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MATRIX OSTROWSKI INEQUALITY VIA THE MATRIX GEOMETRIC MEAN

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Abstract. In this paper, we show a symmetric generalization of the Ostrowski inequality due to Fujii, Lin and Nakamoto. Moreover, we show its two variable extenstion. Inspired by this, we present matrix Ostrowski inequalities via the matrix geometric mean.

1. Introduction

The Cauchy-Schwarz inequality is one of the most useful and fundamental inequalities in functional analysis: Let a and b be two vectors in a Hilbert space $\mathscr{H}$. Then

$$|\langle a,b\rangle| \leqslant \|a\|\,\|b\| \tag{1.1}$$

and the equality holds in (1.1) if and only if a and b are linearly dependent. Many researchers have been studied generalizations and refinements of the Cauchy-Schwarz inequality. The following inequality due to A.M. Ostrowski is regarded as a refinement of the Cauchy-Schwarz inequality, see [5, pp. 92–95]: Let $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$ be two nonproportional sequences of real numbers, $x=(x_1,\dots,x_n)$ any sequence of real numbers for which the following holds:

$$\sum_{i=1}^n a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^n b_i x_i = 1.$$

Then

$$\sum_{i=1}^n x_i^2 \geqslant \frac{\sum_{i=1}^n a_i^2}{\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)-\left(\sum_{i=1}^n a_i b_i\right)^2} \tag{1.2}$$

with equality if and only if

$$x_k = \frac{b_k \sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n a_i b_i}{\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)-\left(\sum_{i=1}^n a_i b_i\right)^2} \qquad \text{for } k=1,\dots,n.$$

In [2], Fujii, Lin and Nakamoto extended the Ostrowski inequality (1.2) to the vector version in a Hilbert space:

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THEOREM A. (Fujii-Lin-Nakamoto) Let a and b be non-zero linearly independent vectors in a Hilbert space $\mathscr{H}$. If $x \in \mathscr{H}$ satisfies $\langle a,x\rangle = 0$ and $\langle b,x\rangle = 1$, then

$$\|x\|^2 \geqslant \frac{\|a\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2} \tag{1.3}$$

and the equality holds in (1.3) if and only if

$$x = \frac{\|a\|^2 b - \langle b,a\rangle a}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}.$$

Though the Cauchy-Schwarz inequality (1.1) is symmetric on vectors a and b in $\mathscr{H}$, the vector version (1.3) of the Ostrowski inequality in Theorem A is not symmetric. Thus, we consider the symmetric generalization of the Ostrowski inequality due to Fujii, Lin and Nakamoto. Moreover, we show its two variable extenstion. Inspired by this, we present matrix Ostrowski inequalities via the matrix geometric mean.

2. Symmetric generalization of Ostrowski inequality

First of all, we show a symmetric generalization of the Ostrowski inequality due to Fujii, Lin and Nakamoto without the conditions on x:

THEOREM 2.1. *Let a and b be non-zero linearly independent vectors in a Hilbert space $\mathscr{H}$. Then for any $x \in \mathscr{H}$*

$$\|x\|^2 \geqslant \frac{\|\langle b,x\rangle a - \langle a,x\rangle b\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2} \tag{2.1}$$

and the equality holds in (2.1) if and only if

$$x = \frac{(\langle x,a\rangle\|b\|^2 - \langle x,b\rangle\langle b,a\rangle)a + (\langle x,b\rangle\|a\|^2 - \langle x,a\rangle\langle a,b\rangle)b}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}.$$

To prove Theorem 2.1, we need the following Lemma:

LEMMA 2.2. *Let a and b be non-zero linearly independent vectors in a Hilbert space $\mathscr{H}$. If $x \in \mathscr{H}$ is a linear combination of a and b, then*

$$\|x\|^2 = \frac{\|\langle b,x\rangle a - \langle a,x\rangle b\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}. \tag{2.2}$$

Proof. We put $x = sa + tb$ for some scalars $s,t \in \mathbb{C}$, and $\Delta = \|a\|^2\|b\|^2 - |\langle a,b\rangle|^2$. Since $\langle x,a\rangle = s\|a\|^2 + t\langle b,a\rangle$ and $\langle x,b\rangle = s\langle a,b\rangle + t\|b\|^2$, we have

$$s = \frac{\langle x,a\rangle\|b\|^2 - \langle x,b\rangle\langle b,a\rangle}{\Delta} \quad \text{and} \quad t = \frac{\langle x,b\rangle\|a\|^2 - \langle x,a\rangle\langle a,b\rangle}{\Delta}.$$

Hence it follows that

$$\begin{aligned}
\|x\|^2 &= \|sa+tb\|^2 = |s|^2\|a\|^2 + 2\mathrm{Re}\, s\bar{t}\langle a,b\rangle + |t|^2\|b\|^2 \\
&= \frac{\|a\|^2}{\Delta^2}\left(|\langle a,x\rangle|^2\|b\|^4 - \langle x,a\rangle\langle b,x\rangle\langle a,b\rangle\|b\|^2\right) \\
&\quad + \frac{\|b\|^2}{\Delta^2}\left(|\langle b,x\rangle|^2\|a\|^4 - \langle x,b\rangle\langle a,x\rangle\langle b,a\rangle\|a\|^2\right) \\
&\quad + \frac{\langle a,b\rangle}{\Delta^2}\left(\langle x,b\rangle\langle b,a\rangle^2\langle a,x\rangle - |\langle a,x\rangle|^2\langle b,a\rangle\|b\|^2\right) \\
&\quad + \frac{\langle b,a\rangle}{\Delta^2}\left(\langle b,x\rangle\langle a,b\rangle^2\langle x,a\rangle - |\langle b,x\rangle|^2\langle a,b\rangle\|a\|^2\right) \\
&= \frac{1}{\Delta}\left(|\langle a,x\rangle|^2\|b\|^2 - \langle x,a\rangle\langle b,x\rangle\langle a,b\rangle - \langle x,b\rangle\langle a,x\rangle\langle b,a\rangle + |\langle b,x\rangle|^2\|a\|^2\right) \\
&= \frac{\|\langle b,x\rangle a - \langle a,x\rangle b\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}
\end{aligned}$$

as desired. $\square$

Proof of Theorem 2.1. For any $x \in \mathscr{H}$, we put $x = sa + tb + c$ for some scalars $s,t \in \mathbb{C}$, where c is orthogonal to a and b. Put $y = sa + tb$. It follows from Lemma 2.2 that

$$\|y\|^2 = \frac{\|\langle b,x\rangle a - \langle a,x\rangle b\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}.$$

Hence we have

$$\|x\|^2 = \|y\|^2 + \|c\|^2 \geqslant \|y\|^2 = \frac{\|\langle b,x\rangle a - \langle a,x\rangle b\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}$$

and the equality condition. $\square$

REMARK 2.3. The inequality (2.1) of Theorem 2.1 is symmetric on a and b, and a generalization of the Ostrowski inequality due to Fujii, Lin and Nakamoto: In fact, if $\langle a,x\rangle = 0$ and $\langle b,x\rangle = 1$ in Theorem 2.1, then we have the inequality (1.3) of Theorem A.

3. Two variable extension of Ostrowski inequality

Next, we show a two variable extension of the Ostrowski inequality (1.3):

THEOREM 3.1. *For a pair of nonzero linearly independent vectors $a,b \in \mathscr{H}$, if $x,y \in \mathscr{H}$ satisfy $\langle a,x\rangle = \langle b,y\rangle = 0$ and $\langle b,x\rangle = \langle a,y\rangle = 1$, then*

$$\|x\|^2\|y\|^2 \geqslant \frac{\|a\|^2\|y\|^2 + \|b\|^2\|x\|^2 - 1}{\|a\|^2\|b\|^2 - \left|\langle a,b\rangle + \frac{\langle y,x\rangle}{\|x\|^2\|y\|^2}\right|^2} \tag{3.1}$$

and the equality holds in (3.1) *if and only if two vectors* $a - \frac{1}{\|y\|^2}y$ *and* $b - \frac{1}{\|x\|^2}x$ *are linearly dependent.*

Proof. It follows from the Cauchy-Schwarz inequality that

$$\left|\left\langle a - \frac{1}{\|y\|^2}y, b - \frac{1}{\|x\|^2}x\right\rangle\right|^2 \leqslant \left\langle a - \frac{1}{\|y\|^2}y, a - \frac{1}{\|y\|^2}y\right\rangle\left\langle b - \frac{1}{\|x\|^2}x, b - \frac{1}{\|x\|^2}x\right\rangle$$

and the conditions $\langle a,x\rangle = \langle b,y\rangle = 0$ and $\langle b,x\rangle = \langle a,y\rangle = 1$ imply

$$\left|\langle a,b\rangle + \frac{\langle y,x\rangle}{\|x\|^2\|y\|^2}\right|^2 \leqslant \left(\|a\|^2 - \frac{1}{\|y\|^2}\right)\left(\|b\|^2 - \frac{1}{\|x\|^2}\right).$$

Therefore, we have the desired inequality (3.1) and the equality conditions. □

REMARK 3.2. Let $\theta \in [0,\pi]$ be the argument between two complex numbers $\langle a,b\rangle$ and $\frac{\langle y,x\rangle}{\|x\|^2\|y\|^2}$. If $-2|\langle a,b\rangle|\cos\theta \leqslant \frac{|\langle y,x\rangle|}{\|x\|^2\|y\|^2}$, then the inequality (3.1) implies (1.3). In fact, this condition is equivalent to the inequality $|\langle a,b\rangle| \leqslant |\langle a,b\rangle + \frac{\langle y,x\rangle}{\|x\|^2\|y\|^2}|$ and we have

$$\|x\|^2\|y\|^2 \geqslant \frac{\|a\|^2\|y\|^2 + \|b\|^2\|x\|^2 - 1}{\|a\|^2\|b\|^2 - |\langle a,b\rangle + \frac{\langle y,x\rangle}{\|x\|^2\|y\|^2}|^2} \geqslant \frac{\|a\|^2\|y\|^2 + \|b\|^2\|x\|^2 - 1}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}.$$

Since $\langle b,x\rangle = 1$, it follows that $\|b\|\,\|x\| \geqslant 1$ and so

$$\|x\|^2 \geqslant \frac{\|a\|^2 + \frac{\|b\|^2\|x\|^2 - 1}{\|y\|^2}}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2} \geqslant \frac{\|a\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}.$$

Hence we have the Ostroiwski inequality (1.3).

Under the orthogonal condition $\langle x,y\rangle = 0$ in Theorem 3.1, we have the following corollary:

COROLLARY 3.3. *For a pair of nonzero linearly independent vectors* $a,b \in \mathscr{H}$, *if* $x,y \in \mathscr{H}$ *satisfy* $\langle a,x\rangle = \langle b,y\rangle = 0$ *and* $\langle b,x\rangle = \langle a,y\rangle = 1$, *and* $\langle x,y\rangle = 0$, *then*

$$\|x\|^2\|y\|^2 \geqslant \frac{\|a\|^2\|y\|^2 + \|b\|^2\|x\|^2 - 1}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2} \tag{3.2}$$

and the equality holds in (3.1) *if and only if two vectors* $a - \frac{1}{\|y\|^2}y$ *and* $b - \frac{1}{\|x\|^2}x$ *are linearly dependent.*

We remark that Corollary 3.3 is a more precise result than Theorem A and the equality holds in (1.3) and (3.2) only if a and b are orthogonal. In fact, suppose that $\|x\|^2 = \frac{\|a\|^2}{\|a\|^2\|b\|^2-|\langle a,b\rangle|^2}$. Then it follows from Remark 3.2 that

$$\|x\|^2 = \frac{\|a\|^2 + \frac{\|b\|^2\|x\|^2-1}{\|y\|^2}}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2} = \frac{\|a\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}$$

and so $\|b\|^2\|x\|^2 - 1 = 0$. Hence we have

$$0 = \|b\|^2\|x\|^2 - 1 = \frac{\|a\|^2\|b\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2} - 1$$

and so $\langle a,b\rangle = 0$.

4. Matrix Ostrowski inequality

Let $\mathbb{M}_{k\times n} = \mathbb{M}_{k\times n}(\mathbb{C})$ be the space of $k\times n$ complex matrices and $\mathbb{M}_n = \mathbb{M}_{n\times n}$, and denote the matrix absolute value of any $A \in \mathbb{M}_{k\times n}$ by $|A| = (A^*A)^{1/2}$. For $A \in \mathbb{M}_n$, we write $A \geqslant 0$ if A is positive semidefinite and $A > 0$ if A is positive definite; that is, $x^*Ax > 0$ for all nonzero column vectors $x \in \mathbb{C}^n$. For two Hermitian matrices A and B of the same size, we write $A \geqslant B$ if $A - B \geqslant 0$, and $A > B$ if $A - B > 0$. For $A \in \mathbb{M}_{k\times n}$, $\ker A$ means the null space of A, and we denote the orthogonal projection on the range space of A by P_A, that is, $P_A = A(A^*A)^\dagger A^*$, where $X^\dagger$ is the Moore-Penrose generalized inverse of X.

First of all, we recall the matrix geometric mean: Let A and B be two positive semidefinite matrices in $\mathbb{M}_n$. Then the matrix geometric mean $A \# B$ is defined by

$$A \# B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2} \tag{4.1}$$

if A is positive definite, also see [6, 4]. By monotonicity, we can uniquely extend the definition of $A \# B$ for all positive semidefinite matrices A and B by setting

$$A \# B = \lim_{\varepsilon\downarrow 0}(A+\varepsilon I) \# (B+\varepsilon I).$$

For the sake of convenience, we cite a useful lemma which we will use frequently in the below.

LEMMA 4.1. *Let A,B,C and D be positive semidefinite matrices.*

(i) Consistency with scalars: If A and B commute, then $A \# B = A^{1/2}B^{1/2}$;

(ii) Monotonicity: $A \leqslant C$ and $B \leqslant D \implies A \# B \leqslant C \# D$;

(iii) Transformer equality: $T^*AT \# T^*BT = T^*(A \# B)T$ for nonsingular T;

(iv) Symmetry: $A \# B = B \# A$*;*

(v) Arithmetic-geometric mean inequality:

$$A \# B \leqslant \frac{A+B}{2}.$$

In [3], we presented matrix Cauchy-Schwarz type inequalities that derived by the matrix geometric mean, also see [1]:

LEMMA 4.2. (Matrix Cauchy-Schwarz inequality) *Let* X *and* Y *be two* $k \times n$ *matrices in* $\mathbb{M}_{k\times n}$ *and* $U \in \mathbb{M}_n$ *a unitary matrix in a polar decomposition of* $Y^*X = U|Y^*X|$*. Then*

$$|Y^*X| \leqslant X^*X \# U^*Y^*YU. \tag{4.2}$$

Under the assumption $\ker X \subset \ker YU$*, the equality holds in* (4.2) *if and only if there exists* $W \in \mathbb{M}_n$ *such that* $YU = XW$*.*

Note that the matrix Cauchy-Schwarz inequality (4.2) is a natural extension of the Cauchy-Schwarz inequality (1.1). In fact, let x and y be $k \times 1$ vectors in $\mathbb{C}^k$. Since $\langle x,y\rangle = e^{i\theta}|\langle x,y\rangle|$ for some real number $\theta \in \mathbb{R}$, it follows from Lemma 4.2 that

$$\begin{aligned} |\langle x,y\rangle| &\leqslant \langle x,x\rangle \# e^{-i\theta}\langle y,y\rangle e^{i\theta} \\ &= \langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}} \\ &= \|x\|\,\|y\|. \end{aligned}$$

The following matrix inequality (4.3) corresponds to the matrix version of (2.1) in Theorem 2.1:

THEOREM 4.3. *Let* A *and* B *be two matrices in* $\mathbb{M}_{k\times n}$*. For any matrix* X *in* $\mathbb{M}_{k\times n}$

$$\begin{aligned} |B^*A - B^*P_XA| &\leqslant A^*(I-P_X)A \sharp U^*B^*(I-P_X)BU \\ &\quad (\leqslant A^*A \sharp U^*B^*BU), \end{aligned} \tag{4.3}$$

where $U \in \mathbb{M}_n$ *is a unitary matrix in a polar decomposition of* $B^*(I-P_X)A = U|B^*(I-P_X)A|$*.*

Under the assumption $\ker(I-P_X)A \subset \ker(I-P_X)BU$*, the equality holds in* (4.3) *if and only if there exists* $W \in \mathbb{M}_n$ *such that* $(I-P_X)BU = (I-P_X)AW$*.*

In fact, we notice that Theorem 4.3 implies Theorem 2.1. Let $a,b,x \in \mathbb{C}^k$. Then we have

$$b^*a - b^*P_xa = \langle a,b\rangle + \frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}$$

and

$$a^*(I-P_x)a = \|a\|^2 - \frac{|\langle a,x\rangle|^2}{\|x\|^2} \quad \text{and} \quad e^{-i\theta}b^*(I-P_x)be^{i\theta} = \|b\|^2 - \frac{|\langle b,x\rangle|^2}{\|x\|^2}.$$

Hence it follows from (4.3) that

$$\left| \langle a,b\rangle + \frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2} \right|^2 \leqslant \left(\|a\|^2 - \frac{|\langle a,x\rangle|^2}{\|x\|^2} \right) \left(\|b\|^2 - \frac{|\langle b,x\rangle|^2}{\|x\|^2} \right)$$

and so we have the desired inequality (2.1) in Theorem 2.1.

Here, Theorem 4.3 is a simple application of the matrix Cauchy-Schwarz inequality in Lemma 4.2. For the readers' convenience, we give a proof:

Proof of Theorem 4.3. By the matrix Cauchy-Schwarz inequality in Lemma 4.2, it follows that

$$\begin{aligned} |B^*A - B^*P_X A| &= |((I-P_X)B)^*(I-P_X)A| \\ &\leqslant A^*(I-P_X)A \sharp U^*B^*(I-P_X)BU \end{aligned}$$

and we have the desired inequality (4.3) and the equality conditions. □

Moreover, the following matrix inequality (4.4) corresponds to the matrix version of a two variable Ostrowski inequality in Theorem 3.1:

THEOREM 4.4. *Let A and B be two matrices in $\mathbb{M}_{k\times n}$, and X and Y two matrices in $\mathbb{M}_{k\times n}$ such that $X^*A = 0$ and $Y^*B = 0$, and $U \in \mathbb{M}_n$ a unitary matrix in a polar decomposition of $B^*A + B^*P_X P_Y A = U|B^*A + B^*P_X P_Y A|$. Then*

$$\begin{aligned} |B^*A + B^*P_X P_Y A| &\leqslant A^*(I-P_Y)A \sharp U^*B^*(I-P_X)BU \\ &\quad (\leqslant A^*A \sharp U^*B^*BU). \end{aligned} \tag{4.4}$$

Under the assumption $\ker(I-P_Y)A \subset \ker(I-P_X)BU$, *the equality holds in* (4.4) *if and only if there exists $W \in \mathbb{M}_n$ such that $(I-P_X)BU = (I-P_Y)AW$.*

Proof. Put $A_1 = (I-P_Y)A$ and $B_1 = (I-P_X)B$. Since $X^*A = 0$ and $Y^*B = 0$, we have $P_X A = 0$ and $B^*P_Y = 0$ and so

$$B_1^*A_1 = B^*(I-P_X)(I-P_Y)A = B^*A + B^*P_X P_Y A.$$

By Lemma 4.2, it follows from $B_1^*A_1 = U|B_1^*A_1|$ that

$$\begin{aligned} |B^*A + B^*P_X P_Y A| &= |B_1^*A_1| \\ &\leqslant A_1^*A_1 \sharp U^*B_1^*B_1 U \\ &= A^*(I-P_Y)A \sharp U^*B^*(I-P_X)BU \end{aligned}$$

and we have the desired inequality (4.4) and the equality conditions. □

REMARK 4.5. Theorem 4.4 implies Theorem 3.1. In fact, let $a,b \in \mathbb{C}^k$ and $x,y \in \mathbb{C}^k$ such that $x^*a = y^*b = 0$ and $x^*b = y^*a = 1$. Then we have

$$b^*a + b^*P_x P_y a = b^*a + \frac{\langle a,y\rangle}{\|y\|^2}\langle y,x\rangle\frac{\langle x,b\rangle}{\|x\|^2} = \langle a,b\rangle + \frac{\langle y,x\rangle}{\|x\|^2\,\|y\|^2}$$

and

$$a^*(1-P_y)a = a^*a - \frac{\langle a,y\rangle}{\|y\|^2}\langle y,a\rangle = \|a\|^2 - \frac{1}{\|y\|^2}(\geqslant 0)$$

and similarly

$$e^{-i\theta}b^*(1-P_x)be^{i\theta} = \|b\|^2 - \frac{1}{\|x\|^2}(\geqslant 0).$$

Therefore, it follows from (4.4) that

$$\left|\langle a,b\rangle + \frac{\langle y,x\rangle}{\|x\|^2\,\|y\|^2}\right| \leqslant \sqrt{\left(\|a\|^2 - \frac{1}{\|y\|^2}\right)\left(\|b\|^2 - \frac{1}{\|x\|^2}\right)}$$

amd by taking the square of both sides, we have the desired inequality (3.1).

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