol{b} \rangle\ 0 < -p \leqslant r \leqslant q$. Then, we notice the following expression holds:

$$\omega(p,q,r) = \frac{q}{r} + \frac{q}{r}\frac{r-p}{q-p} \geqslant 1 + \frac{q}{r}\frac{r-p}{q-p} \geqslant 2 \implies \mathfrak{M} = 2,$$

because $q(r+(-p)) \geqslant r(q+(-p)) \iff q \geqslant r$. $\square$

PROPOSITION 12. *Let $B(p,0) \in [\Pi_{21}]$. If the inequality (10) holds for $B(p,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{c} \rangle\ 0 < -p \leqslant q < r \implies w \leqslant \omega(p,q,r) = 1 + \frac{r-p}{q-p} \tag{54}$$

and in that case

$$\omega(p,q,r) \in (\mathfrak{M}, \infty) \text{ and } \mathfrak{M} = 2. \tag{55}$$

Proof. Let us consider $\langle \boldsymbol{c} \rangle\ 0 < -p \leqslant q < r$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 1 + \frac{r-p}{q-p} > 1 + \frac{q-p}{q-p} = 2 \implies \mathfrak{M} = 2. \quad \square$$

The following three propositions are obtained on the basis of Statement 5.

PROPOSITION 13. *Let $C(q,0) \in [\Pi_{12}]$. If the inequality (10) holds for $C(q,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\ 0 < r \leqslant p < q \implies w \leqslant \omega(p,q,r) = \frac{p}{r}\left(1 + \frac{q+r}{q-p}\right) \tag{56}$$

and in that case

$$\omega(p,q,r) \in (\mathfrak{M}, \infty) \ \text{and}\ \mathfrak{M} = 2. \tag{57}$$

Proof. Let us consider $\langle \boldsymbol{a} \rangle\ 0 < r \leqslant p < q$. Then, we notice the following expression holds:

$$\begin{aligned} \omega(p,q,r) &= \frac{p}{r}\left(1 + \frac{q+r}{q-p}\right) \\ &\geqslant 1 + \frac{q+r}{q-p} \\ &= \frac{2q-2p+r+p}{q-p} > 2 + \frac{r+p}{q-p} > 2 \implies \mathfrak{M} = 2. \quad \square \end{aligned}$$

PROPOSITION 14. *Let $C(q,0) \in [\Pi_{12}]$. If the inequality (10) holds for $C(q,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{b} \rangle\ 0 \leqslant p < r \leqslant q \implies w \leqslant \omega(p,q,r) = 1 + \frac{q+r}{q-p} \tag{58}$$

and in that case

$$\omega(p,q,r) \in (\mathfrak{M}, \infty) \ \text{and}\ \mathfrak{M} = 2. \tag{59}$$

Proof. Let us consider $\langle \boldsymbol{b} \rangle\ 0 \leqslant p < r \leqslant q$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 1 + \frac{q+r}{q-p} > 1 + \frac{q+p}{q-p} \geqslant 2 \implies \mathfrak{M} = 2. \quad \square$$

PROPOSITION 15. *Let $C(q,0) \in [\Pi_{12}]$. If the inequality (10) holds for $C(q,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{c} \rangle\ 0 \leqslant p < q < r \implies w \leqslant \omega(p,q,r) = 1 + \frac{q+r}{q-p} \tag{60}$$

and in that case

$$\omega(p,q,r) \in (\mathfrak{M}, \infty) \ \text{and}\ \mathfrak{M} = 3. \tag{61}$$

Proof. Let us consider $\langle \boldsymbol{c} \rangle\ 0 \leqslant p < q < r$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 1 + \frac{q+r}{q-p} = 2 + \frac{r+p}{q-p} > 2 + \frac{q+p}{q-p} \geqslant 3 \implies \mathfrak{M} = 3. \quad \square$$

Similar to previous propositions, the following three propositions are obtained from Statement 6.

PROPOSITION 16. *Let $C(q,0) \in [\Pi_{22}]$. If the inequality (10) holds for $C(q,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\ 0 < r \leqslant -p \leqslant q \implies w \leqslant \omega(p,q,r) = -\frac{p}{r}\left(1 + \frac{q+r}{q-p}\right) \tag{62}$$

and in that case

$$\omega(p,q,r) \in [\mathfrak{M}, \infty) \text{ and } \mathfrak{M} = 2. \tag{63}$$

Proof. Let us consider $\langle \boldsymbol{a} \rangle\ 0 < r \leqslant -p \leqslant q$. Then, we notice the following expression holds:

$$\begin{aligned} \omega(p,q,r) &= \frac{-p}{r} + \frac{-p}{r}\frac{q+r}{q-p} \\ &\geqslant 1 + \frac{-p}{r}\frac{q+r}{q-p} \\ &= 1 + \frac{-pq + (-p)r}{rq + (-p)r} \geqslant 2 \implies \mathfrak{M} = 2, \end{aligned}$$

because $-pq + (-p)r \geqslant rq + (-p)r \iff -p \geqslant r$. $\square$

PROPOSITION 17. *Let $C(q,0) \in [\Pi_{22}]$. If the inequality (10) holds for $C(q,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{b} \rangle\ 0 < -p \leqslant r \leqslant q \implies w \leqslant \omega(p,q,r) = 1 + \frac{q+r}{q-p} \tag{64}$$

and in that case

$$\omega(p,q,r) \in [\mathfrak{M}, \infty) \text{ and } \mathfrak{M} = 2. \tag{65}$$

Proof. Let us consider $\langle \boldsymbol{b} \rangle\ 0 < -p \leqslant r \leqslant q$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 1 + \frac{q+r}{q-p} \geqslant 1 + \frac{q-p}{q-p} = 2 \implies \mathfrak{M} = 2. \quad \square$$

PROPOSITION 18. *Let* $C(q,0) \in [\Pi_{22}]$. *If the inequality* (10) *holds for* $C(q,0)$, *then the following conclusion holds for the weight coefficient* w:

$$\langle \boldsymbol{c} \rangle\ 0 < -p \leqslant q < r \implies w \leqslant \omega(p,q,r) = 1 + \frac{q+r}{q-p} \tag{66}$$

and in that case

$$\omega(p,q,r) \in (\mathfrak{M}, \infty) \ \text{and} \ \mathfrak{M} = 2. \tag{67}$$

Proof. Let us consider $\langle \boldsymbol{c} \rangle\ 0 < -p \leqslant q < r$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 1 + \frac{q+r}{q-p} > 1 + \frac{2q}{q-p} \geqslant 2 \implies \mathfrak{M} = 2,$$

because $2q \geqslant q-p \iff q \geqslant -p$. □

The following three propositions are obtained on the basis of Statement 7.

PROPOSITION 19. *Let* $D\big(p, \frac{r}{q}(q-p)\big) \in [\Pi_{12}]$. *If the inequality* (10) *holds for* $D\big(p, \frac{r}{q}(q-p)\big) \in [\Pi_{12}]$, *then the following conclusion holds for the weight coefficient* w:

$$\langle \boldsymbol{a} \rangle\ 0 < r \leqslant p < q \implies w \leqslant \omega(p,q,r) = 1 + \frac{q^2 + pr}{2r(q-p)} \tag{68}$$

and in that case

$$\omega(p,q,r) \in [\mathfrak{M}, \infty) \ \text{and} \ \mathfrak{M} = 2 + \sqrt{2}. \tag{69}$$

Proof. Let us consider $\langle \boldsymbol{a} \rangle\ 0 < r \leqslant p < q$. Then, we notice the following expression holds:

$$\begin{aligned}\omega(p,q,r) &= 1 + \frac{q^2}{2r(q-p)} + \frac{p}{2(q-p)}\\ &\geqslant 1 + \frac{q^2}{2p(q-p)} + \frac{p}{2(q-p)}\\ &= 2 + \frac{q-p}{2p} + \frac{p}{q-p} \geqslant 2 + \sqrt{2} \implies \mathfrak{M} = 2 + \sqrt{2},\end{aligned}$$

because $t = \dfrac{p}{q-p} > 0$ for $\dfrac{1}{2t} + t \geqslant \sqrt{2}$. □

PROPOSITION 20. *Let* $D\big(p, \frac{r}{q}(q-p)\big) \in [\Pi_{12}]$. *If the inequality* (10) *holds for* $D\big(p, \frac{r}{q}(q-p)\big) \in [\Pi_{12}]$, *then the following conclusion holds for the weight coefficient* w:

$$\langle \boldsymbol{b} \rangle\ 0 \leqslant p < r \leqslant q \implies w \leqslant \omega(p,q,r) = 1 + \frac{q-p}{r+p} + \frac{q}{q-p} \tag{70}$$

and in that case

$$\omega(p,q,r) \in [\mathfrak{M}, \infty) \ \text{and} \ \mathfrak{M} = \frac{3}{2} + \sqrt{2}. \tag{71}$$

Proof. Let us consider $\langle \boldsymbol{b} \rangle$ $0 \leqslant p < r \leqslant q$. Then, we notice the following expression holds:

$$\begin{aligned} 2\,\omega(p,q,r) &= 2 + 2\frac{q-p}{r+p} + \frac{2q}{q-p} \\ &= 3 + 2\frac{q-p}{r+p} + \frac{q+p}{q-p} \\ &\geqslant 3 + 2\frac{q-p}{q+p} + \frac{q+p}{q-p} \geqslant 3 + 2\sqrt{2} \implies \mathfrak{M} = \frac{3}{2} + \sqrt{2}, \end{aligned}$$

because $t = \dfrac{q-p}{q+p} > 0$ holds $2t + \dfrac{1}{t} \geqslant 2\sqrt{2}$. □

PROPOSITION 21. *Let $D\big(p, \frac{r}{q}(q-p)\big) \in [\Pi_{12}]$. If the inequality* (10) *holds for $D\big(p, \frac{r}{q}(q-p)\big) \in [\Pi_{12}]$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{c} \rangle\ 0 \leqslant p < q < r \implies w \leqslant \omega(p,q,r) = 1 + \frac{q-p}{r+p} + \frac{q}{q-p} \tag{72}$$

and in that case

$$\omega(p,q,r) \in (\mathfrak{M}, \infty) \ \textit{and}\ \mathfrak{M} = 2. \tag{73}$$

Proof. Let us consider $\langle \boldsymbol{c} \rangle$ $0 \leqslant p < q < r$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 1 + \frac{q-p}{r+p} + \frac{q}{q-p} \geqslant 2 + \frac{q-p}{r+p} > 2 \implies \mathfrak{M} = 2,$$

because $\dfrac{q}{q-p} \geqslant 1$. □

Similar to previous propositions, the following three propositions are obtained from Statement 8.

PROPOSITION 22. *Let $O(0,0) \in [\Pi_{22}]$. If the inequality* (10) *holds for $O(0,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{a} \rangle\ 0 < r \leqslant -p \leqslant q \implies w \leqslant \omega(p,q,r) = \frac{1}{2} + \frac{q-p}{2r} \tag{74}$$

and in that case

$$\omega(p,q,r) \in [\mathfrak{M}, \infty) \ \textit{and}\ \mathfrak{M} = \frac{3}{2}. \tag{75}$$

Proof. Let us consider $\langle \boldsymbol{a} \rangle$ $0 < r \leqslant -p \leqslant q$. Then, we notice the following expression holds:

$$\omega(p,q,r) = \frac{1}{2} + \frac{q}{2r} + \frac{-p}{2r} \geqslant \frac{1}{2} + \frac{q}{2(-p)} + \frac{-p}{2(-p)} \geqslant \frac{3}{2} \implies \mathfrak{M} = \frac{3}{2}. \quad \square$$

PROPOSITION 23. *Let $O(0,0) \in [\Pi_{22}]$. If the inequality (10) holds for $O(0,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{b} \rangle \; 0 < -p \leqslant r \leqslant q \implies w \leqslant \omega(p,q,r) = 1 + \frac{q}{r-p} \tag{76}$$

and in that case

$$\omega(p,q,r) \in [\mathfrak{M}, \infty) \text{ and } \mathfrak{M} = \frac{3}{2}. \tag{77}$$

Proof. Let us consider $\langle \boldsymbol{b} \rangle \; 0 < -p \leqslant r \leqslant q$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 1 + \frac{q}{r-p} \geqslant 1 + \frac{q}{2r} \geqslant 1 + \frac{1}{2} = \frac{3}{2} \implies \mathfrak{M} = \frac{3}{2}. \quad \square$$

PROPOSITION 24. *Let $O(0,0) \in [\Pi_{22}]$. If the inequality (10) holds for $O(0,0)$, then the following conclusion holds for the weight coefficient w:*

$$\langle \boldsymbol{c} \rangle \; 0 < -p \leqslant q < r \implies w \leqslant \omega(p,q,r) = 1 + \frac{r}{q-p} \tag{78}$$

and in that case

$$\omega(p,q,r) \in (\mathfrak{M}, \infty) \text{ and } \mathfrak{M} = \frac{3}{2}. \tag{79}$$

Proof. Let us consider $\langle \boldsymbol{c} \rangle \; 0 < -p \leqslant q < r$. Then, we notice the following expression holds:

$$\omega(p,q,r) = 1 + \frac{r}{q-p} > 1 + \frac{q}{q-p} \geqslant 1 + \frac{q}{2q} = \frac{3}{2} \implies \mathfrak{M} = \frac{3}{2}. \quad \square$$

Let us emphasize that the results of the previous three Propositions provide an improvement over some results from paper [5].

3. Summa summarum

Based on the propositions above, a theorem follows:

THEOREM 2. *In taxicab geometry for an interior point of $\triangle ABC$ in an appropriate position, the Erdös-Mordell's inequality holds*

$$R_A + R_B + R_C \geqslant \frac{3}{2}(r_a + r_b + r_c).$$

It is well known that taxicab distance depends on the rotation of the coordinate system, but does not depend on its translation or its reflection over a coordinate axis [20]. For an arbitrary triangle ABC we set the following open problem (illustrated by Fig. 5).

CONJECTURE 1. In taxicab geometry for an interior point of any triangle ABC the Erdös-Mordell's inequality holds

$$R_A + R_B + R_C \geqslant \frac{3}{2}\left(r_a + r_b + r_c\right).$$

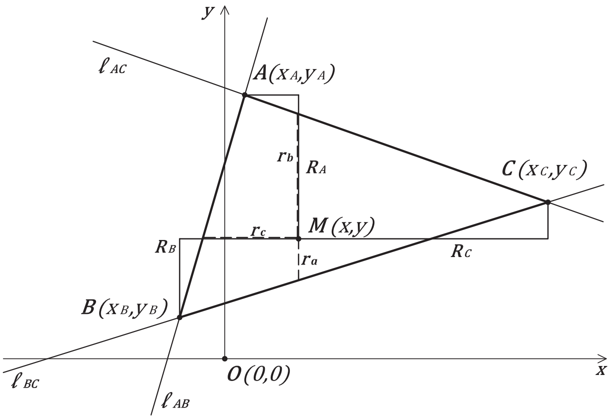

Figure 5. A geometric illustration of Conjecture 1

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