Journal of Mathematical Inequalities
Volume 14, Number 4 (2020), 1389–1414
doi:10.7153/jmi-2020-14-91

THE PARTICULAR SOLUTION AND ULAM STABILITY OF LINEAR RIEMANN–LIOUVILLE FRACTIONAL DYNAMIC EQUATIONS ON ISOLATED TIME SCALES

YONGHONG SHEN* AND YONGJIN LI

(*Communicated by J. Pečarić*)

Abstract. By using the Laplace transform method on isolated time scales, this paper deals with the particular solution and the Ulam stability of linear Riemann-Liouville fractional dynamic equations with constant coefficients.

1. Introduction

The Ulam stability has been studied by a vast number of scientists for a few decades now, tracing back to Hyers' paper in 1941 [14]. This type of stability originated from a question about group homomorphism proposed by Ulam at university of Wisconsin in 1940 [35]. Essentially, the Ulam stability problem can be summarized as follows: "Under what conditions can a solution of a perturbed equation be close to a solution of the original equation?" Soon after, Hyers [14] gave a first affirmative answer to the previous question in Banach spaces. After many years, Rassias [28] further generalized the work of Hyers for linear mappings by considering an unbounded Cauchy difference. In the ensuing two decades, almost all research related to such stability focused on different types of functional equations and different abstract spaces. For more details, we refer to [10, 11, 16, 20, 21, 22, 29] and reference therein.

The Ulam stability of differential equations was initiated by Obloza [27] in 1993. A few years later, Alsina and Ger [3] considered the Hyers-Ulam stability of the differential equation $y'(x) = y(x)$. Afterward, Miura and Takahasi et al. [23, 25, 33] further studied the Ulam stability of the differential equation $y'(x) = \lambda y(x)$ in various abstract spaces. By using the same method as in [3], Jung [17] considered the Hyers-Ulam stability of the differential equation $\varphi(x)y'(x) = y(x)$. In particular, the Hyers-Ulam stability of the linear differential equation was extensively investigated by many authors [2, 18, 19, 24, 26, 34].

Fractional differential equations serve as an excellent tool for the description of memory and hereditary properties of various materials and processes. The theory of

Mathematics subject classification (2010): 39A30, 34N05, 34A08.
Keywords and phrases: Ulam stability, Laplace transform, Riemann-Liouville fractional Δ-integral (derivative), Fractional dynamic equations, Isolate time scales.
* Corresponding author.

Paper JMI-14-91

fractional differential equations has gained considerable popularity and importance over the past three decades. In 2012, Wang et al. [40] first studied the Ulam stability of the fractional differential equation ${}^cD^{\alpha}y(x) = f(x,y(x))$ with the Caputo derivative by using the fixed point method in a generalized complete metric space. In the same year, Wang et al. [41] also considered the Ulam stability of the previous equation with impulsive conditions. Afterwards, the Ulam stability problem of various types of fractional differential equations were investigated by Wang et al. [36, 37, 38, 39]. In 2016, Jiang et al. [15] studied the Ulam stability of the Caputo fractional differential equation ${}^cD^{\alpha}u(t) = (Qu)(t)$ with the causal operator Q by a fixed point theorem for a class of operators satisfying suitable conditions. Recently, Cuong [9] obtained a result of Hyers-Ulam stability of Riemann-Liouville multi-order fractional differential equations by using the Bielecki's type norm and Banach fixed point theorem. By employing some fixed point theorems in Banach spaces, Abbas et al. [1] established the existence and Ulam stability of implicit Caputo fractional q-difference equations. At the same time, Butt et al. [8] considered the Ulam stability of Caputo q-fractional delay difference equations based on the q-fractional Gronwall inequality.

In 1988, Hilger [13] proposed the time scale and established the theory of time scale calculus in order to unify discrete and continuous analysis. Accordingly, these results provide an important basis for studying differential equations and difference equations in a uniform way. In the past few decades, the time scale calculus and the corresponding dynamic equations have been deeply and systematically studied (see Bohner and Peterson [4, 6]). With the rapid development of the theory of fractional differential equations, Georgiev [12] established the theory of fractional dynamic calculus and considered the solution of fractional dynamic equations on time scales. The Laplace transform method is a significant tool in solving linear differential equations with constant coefficients. In 2013, Rezaei et al. [30] studied the Ulam stability of linear differential equations with constant coefficients by using the Laplace transform. Inspired by this paper, Shen and Chen [31] discussed the Ulam stability of Riemann-Liouville fractional differential equations with general form by the Laplace transform method. Recently, Shen and Li [32] also studied the Ulam stability of linear difference equations with constant coefficients by the z-transform method. More generally, Bohner and Peterson [5] introduced the Laplace transform on time scales to solve linear dynamic equations. However, the inverse Laplace transform and the convolution of two arbitrary regulated functions on time scales are not properly defined. Until 2010, Bohner and Guseinov [7] proposed the inverse Laplace transform and the convolution of two regulated functions on isolated time scales. Up to now, it is still an open question to give an appropriate definition of the inverse Laplace transform and convolution on an arbitrary time scale.

Similar to the idea of our previous paper [31], the aim of this paper is, using the Laplace transform method, to discuss the particular solution and Ulam stability problem of linear Riemman-Liouville fractional dynamic equations with constant coefficients on isolated time scales. The general form of the fractional dynamic equation is as follows:

$$\sum_{k=1}^{l} A_k D^{\alpha_k}_{\Delta,t_0} y(t) + A_0 y(t) = f(t), \qquad t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}},$$

where $l \in \mathbb{N}$, $0 < \alpha_1 < \cdots < \alpha_l$ and $A_0, A_1, \cdots, A_l \in \mathbb{R}$. It should be pointed out that the results obtain in this paper do not include the results in [31], which can be regarded a proper supplement to the corresponding results.

2. Preliminaries

For the sake of completeness, in this section, we review some basic notions and results associated with the Laplace transform, fractional derivative and fractional integral on isolated time scales. Let $\mathbb{N}$, $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers, respectively.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Obviously, $\mathbb{R}$ and $\mathbb{Z}$ are two typical examples of time scales.

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ can be defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

for all $t \in \mathbb{T}$, where we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

DEFINITION 2.1. (Bohner and Guseinov [7]) An isolated time scale $\widetilde{\mathbb{T}} = \{t_n : n \in \mathbb{N}_0\}$ is a countable subset of the real line $\mathbb{R}$ with

$$\lim_{n\to\infty} t_n = \infty, \qquad \omega := \inf_{n\in\mathbb{N}_0} \mu(t_n) > 0,$$

where $\mu(t_n) = \sigma(t_n) - t_n = t_{n+1} - t_n$, $n \in \mathbb{N}_0$.

From Definition 2.1, it is easy to see that the number sets $\mathbb{N}_0$, $\mathbb{N}_0^2$, $h\mathbb{N}_0 = \{hn : n \in \mathbb{N}_0\}$ $(h > 0)$ and $q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$ $(q > 1)$ are examples of isolated time scales, while the number sets $\{\sqrt{n} : n \in \mathbb{N}_0\}$, $\{\ln n : n \in \mathbb{N}\}$ and $\{\sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\}$ are not isolated time scales.

DEFINITION 2.2. (Bohner and Guseinov [7]) Let $f : \widetilde{\mathbb{T}} \to \mathbb{C}$ be a function. Then the Laplace transform of f is defined by

$$\mathscr{L}(f(t))(z) := \sum_{n=0}^{\infty} \frac{\mu(t_n) f(t_n)}{\prod_{k=0}^{n}(1+\mu(t_k)z)} \tag{1}$$

for all $z \in \mathbb{C}$ satisfying $1 + z\mu(t_n) \neq 0$, $n \in \mathbb{N}_0$ for which the series (1) converges (or exists).

For an isolated time scale, we note that $\mu(t_n) \geqslant \omega$ for all $n \in \mathbb{N}_0$. Then, we have $-\frac{1}{\mu(t_n)} \in [-\frac{1}{\omega}, 0)$. Given an arbitrary $\delta > 0$, we define

$$D_\delta = \mathbb{C} \setminus \bigcup_{n=0}^{\infty} D_\delta^n,$$

where $D_{\delta}^{n} = \left\{ z \in \mathbb{C} : \left| z + \frac{1}{\mu(t_n)} \right| < \delta \right\}$, $n \in \mathbb{N}_0$.

Furthermore, we set

$$\mathscr{F}_{\delta} = \left\{ f : \widetilde{\mathbb{T}} \to \mathbb{C} : \sum_{n=0}^{\infty} (\delta\omega)^{-n} |f(t_n)| < \infty \right\}.$$

DEFINITION 2.3. (Bohner and Guseinov [7]) Let $f \in \mathscr{F}_{\delta}$ and let $\mathscr{L}(f(t))(z)$ be its Laplace transform. Then the inverse Laplace transform $f(t_n)$ of $\mathscr{L}(f(t))(z)$ is defined by

$$f(t_n) = \mathscr{L}^{-1}(\mathscr{L}(f)(z))(t_n) = \frac{1}{2\pi i} \int_{\gamma} \mathscr{L}(f)(z) \prod_{k=0}^{n-1} (1 + \mu(t_k) z) dz, \quad n \in \mathbb{N}_0,$$

where γ is any positively oriented closed curve in the region D_{δ} that encloses all the points $-\frac{1}{\mu(t_n)}$ for all $n \in N_0$.

For given two functions $f, g : \widetilde{\mathbb{T}} \to \mathbb{R}$, the convolution $f \star g$ can be defined by

$$(f \star g)(t) = \int_{t_0}^{t} \widehat{f}(t, \sigma(s)) g(s) \Delta s, \qquad t \in \widetilde{\mathbb{T}},\ t \geqslant t_0,$$

where $\widehat{f}$ is the shift or delay of f, i.e., f is the solution of the shifting problem

$$\begin{cases} u^{\Delta_t}(t, \sigma(s)) = -u^{\Delta_s}(t, s), \\ u(t, t_0) = f(t), \qquad t, s \in \widetilde{\mathbb{T}},\ t \geqslant s \geqslant t_0. \end{cases}$$

Suppose that $f, g : \widetilde{\mathbb{T}} \to \mathbb{R}$ are two functions such that $\mathscr{L}(f)(z)$, $\mathscr{L}(g)(z)$ and $\mathscr{L}(f \star g)(z)$ exist for a $z \in \mathbb{C}$. Then we have

$$\mathscr{L}(f \star g)(z) = \mathscr{L}(f)(z) \mathscr{L}(g)(z).$$

Using the inverse Laplace transform, the generalized Δ-power function $h_{\alpha}(t, t_0)$ on $\widetilde{\mathbb{T}}$ can be introduced as following:

$$h_{\alpha}(t, t_0) = \mathscr{L}^{-1}\left(\frac{1}{z^{\alpha+1}} \right)(t),\ t \geqslant t_0,$$

for all $z \in \mathbb{C} \setminus \{0\}$ such that $\mathscr{L}^{-1}$ exists for any $t \geqslant t_0$, where $\alpha \in \mathbb{R}$.

DEFINITION 2.4. (Georgiev [12]) Let $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ be a function and let $\alpha \geqslant 0$. The Riemann-Liouville fractional Δ-integral of order α is defined in the following form

$$\begin{aligned} I_{\Delta, t_0}^{\alpha} f(t) &:= (h_{\alpha-1}(\cdot, t_0) \star f)(t) \\ &= \int_{t_0}^{t} h_{\alpha-1}(t, \sigma(\tau)) f(\tau) \Delta\tau \end{aligned}$$

for $t > t_0$. In particular, $I_{\Delta, t_0}^{0} f(t) = f(t)$ if $\alpha = 0$.

DEFINITION 2.5. (Georgiev [12]) Let $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ be a function and let $m = -\overline{[-\alpha]}$ with $\alpha \geqslant 0$. The Riemann-Liouville fractional Δ-derivative of order α is defined by

$$D^{\alpha}_{\Delta,t_0} f(t) = D^m_{\Delta} I^{m-\alpha}_{\Delta,t_0} f(t)$$

provided that the right-hand side exists for $t \in \widetilde{\mathbb{T}}$ and $t > t_0$.

THEOREM 2.1. (Georgiev [12]) *Let $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ be locally Δ-integrable, i.e., f is Δ-integrable on each compact subinterval of $\widetilde{\mathbb{T}}$, and let $m = -\overline{[-\alpha]}$ with $\alpha > 0$. Then*

$$\mathscr{L}(D^{\alpha}_{\Delta,t_0} f(t))(z,t_0) = z^{\alpha} \mathscr{L}(f(t))(z,t_0) - \sum_{k=1}^{m} z^{k-1} D^{\alpha-k}_{\Delta,t_0} f(t_0)$$

for all $t \geqslant t_0$.

The Δ-Mittage-Leffler function is defined by

$${}_{\Delta}F_{\alpha,\beta}(\lambda,t,t_0) = \sum_{k=0}^{\infty} \lambda^k h_{\alpha k+\beta-1}(t,t_0)$$

provided that the right-hand side converges, where $\alpha, \beta > 0$ and $\lambda \in \mathbb{R}$.

LEMMA 2.2. *Let $g : [t_0,+\infty)_{\widetilde{\mathbb{T}}} = [t_0,+\infty) \bigcap \widetilde{\mathbb{T}})$ be a function. Then the convolution of the Δ-Mittage-Leffler function ${}_{\Delta}F_{\alpha,\beta}$ and the function g is*

$$({}_{\Delta}F_{\alpha,\beta} \star g)(t) = \sum_{k=0}^{\infty} \lambda^k \int_{t_0}^{t} h_{\alpha k+\beta-1}(t,\sigma(s)) g(s) \Delta s.$$

Proof. *By Theorem 3.4 in* [12], *we get*

$${}_{\Delta}\widehat{F}_{\alpha,\beta}(\lambda,t,s) = \sum_{k=0}^{\infty} \lambda^k h_{\alpha k+\beta-1}(t,s).$$

Furthmore, we have

$${}_{\Delta}\widehat{F}_{\alpha,\beta}(\lambda,t,\sigma(s)) = \sum_{k=0}^{\infty} \lambda^k h_{\alpha k+\beta-1}(t,\sigma(s)).$$

Then, we can infer that

$$\begin{aligned}({}_{\Delta}F_{\alpha,\beta} \star g)(t) &= \int_{t_0}^{t} {}_{\Delta}\widehat{F}_{\alpha,\beta}(\lambda,t,\sigma(s)) g(s) \Delta s \\ &= \int_{t_0}^{t} \sum_{k=0}^{\infty} \lambda^k h_{\alpha k+\beta-1}(t,\sigma(s)) g(s) \Delta s \\ &= \sum_{k=0}^{\infty} \lambda^k \int_{t_0}^{t} h_{\alpha k+\beta-1}(t,\sigma(s)) g(s) \Delta s. \quad \square\end{aligned}$$

For the Laplace transform of the Δ-Mittage-Leffler function, we have the following result.

THEOREM 2.3. (Georgiev [12]) *Let* ${}_\Delta F_{\alpha,\beta}(\lambda,t,t_0)$ *be* Δ*-Mittage-Leffler function. Then*

$$\mathscr{L}({}_\Delta F_{\alpha,\beta}(\lambda,t,t_0))(z,t_0)=\frac{z^{\alpha-\beta}}{z^\alpha-\lambda},$$
$$\mathscr{L}\Big(\frac{\partial^n}{\partial\lambda^n}{}_\Delta F_{\alpha,\beta}(\lambda,t,t_0)\Big)(z,t_0)=\frac{n!z^{\alpha-\beta}}{(z^\alpha-\lambda)^{n+1}},$$

where $\alpha,\beta>0$, $n\in\mathbb{N}$ *and* $|\lambda|<|z|^\alpha$.

3. Particular solutions of inhomogeneous Riemann-Liouville fractional dynamic equations

In [12], the explicit solutions to homogeneous Riemann-Liouville fractional dynamic equations with constant coefficients have derived by applying the Laplace transform method. Here we apply this method to find particular solutions to the corresponding inhomogeneous equations

$$\sum_{k=1}^m A_k D_{\Delta,t_0}^{\alpha_k}y(t)+A_0y(t)=f(t),\quad t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}, \tag{2}$$

where $0<\alpha_1<\alpha_2<\cdots<\alpha_m$, $m\in\mathbb{N}$, $A_0,A_k\in\mathbb{R}$, $k=1,2,\cdots,m$.

The Laplace fractional analogue of the Green function is introduced as follows

$$G_{\alpha_1,\alpha_2,\cdots,\alpha_m}(t)=\mathscr{L}^{-1}\Big(\frac{1}{P_{\alpha_1,\alpha_2,\cdots,\alpha_m}(z)}\Big)(t),\quad t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}, \tag{3}$$

where $P_{\alpha_1,\alpha_2,\cdots,\alpha_m}(z)=A_0+\sum_{k=1}^m A_kz^{\alpha_k}$.

For a particular solution $y_p(t)$ of (2) subject to the initial conditions $D_{\Delta,t_0}^{\alpha_k-j}(t_0)=0$, $j=1,2,\cdots,m_k$, $k=1,2,\cdots,m$, $m_k=[\alpha_k]+1$. By taking the Laplace transform of both sides of (2), we get

$$\mathscr{L}(y_p(t))(z,t_0)=\frac{\mathscr{L}(f(t))(z,t_0)}{P_{\alpha_1,\alpha_2,\cdots,\alpha_m}(z)}.$$

Then, the particular solution $y_p(t)$ of (2) can be obtained by the convolution of $f(t)$ and $G_{\alpha_1,\alpha_2,\cdots,\alpha_m}(t)$, i.e., $y_p(t)=(G_{\alpha_1,\alpha_2,\cdots,\alpha_m}\star f)(t)$.

For convenience, we will use the Δ-Mittag-Lefler function to derive a particular solution to (2).

THEOREM 3.1. *Let $\alpha > 0$, $\lambda \in \mathbb{R}$ and let $f(t)$ be a function defined on the isolated time scale $[t_0, +\infty)_{\widetilde{\mathbb{T}}}$. Then the particular solution to the fractional dynamic equation*

$$D^{\alpha}_{\Delta,t_0} y(t) - \lambda y(t) = f(t) \tag{4}$$

has the form

$$y(t) = \sum_{k=0}^{\infty} \lambda^k \int_{t_0}^{t} h_{(k+1)\alpha-1}(t, \sigma(s)) f(s) \Delta s \tag{5}$$

provided that the integral in the right-hand side of (5) is convergent.

Proof. From (3), we can obtain

$$G_\alpha(t) = \mathscr{L}^{-1}\Big(\frac{1}{z^\alpha - \lambda}\Big)(t).$$

By Theorem 2.3 with $\beta = \alpha$, we get

$$\mathscr{L}({}_{\Delta}F_{\alpha,\alpha}(\lambda, t, t_0))(z, t_0) = \frac{1}{z^\alpha - \lambda}, \qquad |\lambda| < |z|^\alpha.$$

Then, we have

$$G_\alpha(t) = {}_{\Delta}F_{\alpha,\alpha}(\lambda, t, t_0).$$

Hence $y(t) = (G_\alpha \star f)(t)$ and thus (5) is valid from Lemma 2.2. □

THEOREM 3.2. *Let $\alpha > \beta > 0$, $\lambda, \mu \in \mathbb{R}$ with $\mu \neq 0$ and let $f(t)$ be a function defined on the isolated time scale $[t_0, +\infty)_{\widetilde{\mathbb{T}}}$. Then the particular solution to the fractional dynamic equation*

$$D^{\alpha}_{\Delta,t_0} y(t) - \lambda D^{\beta}_{\Delta,t_0} y(t) - \mu y(t) = f(t) \tag{6}$$

has the form

$$y(t) = (G_{\alpha,\beta} \star f)(t) \tag{7}$$

provided that the right-hand side of (7) exists, where

$$G_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\Delta}F_{\alpha-\beta,\alpha+\beta n}(\lambda, t, t_0).$$

Proof. From (3), we get

$$G_{\alpha,\beta}(t) = \mathscr{L}^{-1}\Big(\frac{1}{z^\alpha - \lambda z^\beta - \mu}\Big)(t).$$

For $z \in \mathbb{C}$ with $\left|\frac{\mu z^{-\beta}}{z^{\alpha-\beta} - \lambda}\right| < 1$, we have

$$\begin{aligned}\frac{1}{z^\alpha - \lambda z^\beta - \mu} &= \frac{z^{-\beta}}{z^{\alpha-\beta} - \lambda} \frac{1}{1 - \frac{\mu z^{-\beta}}{z^{\alpha-\beta}-\lambda}} \\ &= \sum_{n=0}^{\infty} \frac{\mu^n z^{-(n+1)\beta}}{(z^{\alpha-\beta} - \lambda)^{n+1}}.\end{aligned}$$

Then, we can obtain that

$$G_{\alpha,\beta}(t)=\mathscr{L}^{-1}\Big(\sum_{n=0}^{\infty}\frac{\mu^n z^{-(n+1)\beta}}{(z^{\alpha-\beta}-\lambda)^{n+1}}\Big)(t).$$

By Theorem 2.3, for $|\lambda|<|z|^{\alpha-\beta}$, replacing α by $\alpha-\beta$ and β by $\alpha+\beta n$, it follows that

$$\begin{aligned}\frac{z^{-(n+1)\beta}}{(z^{\alpha-\beta}-\lambda)^{n+1}}&=\frac{z^{(\alpha-\beta)-(\alpha+\beta n)}}{(z^{\alpha-\beta}-\lambda)^{n+1}}\\&=\frac{1}{n!}\mathscr{L}\Big(\frac{\partial^n}{\partial\lambda^n}{}_{\Delta}F_{\alpha-\beta,\alpha+\beta n}(\lambda,t,t_0)\Big)(z,t_0).\end{aligned}$$

Then, we can infer that

$$G_{\alpha,\beta}(t)=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\Delta}F_{\alpha-\beta,\alpha+\beta n}(\lambda,t,t_0).$$

Hence $y(t)=(G_{\alpha,\beta}\star f)(t)$ and thus (7) is valid. □

THEOREM 3.3. *Let $\alpha>\beta>0$, $\lambda\in\mathbb{R}$ and let $f(t)$ be a function defined on the isolated time scale $[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. Then the particular solution to the fractional dynamic equation*

$$D^{\alpha}_{\Delta,t_0}y(t)-\lambda D^{\beta}_{\Delta,t_0}y(t)=f(t)$$

has the form

$$y(t)=\sum_{k=0}^{\infty}\lambda^k\int_{t_0}^{t}h_{(k+1)\alpha-k\beta-1}(t,\sigma(s))f(s)\Delta s$$

provided that the integral in the right-hand side of the forgoing equality is convergent.

Proof. Using the similar argument as in Theorem 3.3, by (3), we get

$$G_{\alpha,\beta}(t)=\mathscr{L}^{-1}\Big(\frac{1}{z^{\alpha}-\lambda z^{\beta}}\Big)(t)$$

Moreover, we have

$$\frac{1}{z^{\alpha}-\lambda z^{\beta}}=\frac{z^{-\beta}}{z^{\alpha-\beta}-\lambda}.$$

According to Theorem 2.3, replacing α and β by $\alpha-\beta$ and α, respectively. So we can obtain

$$\mathscr{L}({}_{\Delta}F_{\alpha-\beta,\alpha}(\lambda,t,t_0))(z,t_0)=\frac{z^{-\beta}}{z^{\alpha-\beta}-\lambda}.$$

Taking the inverse Laplace transform and using the previous inequalities, we obtain

$$G_{\alpha,\beta}(t)={}_{\Delta}F_{\alpha-\beta,\alpha}(\lambda,t,t_0).$$

Therefore, the conclusion is valid from Lemma 2.2. □

THEOREM 3.4. *Let* $\alpha > \beta > \alpha_{m-2} > \cdots > \alpha_1 > \alpha_0 = 1$, $m \in \mathbb{N} \setminus \{1,2\}$, *and let* $\lambda, A_0, \cdots, A_{m-2} \in \mathbb{R}$. *Suppose that* $f(t)$ *is a function defined on the isolated time scale* $[t_0, +\infty)_{\widetilde{\mathbb{T}}}$. *Then the particular solution to the fractional dynamic equation*

$$D^{\alpha}_{\Delta,t_0} y(t) - \lambda D^{\beta}_{\Delta,t_0} y(t) - \sum_{k=0}^{m-2} A_k D^{\alpha_k}_{\Delta,t_0} y(t) = f(t) \tag{8}$$

has the form

$$y(t) = (G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}} \star f)(t) \tag{9}$$

provided that the right-hand side of (9) exists, where

$$\begin{aligned} G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}(t) = \sum_{n=0}^{\infty} \sum_{k_0+k_1+\cdots+k_{m-2}=n} \frac{1}{k_0! k_1! \cdots k_{m-2}!} \\ \times \prod_{v=0}^{m-2} A_v^{k_v} \frac{\partial}{\partial \lambda^n} {}_{\Delta}F_{\alpha-\beta,\alpha+\sum_{v=0}^{m-2}(\beta-\alpha_v)k_v}(\lambda, t, t_0) \end{aligned} \tag{10}$$

Proof. From (3), we get

$$G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}(t) = \mathscr{L}^{-1}\left(\frac{1}{z^{\alpha} - \lambda z^{\beta} - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} \right)(t).$$

For $z \in \mathbb{C}$ with $\left| \sum_{k=0}^{m-2} A_k \frac{z^{\alpha_k-\beta}}{z^{\alpha-\beta} - \lambda} \right| < 1$, it follows that

$$\begin{aligned} &\frac{1}{z^{\alpha} - \lambda z^{\beta} - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} \\ &= \frac{z^{-\beta}}{z^{\alpha-\beta} - \lambda} \frac{1}{1 - \sum_{k=0}^{m-2} A_k \frac{z^{\alpha_k-\beta}}{z^{\alpha-\beta}-\lambda}} \\ &= \frac{z^{-\beta}}{z^{\alpha-\beta} - \lambda} \sum_{n=0}^{\infty} \left(\frac{\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta}}{z^{\alpha-\beta}-\lambda} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{z^{-\beta}}{(z^{\alpha-\beta}-\lambda)^{n+1}} \left(\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta} \right)^n \\ &= \sum_{n=0}^{\infty} \sum_{k_0+k_1+\cdots+k_{m-2}=n} \frac{n!}{k_0! k_1! \cdots k_{m-2}!} \prod_{v=0}^{m-2} A_v^{k_v} \frac{z^{-\beta-\sum_{v=0}^{m-2}(\beta-\alpha_v)k_v}}{(z^{\alpha-\beta}-\lambda)^{n+1}}. \end{aligned}$$

According to Theorem 2.3, for $|\lambda| < |z|^{\alpha-\beta}$, replacing α by $\alpha-\beta$ and β by $\alpha+\sum_{l=0}^{m-2}(\beta-\alpha_l)k_l$, we can obtain

$$\begin{aligned}\frac{z^{-\beta-\sum_{l=0}^{m-2}(\beta-\alpha_l)k_l}}{(z^{\alpha-\beta}-\lambda)^{n+1}} &= \frac{z^{\alpha-\beta-(\alpha+\sum_{l=0}^{m-2}(\beta-\alpha_l)k_l)}}{(z^{\alpha-\beta}-\lambda)^{n+1}}\\ &= \frac{1}{n!}\mathscr{L}\Big(\frac{\partial}{\partial\lambda^n}{}_\Delta F_{\alpha-\beta,\alpha+\sum_{v=0}^{m-2}(\beta-\alpha_v)k_v}(\lambda,t,t_0)\Big)(z,t_0)\Big).\end{aligned}$$

Then, we can infer that (10) is true and hence (9) holds. □

4. Ulam stability of Riemann-Liouville fractional dynamic equations

By employing the Laplace transform method, in this section, we shall deal with the Ulam stability problem of linear Riemann-Liouville fractional dynamic equations with constant coefficients on isolated time scales.

THEOREM 4.1. *Let $l-1<\alpha\leqslant l$, $l\in\mathbb{N}$, $n=-\overline{[-\alpha]}$ and let $\lambda\in\mathbb{R}$. Assume that $\varphi:[t_0,+\infty)_{\widetilde{\mathbb{T}}}\to[0,+\infty)$ is a function such that the series*

$$\sum_{k=0}^{\infty}|\lambda|^k\int_{t_0}^{t}|h_{(k+1)\alpha-1}(t,\sigma(s))|\varphi(s)\Delta s$$

converges for all $t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. If a function $y:[t_0,+\infty)_{\widetilde{\mathbb{T}}}\to\mathbb{R}$ satisfies the inequality

$$|D^{\alpha}_{\Delta,t_0}y(t)-\lambda y(t)-f(t)|\leqslant\varphi(t) \tag{11}$$

for all $t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}$, then there exists a solution $y_\varphi:[t_0,+\infty)_{\widetilde{\mathbb{T}}}\to\mathbb{R}$ of the fractional dynamic equation

$$D^{\alpha}_{\Delta,t_0}y(t)-\lambda y(t)=f(t) \tag{12}$$

such that

$$|y(t)-y_\varphi(t)|\leqslant\sum_{k=0}^{\infty}|\lambda|^k\int_{t_0}^{t}|h_{(k+1)\alpha-1}(t,\sigma(s))|\varphi(s)\Delta s$$

for all $t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}$.

Proof. Set $g(t)=D^{\alpha}_{\Delta,t_0}y(t)-\lambda y(t)-f(t)$, $t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. In view of Theorem 2.1, by using the Laplace transform to $g(t)$, we can obtain

$$\begin{aligned}\mathscr{L}(g(t))(z,t_0)=z^{\alpha}\mathscr{L}(y(t))(z,t_0)-\sum_{k=1}^{m}d_kz^{k-1}\\ -\lambda\mathscr{L}(y(t))(z,t_0)-\mathscr{L}(f(t))(z,t_0),\end{aligned}$$

where $d_k=D^{\alpha-k}_{\Delta,t_0}f(t_0)$. Then, it follows from the previous equality that

$$\mathscr{L}(y(t))(z,t_0)=\sum_{k=1}^{l}d_k\frac{z^{k-1}}{z^{\alpha}-\lambda}+\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda}+\frac{\mathscr{L}(g(t))(z,t_0)}{z^{\alpha}-\lambda}. \tag{13}$$

Define

$$y_\varphi(t) = \sum_{k=1}^{l} d_k y_k(t) + (G_\alpha \star f)(t),$$

where

$$\begin{aligned} &y_k(t) = {}_\Delta F_{\alpha,\alpha-k+1}(\lambda,t,t_0),\\ &G_\alpha(t) = {}_\Delta F_{\alpha,\alpha}(\lambda,t,t_0),\\ &(G_\alpha \star f)(t) = \sum_{k=0}^{\infty} \lambda^k \int_{t_0}^{t} h_{(k+1)\alpha-1}(t,\sigma(s)) f(s)\Delta s. \end{aligned}$$

According to Theorem 2.3, replacing β by $\alpha-k+1$, we can obtain that

$$\mathscr{L}(y_k(t))(z,t_0) = \mathscr{L}({}_\Delta F_{\alpha,\alpha-k+1}(\lambda,t,t_0))(z,t_0) = \frac{z^{k-1}}{z^\alpha-\lambda}, \qquad |\lambda| < |z|^\alpha. \tag{14}$$

Again, we replace β by α, it follows that

$$\mathscr{L}(G_\alpha(t))(z,t_0) = \mathscr{L}({}_\Delta F_{\alpha,\alpha}(\lambda,t,t_0))(z,t_0) = \frac{1}{z^\alpha-\lambda}, \qquad |\lambda| < |z|^\alpha. \tag{15}$$

From (14) and (15), we can infer that

$$\begin{aligned} \mathscr{L}(y_\varphi(t))(z,t_0) &= \sum_{k=1}^{l} d_k \mathscr{L}(y_k(t))(z,t_0) + \mathscr{L}((G_\alpha \star f)(t))(z,t_0)\\ &= \sum_{k=1}^{l} d_k \frac{z^{k-1}}{z^\alpha-\lambda} + \frac{\mathscr{L}(f(t))(z,t_0)}{z^\alpha-\lambda}. \end{aligned} \tag{16}$$

By Theorem 3.1, we know that $(G_\alpha \star f)(t)$ is a particular solution of Eq. (12). Then, it follows from Lemma 2.1 that

$$\begin{aligned} \mathscr{L}(f(t))(z,t_0) &= \mathscr{L}(D^\alpha_{\Delta,t_0}(G_\alpha \star f)(t))(z,t_0) - \lambda \mathscr{L}(G_\alpha \star f)(t)(z,t_0)\\ &= z^\alpha \mathscr{L}(G_\alpha \star f)(t)(z,t_0) - \sum_{k=1}^{l} h_k z^{k-1} - \lambda \mathscr{L}(G_\alpha \star f)(t)(z,t_0)\\ &= (z^\alpha-\lambda)\mathscr{L}(G_\alpha \star f)(t)(z,t_0) - \sum_{k=1}^{l} h_k z^{k-1}\\ &= (z^\alpha-\lambda)\frac{\mathscr{L}(f(t))(z,t_0)}{z^\alpha-\lambda} - \sum_{k=1}^{l} h_k z^{k-1}, \end{aligned} \tag{17}$$

where $h_k = D^{\alpha-k}_{\Delta,t_0} f(t_0)$. From the previous equality, we can obtain $\sum_{k=1}^{m} h_k z^{k-1} = 0$. Moreover, by Theorem 6.2 in [12], we know that $y_k(t)$ is a solution of the associated homogeneous equation of Eq. (12). So we have

$$D^\alpha_{\Delta,t_0} {}_\Delta F_{\alpha,\alpha-k+1}(\lambda,t,t_0) = \lambda {}_\Delta F_{\alpha,\alpha-k+1}(\lambda,t,t_0). \tag{18}$$

Since the operators $\mathscr{L}$ and D^{α}_{Δ,t_0} are linear, we can from (14) and (16)–(18) that

$$\begin{aligned}
&\mathscr{L}(D^{\alpha}_{\Delta,t_0}y_{\varphi}(t)-\lambda y_{\varphi}(t))(z,t_0)\\
&=\mathscr{L}\Big[\big(D^{\alpha}_{\Delta,t_0}\big(\sum_{k=1}^{m}d_k y_k+D^{\alpha}_{\Delta,t_0}(G_{\alpha}\star f)-\lambda y_{\varphi}\big)(t)\Big](z,t_0)\\
&=\sum_{k=1}^{l}d_k\mathscr{L}(D^{\alpha}_{\Delta,t_0}y_k(t))(z,t_0)+\mathscr{L}(D^{\alpha}_{\Delta,t_0}(G_{\alpha}\star f)(t))(z,t_0)-\lambda\mathscr{L}(y_{\varphi}(t))(z,t_0)\\
&=\lambda\sum_{k=1}^{l}d_k\mathscr{L}(y_k(t))(z,t_0)+z^{\alpha}\mathscr{L}(G_{\alpha}\star f)(t))(z,t_0)-\sum_{k=1}^{l}h_k z^{k-1}\\
&\qquad-\lambda\Big(\sum_{k=1}^{l}d_k\frac{z^{k-1}}{z^{\alpha}-\lambda}+\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda}\Big)\\
&=\mathscr{L}(f(t))(z,t_0),
\end{aligned}\tag{19}$$

which implies that $y_{\varphi}(t)$ is a solution of Eq. (12), since the Laplace transform operator $\mathscr{L}$ is one-to-one on the isolated time scale.

Additionally, it follows from (13) and (16) that

$$\mathscr{L}(y(t))(z,t_0)-\mathscr{L}(y_{\varphi}(t))(z,t_0)=\frac{\mathscr{L}(g(t))(z,t_0)}{z^{\alpha}-\lambda}=\mathscr{L}((G_{\alpha}\star g)(t))(z,t_0).$$

Using the linearity and the inverse Laplace transform $\mathscr{L}^{-1}$, we can obtain

$$y(t)-y_{\varphi}(t)=(G_{\alpha}\star g)(t),\qquad t\in\widetilde{\mathbb{T}}.$$

From the inequality (11), we know that $|g(t)|\leqslant\varphi(t)$ for all $t\in\widetilde{\mathbb{T}}$. Therefore, we can get

$$\begin{aligned}
|y(t)-y_{\varphi}(t)|&=|(G_{\alpha}\star g)(t)|\\
&=\Big|\sum_{k=0}^{\infty}\lambda^k\int_{t_0}^{t}h_{(k+1)\alpha-1}(t,\sigma(s))g(s)\Delta s\Big|\\
&\leqslant\sum_{k=0}^{\infty}|\lambda|^k\int_{t_0}^{t}|h_{(k+1)\alpha-1}(t,\sigma(s))g(s)|\Delta s\\
&\leqslant\sum_{k=0}^{\infty}|\lambda|^k\int_{t_0}^{t}|h_{(k+1)\alpha-1}(t,\sigma(s))|\varphi(s)\Delta s
\end{aligned}$$

for all $t\in\widetilde{\mathbb{T}}$. The theorem is now completed. □

As a direct consequence of Theorem 4.1, we can obtain the Hyers-Ulam stability of Eq. (12).

COROLLARY 4.2. *Let $l-1<\alpha\leqslant l$, $l\in\mathbb{N}$, $n=-\lceil-\alpha\rceil$ and let $\lambda\in\mathbb{R}$. Assume that the series $\sum_{k=0}^{\infty}|\lambda|^k\int_{t_0}^{t}|h_{(k+1)\alpha-1}(t,\sigma(s))|\Delta s$ converges for all $t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. For a given $\varepsilon>0$, if a function $y:[t_0,+\infty)_{\widetilde{\mathbb{T}}}\to\mathbb{R}$ satisfies the inequality*

$$|D^{\alpha}_{\Delta,t_0}y(t)-\lambda y(t)-f(t)|\leqslant\varepsilon$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$, *then there exists a solution* $y_\varepsilon : [t_0,+\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ *of the fractional dynamic equation (12) such that*

$$|y(t) - y_\varphi(t)| \leqslant K_1\varepsilon$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$, *where*

$$K_1 = \sup_{t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}} \sum_{k=0}^{\infty} |\lambda|^k \int_{t_0}^{t} |h_{(k+1)\alpha-1}(t,\sigma(s))|\Delta s.$$

REMARK 1. When K_1 is finite, the result shows that the fractional dynamic equation (12) is Hyers-Ulam stable. Otherwise, we can say that the equation (12) is generalized Hyers-Ulam stable.

THEOREM 4.3. *Let* $l-1<\alpha\leqslant l$, $l\in\mathbb{N}$, $\alpha>\beta>0$, $n=-\overline{[-\alpha]}$, $\lambda,\mu\in\mathbb{R}$ *with* $\mu\neq 0$. *Let* $f(t)$ *be a function defined on the isolated time scale* $[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. *Assume that* $\varphi : [t_0,+\infty)_{\widetilde{\mathbb{T}}} \to [0,+\infty)$ *is a function such that the integral* $\int_{t_0}^{t} |\widehat{G_{\alpha,\beta}}(t,\sigma(s))|\varphi(s)\Delta s$ *exists for all* $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$. *If a function* $y : [t_0,+\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ *satisfies the inequality*

$$\left|D^{\alpha}_{\Delta,t_0}y(t) - \lambda D^{\beta}_{\Delta,t_0}y(t) - \mu y(t) - f(t)\right| \leqslant \varphi(t) \tag{20}$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$, *then there exists a solution* $y_\varphi : [t_0,+\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ *of the fractional dynamic equation*

$$D^{\alpha}_{\Delta,t_0}y(t) - \lambda D^{\beta}_{\Delta,t_0}y(t) - \mu y(t) = f(t) \tag{21}$$

such that

$$|y(t) - y_\varphi(t)| \leqslant \int_{t_0}^{t} |\widehat{G_{\alpha,\beta}}(t,\sigma(s))|\varphi(s)\Delta s$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$.

Proof. Define $g(t) = D^{\alpha}_{\Delta,t_0}y(t) - \lambda D^{\beta}_{\Delta,t_0}y(t) - \mu y(t) - f(t)$, $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$. Set $m-1<\beta\leqslant m$, $m\in\mathbb{N}$. Clearly, we can see $m\leqslant l$ due to $0<\beta<\alpha$. Applying the Laplace transform $\mathscr{L}$ and using Lemma 2.1, we get

$$\begin{aligned}\mathscr{L}(g(t))(z,t_0) = {} & z^{\alpha}\mathscr{L}(y(t))(z,t_0) - \sum_{k=1}^{l} D^{\alpha-k}_{\Delta,t_0}y(t_0)z^{k-1} - \lambda z^{\beta}\mathscr{L}(y(t))(z,t_0) \\ & + \lambda\sum_{j=1}^{m} D^{\beta-j}_{\Delta,t_0}y(t_0)z^{j-1} - \mu\mathscr{L}(y(t))(z,t_0) - \mathscr{L}(f(t))(z,t_0).\end{aligned} \tag{22}$$

Combining the same items yields

$$\mathscr{L}(y(t))(z,t_0) = \sum_{k=1}^{l} d_k \frac{z^{k-1}}{z^{\alpha}-\lambda z^{\beta}-\mu} + \frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\mu} + \frac{\mathscr{L}(g(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\mu}, \tag{23}$$

where

$$d_k = \begin{cases} D_{\Delta,t_0}^{\alpha-k}y(t_0) - \lambda D_{\Delta,t_0}^{\beta-k}y(t_0), & k=1,\cdots,m, \\ D_{\Delta,t_0}^{\alpha-k}y(t_0), & k=m+1,\cdots,l. \end{cases}$$

Now we set

$$y_\varphi(t) = \sum_{k=1}^{l} d_k y_k(t) + (G_{\alpha,\beta} \star f)(t),$$

where

$$y_k(t) = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \frac{\partial^j}{\partial\lambda^j} {}_{\Delta}F_{\alpha-\beta,\alpha+j\beta+1-k}(\lambda,t,t_0).$$

By Theorem 3.2, we get

$$\mathscr{L}(G_{\alpha,\beta}(t))(z,t_0) = \frac{1}{z^\alpha - \lambda z^\beta - \mu}.$$

Then, we can obtain

$$\mathscr{L}[(G_{\alpha,\beta} \star f)(t)](z,t_0) = \frac{\mathscr{L}(f(t))(z,t_0)}{z^\alpha - \lambda z^\beta - \mu}. \tag{24}$$

According to the proof of Theorem 3.2, for $z \in \mathbb{C}$ and $\left|\frac{\mu z^{-\beta}}{z^{\alpha-\beta}-\lambda}\right| < 1$, we have

$$\frac{1}{z^\alpha - \lambda z^\beta - \mu} = \sum_{j=0}^{\infty} \frac{\mu^j z^{-\beta-j\beta}}{(z^{\alpha-\beta}-\lambda)^{j+1}}. \tag{25}$$

Furthermore, using Lemma 2.3, for $|\lambda z^{\beta-\alpha}| < 1$, we can infer that

$$\begin{aligned} \frac{z^{k-1-\beta-j\beta}}{(z^{\alpha-\beta}-\lambda)^{j+1}} &= \frac{z^{\alpha-\beta-(\alpha-j\beta+a-k)}}{(z^{\alpha-\beta}-\lambda)^{k+1}} \\ &= \mathscr{L}\left(\frac{1}{j!}\frac{\partial^j}{\partial\lambda^j} {}_{\Delta}F_{\alpha-\beta,\alpha+j\beta+1-k}(\lambda,t,t_0)\right)(z,t_0). \end{aligned} \tag{26}$$

From (25) and (26), it follows that

$$\begin{aligned} \mathscr{L}(y_k(t))(z,t_0) &= \sum_{j=0}^{\infty} \mathscr{L}\left(\frac{\mu^j}{j!}\frac{\partial^j}{\partial\lambda^j} {}_{\Delta}F_{\alpha-\beta,\alpha+j\beta+1-k}(\lambda,t,t_0)\right) \\ &= \sum_{j=0}^{\infty} \frac{\mu^j z^{k-1-\beta-j\beta}}{(z^{\alpha-\beta}-\lambda)^{j+1}} \\ &= z^{k-1}\sum_{j=0}^{\infty} \frac{\mu^j z^{-\beta-j\beta}}{(z^{\alpha-\beta}-\lambda)^{j+1}} \\ &= \frac{z^{k-1}}{z^\alpha - \lambda z^\beta - \mu}. \end{aligned} \tag{27}$$

Therefore, it can be inferred from (24) and (27) that

$$\begin{aligned}\mathscr{L}(y_\varphi(t))(z,t_0) &= \sum_{k=1}^{l} d_k \mathscr{L}(y_k(t))(z,t_0) + \mathscr{L}[(G_{\alpha,\beta} \star f)(t)](z,t_0) \\ &= \sum_{k=1}^{l} d_k \frac{z^{k-1}}{z^\alpha - \lambda z^\beta - \mu} + \frac{\mathscr{L}(f(t))(z,t_0)}{z^\alpha - \lambda z^\beta - \mu}.\end{aligned} \tag{28}$$

In view of Theorem 3.2, we see that $(G_{\alpha,\beta} \star f)(t)$ is a particular solution of Eq. (21). Then, it follows from Lemma 2.1 that

$$\begin{aligned}\mathscr{L}(f(t))(z,t_0) &= \mathscr{L}\left[D^{\alpha}_{\Delta,t_0}(G_{\alpha,\beta} \star f)(t)\right](z,t_0) - \lambda \mathscr{L}\left[D^{\beta}_{\Delta,t_0}(G_{\alpha,\beta} \star f)(t)\right](z,t_0) \\ &\quad - \mu \mathscr{L}(y(t))(z,t_0) \\ &= z^\alpha \mathscr{L}[(G_{\alpha,\beta} \star f)(t)](z,t_0) - \sum_{k=1}^{l} D^{\alpha-k}_{\Delta,t_0}(G_{\alpha,\beta} \star f)(t_0) z^{k-1} \\ &\quad - \lambda z^\beta \mathscr{L}[(G_{\alpha,\beta} \star f)(t)](z,t_0) + \lambda \sum_{j=1}^{m} D^{\alpha-j}_{\Delta,t_0}(G_{\alpha,\beta} \star f)(t_0) z^{j-1} \\ &\quad - \mu \mathscr{L}[(G_{\alpha,\beta} \star f)(t)](z,t_0) \\ &= (z^\alpha - \lambda z^\beta - \mu)\mathscr{L}[(G_{\alpha,\beta} \star f)(t)](z,t_0) - \sum_{k=1}^{l} h_k z^{k-1} \\ &= (z^\alpha - \lambda z^\beta - \mu)\frac{\mathscr{L}(f(t))(z,t_0)}{z^\alpha - \lambda z^\beta - \mu} - \sum_{k=1}^{l} h_k z^{k-1} \\ &= \mathscr{L}(f(t))(z,t_0) - \sum_{k=1}^{l} h_k z^{k-1},\end{aligned}$$

where

$$h_k = \begin{cases} D^{\alpha-k}_{\Delta,t_0}(G_{\alpha,\beta} \star f)(t_0) - \lambda D^{\alpha-k}_{\Delta,t_0}(G_{\alpha,\beta} \star f)(t_0), & k = 1, \cdots, m, \\ D^{\alpha-k}_{\Delta,t_0}(G_{\alpha,\beta} \star f)(t_0), & k = m+1, \cdots, l. \end{cases}$$

From the above equality, we can obtain $\sum_{k=1}^{l} h_k z^{k-1} = 0$. According to Theorem 6.3 in [12], we know that $y_k(t)$ is a solution of the corresponding homogeneous equation of Eq.(21). Thus, we have

$$D^{\alpha}_{\Delta,t_0} y_k(t) - \lambda D^{\beta}_{\Delta,t_0} y_k(t) - \mu y_k(t) = 0, \qquad k = 1, 2, \cdots, l. \tag{29}$$

Using the linearity of the operators $\mathscr{L}$, D^{α}_{Δ,t_0} and D^{β}_{Δ,t_0}, we can infer from (24), (27)–(29) that

$$\begin{aligned}
&\mathscr{L}\Big[D^{\alpha}_{\Delta,t_0}y_{\varphi}(t)-\lambda D^{\beta}_{\Delta,t_0}y_{\varphi}(t)-\mu y_{\varphi}(t)\Big](z,t_0)\\
&=\mathscr{L}\Big[D^{\alpha}_{\Delta,t_0}\Big(\sum_{k=1}^{l}d_k y_k(t)\Big)+D^{\alpha}_{\Delta,t_0}\Big(G_{\alpha,\beta}\star f\Big)(t)-\lambda D^{\beta}_{\Delta,t_0}\Big(\sum_{k=1}^{l}d_k y_k(t)\Big)\\
&\qquad-\lambda D^{\beta}_{\Delta,t_0}\Big(G_{\alpha,\beta}\star f\Big)(t)-\mu y_{\varphi}(t)\Big](z,t_0)\\
&=\sum_{k=1}^{l}d_k\mathscr{L}\Big[D^{\alpha}_{\Delta,t_0}y_k(t)-\lambda D^{\beta}_{\Delta,t_0}y_k(t)\Big](z,t_0)\\
&\qquad+\mathscr{L}\Big[D^{\alpha}_{\Delta,t_0}\Big(G_{\alpha,\beta}\star f\Big)(t)-\lambda D^{\beta}_{\Delta,t_0}\Big(G_{\alpha,\beta}\star f\Big)(t)\Big](z,t_0)-\mu\mathscr{L}(y_{\varphi}(t))(z,t_0)\\
&=\mu\sum_{k=1}^{l}d_k\mathscr{L}(y_k(t))(z,t_0)+(z^{\alpha}-\lambda z^{\beta})\mathscr{L}\Big((G_{\alpha,\beta}\star f)(t)\Big)(z,t_0)\\
&\qquad-\sum_{k=1}^{l}h_k z^{k-1}-\mu\mathscr{L}(y_{\varphi}(t))(z,t_0)\\
&=\mu\sum_{k=1}^{l}d_k\frac{z^{k-1}}{z^{\alpha}-\lambda z^{\beta}-\mu}+(z^{\alpha}-\lambda z^{\beta})\mathscr{L}\Big((G_{\alpha,\beta}\star f)(t)\Big)(z,t_0)\\
&\qquad-\mu\Big(\sum_{k=1}^{l}d_k\frac{z^{k-1}}{z^{\alpha}-\lambda z^{\beta}-\mu}+\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\mu}\Big)\\
&=(z^{\alpha}-\lambda z^{\beta})\mathscr{L}((G_{\alpha,\beta}(t))(z,t_0)\mathscr{L}(f(t))(z,t_0)-\mu\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\mu}\\
&=(z^{\alpha}-\lambda z^{\beta})\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\mu}-\mu\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\mu}\\
&=\mathscr{L}(f(t))(z,t_0).
\end{aligned}$$

Obviously, $y_{\varphi}(t)$ is a solution of Eq. (21), since the Laplace transform operator is one-to-one on the isolated time scale. From (23) and (28), it follows that

$$\mathscr{L}(y(t))(z,t_0)-\mathscr{L}(y_{\varphi}(t))(z,t_0)=\frac{\mathscr{L}(g(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\mu}=\mathscr{L}((G_{\alpha,\beta}\star g)(t))(z,t_0).$$

Owing to the linearity of the Laplace transform operator and we apply the inverse Laplace transform to both sides of the previous equality, we get

$$y(t)-y_{\varphi}(t)=(G_{\alpha,\beta}\star g)(t),\qquad t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}.$$

By the condition (20), we have $|g(t)|\leqslant\varphi(t)$ for all $t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. Then, we can obtain

$$\begin{aligned}
|y(t)-y_{\varphi}(t)|&=|(G_{\alpha,\beta}\star g)(t)|=\Big|\int_{t_0}^{t}\widehat{G_{\alpha,\beta}}(t,\sigma(s))g(s)\Delta s\Big|\\
&\leqslant\int_{t_0}^{t}|\widehat{G_{\alpha,\beta}}(t,\sigma(s))g(s)|\Delta s\leqslant\int_{t_0}^{t}|\widehat{G_{\alpha,\beta}}(t,\sigma(s))|\varphi(s)\Delta s
\end{aligned}$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$. This completes the proof of the theorem. $\square$

From Theorem 4.3, we can establish the Hyers-Ulam stability of the inhomogeneous linear fractional dynamic equation (21) on an isolated time scale.

COROLLARY 4.4. *Let $l-1<\alpha \leqslant l$, $l \in \mathbb{N}$, $\alpha > \beta > 0$, $n = -\overline{[-\alpha]}$, $\lambda,\mu \in \mathbb{R}$ with $\mu \neq 0$. Let $f(t)$ be a function defined on the isolated time scale $[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. Assume that the integral $\int_{t_0}^{t} |\widehat{G_{\alpha,\beta}}(t,\sigma(s))|\Delta s$ exists for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$. For a given $\varepsilon > 0$, if a function $y : [t_0,+\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ satisfies the inequality*

$$\left| D^{\alpha}_{\Delta,t_0} y(t) - \lambda D^{\beta}_{\Delta,t_0} y(t) - \mu y(t) - f(t) \right| \leqslant \varepsilon \tag{30}$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$, then there exists a solution $y_\varepsilon : [t_0,+\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ of the fractional dynamic equation (21) such that

$$|y(t) - y_\varphi(t)| \leqslant K_2 \varepsilon$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$, where

$$K_2 = \sup_{t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}} \int_{t_0}^{t} |\widehat{G_{\alpha,\beta}}(t,\sigma(s))|\Delta s.$$

As a complement to Theorem 4.3, we further consider the Ulam stability of the fractional dynamic equation (21) with the coefficient $\mu = 0$.

THEOREM 4.5. *Let $l-1<\alpha \leqslant l$, $l \in \mathbb{N}$, $\alpha > \beta > 0$, $n = -\overline{[-\alpha]}$, $\lambda \in \mathbb{R}$. Let $f(t)$ be a function defined on the isolated time scale $[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. Assume that $\varphi : [t_0,+\infty)_{\widetilde{\mathbb{T}}} \to [0,+\infty)$ is a function such that the series*

$$\sum_{k=0}^{\infty} |\lambda|^k \int_{t_0}^{t} |h_{(k+1)\alpha - k\beta - 1}(t,\sigma(s))|\varphi(s)\Delta s$$

converges for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$. If a function $y : [t_0,+\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ satisfies the inequality

$$\left| D^{\alpha}_{\Delta,t_0} y(t) - \lambda D^{\beta}_{\Delta,t_0} y(t) - f(t) \right| \leqslant \varphi(t)$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$, then there exists a solution $y_\varphi : [t_0,+\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ of the fractional dynamic equation

$$D^{\alpha}_{\Delta,t_0} y(t) - \lambda D^{\beta}_{\Delta,t_0} y(t) = f(t) \tag{31}$$

such that

$$|y(t) - y_\varphi(t)| \leqslant \sum_{k=0}^{\infty} |\lambda|^k \int_{t_0}^{t} |h_{(k+1)\alpha - k\beta - 1}(t,\sigma(s))|\varphi(s)\Delta s$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$.

Proof. Define $g(t) = D^{\alpha}_{\Delta,t_0} y(t) - \lambda D^{\beta}_{\Delta,t_0} y(t) - f(t)$, $t \in [t_0, +\infty)_{\widetilde{\mathbb{T}}}$. Using the similar argument as in Theorem 4.3, we can obtain

$$\mathscr{L}(y(t))(z,t_0) = \sum_{k=1}^{l} d_k \frac{z^{k-1}}{z^{\alpha} - \lambda z^{\beta}} + \frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha} - \lambda z^{\beta}} + \frac{\mathscr{L}(g(t))(z,t_0)}{z^{\alpha} - \lambda z^{\beta}},$$

where d_k, $k = 1, 2, \cdots, l$ are defined as in (23). Similar to Theorem 4.3, we define

$$y_{\varphi}(t) = \sum_{k=1}^{l} d_k y_k(t) + (G_{\alpha,\beta} \star f)(t),$$

where

$$\begin{aligned}
&y_k(t) = {}_{\Delta}F_{\alpha-\beta,\alpha-k+1}(\lambda, t, t_0),\\
&G_{\alpha,\beta}(t) = {}_{\Delta}F_{\alpha-\beta,\alpha}(\lambda, t, t_0),\\
&(G_{\alpha,\beta} \star f)(t) = \sum_{k=0}^{\infty} \lambda^k \int_{t_0}^{t} h_{(k+1)\alpha-k\beta-1}(t, \sigma(s)) f(s) \Delta s.
\end{aligned}$$

Furthermore, we can infer that

$$\mathscr{L}(y_{\varphi}(t))(z,t_0) = \sum_{k=1}^{l} d_k \frac{z^{k-1}}{z^{\alpha} - \lambda z^{\beta}} + \frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha} - \lambda z^{\beta}}.$$

Similar to the proof of Theorem 4.1 and Theorem 4.3, we can obtain

$$\begin{aligned}
|y(t) - y_{\varphi}(t)| &= |(G_{\alpha,\beta} \star g)(t)|\\
&\leqslant \sum_{k=0}^{\infty} |\lambda|^k \int_{t_0}^{t} |h_{(k+1)\alpha-k\beta-1}(t, \sigma(s))| \varphi(s) \Delta s
\end{aligned}$$

for all $[t_0, +\infty)_{\widetilde{\mathbb{T}}}$. □

COROLLARY 4.6. *Let $l - 1 < \alpha \leqslant l$, $l \in \mathbb{N}$, $\alpha > \beta > 0$, $n = -\overline{[-\alpha]}$, $\lambda \in \mathbb{R}$. Let $f(t)$ be a function defined on the isolated time scale $[t_0, +\infty)_{\widetilde{\mathbb{T}}}$. Assume that the series $\sum_{k=0}^{\infty} \lambda^k \int_{t_0}^{t} |h_{(k+1)\alpha-k\beta-1}(t, \sigma(s))| \Delta s$ converges for all $t \in [t_0, +\infty)_{\widetilde{\mathbb{T}}}$. For a given $\varepsilon > 0$, if a function $y : [t_0, +\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ satisfies the inequality*

$$\left| D^{\alpha}_{\Delta,t_0} y(t) - \lambda D^{\beta}_{\Delta,t_0} y(t) - f(t) \right| \leqslant \varepsilon$$

for all $t \in [t_0, +\infty)_{\widetilde{\mathbb{T}}}$, then there exists a solution $y_{\varepsilon} : [t_0, +\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ of the fractional dynamic equation (31) such that

$$|y(t) - y_{\varepsilon}(t)| \leqslant K_3 \varepsilon$$

for all $t \in [t_0, +\infty)_{\widetilde{\mathbb{T}}}$, where

$$K_3 = \sup_{t \in [t_0, +\infty)_{\widetilde{\mathbb{T}}}} \sum_{k=0}^{\infty} |\lambda|^k \int_{t_0}^{t} |h_{(k+1)\alpha-k\beta-1}(t, \sigma(s))| \Delta s.$$

More generally, we shall consider the Ulam stability problem of the linear Riemann-Liouville fractional dynamic equation with general form on the isolate time scale.

THEOREM 4.7. *Let $m\in\mathbb{N}\setminus\{1,2\}$, $\lambda, A_0,\cdots,A_{m-2}\in\mathbb{R}$, and let $\alpha>\beta>\alpha_{m-2}>\cdots>\alpha_1>\alpha_0=0$ with $l-1<\alpha\leqslant l$, $l\in\mathbb{N}$, $l_{m-1}-1<\beta\leqslant l_{m-1}$, $l_k-1<\alpha_k\leqslant l_k$, $l_k\in\mathbb{N}$, $k\in\{1,\cdots,m-2\}$. Let $f(t)$ be a function defined on the isolated time scale $[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. Assume that $\varphi:[t_0,+\infty)_{\widetilde{\mathbb{T}}}\to[0,+\infty)$ is a function such that the integral $\int_{t_0}^{t}|G_{\alpha,\widehat{\beta,\alpha_1,\cdots,\alpha_{m-2}}}(t,\sigma(s))|\varphi(s)\Delta s$ exists for all $t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}$. If a function $y:[t_0,+\infty)_{\widetilde{\mathbb{T}}}\to\mathbb{R}$ satisfies the inequality*

$$\left|D_{\Delta,t_0}^{\alpha}y(t)-\lambda D_{\Delta,t_0}^{\beta}y(t)-\sum_{k=0}^{m-2}A_kD_{\Delta,t_0}^{\alpha_k}y(t)-f(t)\right|\leqslant\varphi(t) \tag{32}$$

for all $t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}$, then there exists a solution $y_\varphi:[t_0,+\infty)_{\widetilde{\mathbb{T}}}\to\mathbb{R}$ of the fractional dynamic equation

$$D_{\Delta,t_0}^{\alpha}y(t)-\lambda D_{\Delta,t_0}^{\beta}y(t)-\sum_{k=0}^{m-2}A_kD_{\Delta,t_0}^{\alpha_k}y(t)=f(t) \tag{33}$$

such that

$$|y(t)-y_\varphi(t)|\leqslant\int_{t_0}^{t}|G_{\alpha,\widehat{\beta,\alpha_1,\cdots,\alpha_{m-2}}}(t,\sigma(s))|\varphi(s)\Delta s$$

for all $t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}$.

Proof. First, we define

$$g(t)=D_{\Delta,t_0}^{\alpha}y(t)-\lambda D_{\Delta,t_0}^{\beta}y(t)-\sum_{k=0}^{m-2}A_kD_{\Delta,t_0}^{\alpha_k}y(t)-f(t),\qquad t\in[t_0,+\infty)_{\widetilde{\mathbb{T}}}.$$

Applying the Laplace transform to $g(t)$ and using Lemma 2.1, we can obtain

$$\begin{aligned}\mathscr{L}(g(t))(z,t_0)&=z^{\alpha}\mathscr{L}(y(t))(z,t_0)-\sum_{j=1}^{l}D_{\Delta,t_0}^{\alpha-j}y(t_0)z^{j-1}-\lambda z^{\beta}\mathscr{L}(y(t))(z,t_0)\\&\quad+\lambda\sum_{j=1}^{l_{m-1}}D_{\Delta,t_0}^{\beta-j}y(t_0)z^{j-1}-\sum_{k=0}^{m-2}A_kz^{\alpha_k}\mathscr{L}(y(t))(z,t_0))\\&\quad+\sum_{k=1}^{m-2}A_k\Big(\sum_{j=1}^{l_k}D_{\Delta,t_0}^{\alpha_k-j}y(t_0)z^{j-1}\Big)-\mathscr{L}(f(t))(z,t_0)\\&=\Big(z^{\alpha}-\lambda z^{\beta}-\sum_{k=0}^{m-2}A_kz^{\alpha_k}\Big)\mathscr{L}(y(t))(z,t_0))-\sum_{j=1}^{l}d_jz^{j-1},\end{aligned} \tag{34}$$

where

$$d_j = \begin{cases} D_{\Delta,t_0}^{\alpha-j} y(t_0) - \lambda D_{\Delta,t_0}^{\beta-j} y(t_0) - \sum_{k=1}^{m-2} A_k D_{\Delta,t_0}^{\alpha_k-j} y(t_0), & j = 1,\cdots,l_1, \\ D_{\Delta,t_0}^{\alpha-j} y(t_0) - \lambda D_{\Delta,t_0}^{\beta-j} y(t_0) - \sum_{k=2}^{m-2} A_k D_{\Delta,t_0}^{\alpha_k-j} y(t_0), & j = l_1+1,\cdots,l_2, \\ \quad \cdots & \\ D_{\Delta,t_0}^{\alpha-j} y(t_0) - \lambda D_{\Delta,t_0}^{\beta-j} y(t_0) - A_{m-2} D_{\Delta,t_0}^{\alpha_{m-2}-j} y(t_0), & j = l_{m-3}+1,\cdots,l_{m-2}, \\ D_{\Delta,t_0}^{\alpha-j} y(t_0) - \lambda D_{\Delta,t_0}^{\beta-j} y(t_0), & j = l_{m-2}+1,\cdots,l_{m-1}, \\ D_{\Delta,t_0}^{\alpha-j} y(t_0) & j = l_{m-1}+1,\cdots,l. \end{cases}$$

From (34), we can obtain

$$\mathscr{L}(y(t))(z,t_0) = \sum_{j=1}^{l} \frac{z^{j-1}}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} + \frac{\mathscr{L}(f(t))(z,t_0) + \mathscr{L}(g(t))(z,t_0)}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}}. \quad (35)$$

Set

$$y_\varphi(t) = \sum_{k=1}^{l} d_k y_k(t) + (G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}} \star g)(t),$$

where

$$y_k(t) = \sum_{r=0}^{\infty} \Big(\sum_{k_0+k_1+\cdots+k_{m-2}=r} \Big) \frac{1}{k_0!k_1!\cdots k_{m-2}!} \cdot \Big(\prod_{v=0}^{m-2} A_v{}^{k_v} \Big) \frac{\partial^r}{\partial \lambda^r} {}_\Delta F_{\alpha-\beta,\alpha-k+1+\sum_{v=0}^{m-2}(\beta-\alpha_v k_v)}(\lambda,t,t_0).$$

By the proof of Theorem 3.4, for $z \in \mathbb{C}$ with $\left| \sum_{k=0}^{m-2} A_k \frac{z^{\alpha_k-\beta}}{z^{\alpha-\beta}-\lambda} \right| < 1$, we get

$$\frac{1}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} = \sum_{r=0}^{\infty} \sum_{k_0+k_1+\cdots+k_{m-2}=r} \frac{r!}{k_0!k_1!\cdots k_{m-2}!} \prod_{v=0}^{m-2} A_v^{k_v} \frac{z^{-\beta-\sum_{v=0}^{m-2}(\beta-\alpha_v)k_v}}{(z^{\alpha-\beta}-\lambda)^{r+1}}.$$

Then, it follows from Lemma 2.3 that

$$\begin{aligned} &\frac{z^{j-1-\beta-\sum_{v=0}^{m-2}(\beta-\alpha_v)k_v}}{(z^{\alpha-\beta}-\lambda)^{r+1}} \\ &= \frac{z^{\alpha-\beta-\left(\alpha-j+1+\sum_{v=0}^{m-2}(\beta-\alpha_v)k_v\right)}}{(z^{\alpha-\beta}-\lambda)^{r+1}} \\ &= \mathscr{L}\Big(\frac{1}{r!} \frac{\partial^r}{\partial \lambda^r} {}_\Delta F_{\alpha-\beta,\alpha-j+1+\sum_{v=0}^{m-2}(\beta-\alpha_v)k_v}(\lambda,t,t_0) \Big)(z,t_0). \end{aligned} \quad (36)$$

Moreover, by Theorem 3.4, we can obtain

$$\mathscr{L}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}(t))(z,t_0)=\frac{1}{z^{\alpha}-\lambda z^{\beta}-\sum\limits_{k=0}^{m-2}A_k z^{\alpha_k}}.$$

Thus, we get

$$\mathscr{L}((G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t))(z,t_0)=\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\sum\limits_{k=0}^{m-2}A_k z^{\alpha_k}}. \tag{37}$$

From (36), we can infer that

$$\begin{aligned}&\mathscr{L}(y_k(t))(z,t_0)\\&=\sum_{r=0}^{\infty}\Big(\sum_{k_0+k_1+\cdots+k_{m-2}=r}\Big)\frac{1}{k_0!k_1!\cdots k_{m-2}!}\Big(\prod_{v=0}^{m-2}A_v{}^{k_v}\Big)\frac{z^{j-1-\beta-\sum_{v=0}^{m-2}(\beta-\alpha_v)k_v}}{(z^{\alpha-\beta}-\lambda)^{r+1}}\\&=\frac{j-1}{z^{\alpha}-\lambda z^{\beta}-\sum_{k=0}^{m-2}A_k z^{\alpha_k}}.\end{aligned} \tag{38}$$

By (37) and (38), we can obtain

$$\begin{aligned}&\mathscr{L}(y_{\varphi}(t))(z,t_0)\\&=\sum_{j=1}^{l}d_j\mathscr{L}(y_j(t))(z,t_0)+\mathscr{L}((G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t))(z,t_0)\\&=\sum_{j=1}^{l}d_j\frac{j-1}{z^{\alpha}-\lambda z^{\beta}-\sum_{k=0}^{m-2}A_k z^{\alpha_k}}+\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\sum\limits_{k=0}^{m-2}A_k z^{\alpha_k}}.\end{aligned} \tag{39}$$

In view of Theorem 3.4, it is easy to know that $(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)$ is a particular solution of Eq. (33). From Lemma 2.1, it follows that

$$\begin{aligned}&\mathscr{L}(y_{\varphi}(t))(z,t_0)\\&=\mathscr{L}\Big[D^{\alpha}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)-\lambda D^{\beta}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)\\&\qquad-\sum_{k=0}^{m-2}A_k D^{\alpha_k}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)\Big](z,t_0)\\&=\Big(z^{\alpha}-\lambda z^{\beta}-\sum_{k=0}^{m-2}A_k z^{\alpha_k}\Big)\mathscr{L}((G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t))(z,t_0))-\sum_{j=1}^{l}h_j z^{j-1}\\&=\Big(z^{\alpha}-\lambda z^{\beta}-\sum_{k=0}^{m-2}A_k z^{\alpha_k}\Big)\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha}-\lambda z^{\beta}-\sum\limits_{k=0}^{m-2}A_k z^{\alpha_k}}-\sum_{j=1}^{l}h_j z^{j-1},\end{aligned}$$

where

$$h_j=\begin{cases} D^{\alpha-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0)-\lambda D^{\beta-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0)\\ \quad -\sum_{k=1}^{m-2}A_kD^{\alpha_k-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0), \qquad j=1,\cdots,l_1,\\ D^{\alpha-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0)-\lambda D^{\beta-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0)\\ \quad -\sum_{k=2}^{m-2}A_kD^{\alpha_k-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0), \qquad j=l_1+1,\cdots,l_2,\\ \quad \cdots\\ D^{\alpha-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0)-\lambda D^{\beta-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0)\\ \quad -A_{m-2}D^{\alpha_{m-2}-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0), \qquad j=l_{m-3}+1,\cdots,l_{m-2},\\ D^{\alpha-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0)-\lambda D^{\beta-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0),\\ \qquad\qquad j=l_{m-2}+1,\cdots,l_{m-1},\\ D^{\alpha-j}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t_0) \qquad j=l_{m-1}+1,\cdots,l. \end{cases}$$

Obviously, the preceding equality implies $\sum_{j=1}^{l}h_jz^{j-1}=0$. According to Theorem 6.4 in [12], $y_j(t)$ is a solution of the corresponding homogeneous equation of Eq. (33). Hence, we obtain

$$D^{\alpha}_{\Delta,t_0}y_j(t)-\lambda D^{\beta}_{\Delta,t_0}y_j(t)-\sum_{j=1}^{m-2}A_jD^{\alpha_j}_{\Delta,t_0}y_j(t)=A_0y_j(t), \qquad j=1,2,\cdots,l.$$

Using the linearity of the operators $\mathscr{L}$, D^{α}_{Δ,t_0}, D^{β}_{Δ,t_0} and $D^{\alpha_j}_{\Delta,t_0}$ $(j=1,2,\cdots,l)$, we can infer that

$$\begin{aligned} &\mathscr{L}\Big[D^{\alpha}_{\Delta,t_0}y_\varphi(t)-\lambda D^{\beta}_{\Delta,t_0}y_\varphi(t)-\sum_{j=0}^{m-2}A_jD^{\alpha_j}_{\Delta,t_0}y_\varphi(t)\Big](z,t_0)\\ &=\mathscr{L}\Big[D^{\alpha}_{\Delta,t_0}\Big(\sum_{j=1}^{l}d_jh_j(t)\Big)+D^{\alpha}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)\\ &\quad-\lambda D^{\beta}_{\Delta,t_0}\Big(\sum_{j=1}^{l}d_jh_j(t)\Big)-\lambda D^{\beta}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)\\ &\quad-\sum_{k=0}^{m-2}A_kD^{\alpha_k}_{\Delta,t_0}\Big(\sum_{j=1}^{l}d_jh_j(t)\Big)-\sum_{k=0}^{m-2}A_kD^{\alpha_k}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)\Big](z,t_0)\\ &=\sum_{j=1}^{l}d_j\mathscr{L}\Big[D^{\alpha}_{\Delta,t_0}h_j(t)-\lambda D^{\beta}_{\Delta,t_0}h_j(t)-\sum_{k=1}^{m-2}A_kD^{\alpha_k}_{\Delta,t_0}h_j(t)\Big](z,t_0)\\ &\quad+\mathscr{L}\Big[D^{\beta}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)-\lambda D^{\beta}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)\\ &\quad-\sum_{k=1}^{m-2}A_kD^{\alpha_k}_{\Delta,t_0}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}}\star f)(t)\Big](z,t_0)-A_0\mathscr{L}[y_\varphi(t)](z,t_0) \end{aligned}$$

$$
\begin{aligned}
&= A_0 \sum_{j=1}^{l} d_j \mathscr{L}(h_j(t))(z,t_0) + \Big(z^{\alpha} - \lambda z^{\beta} - \sum_{k=0}^{m-2} A_k z^{\alpha_k}\Big)\mathscr{L}((G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}} \star f)(t))(z,t_0) \\
&\quad - \sum_{j=1}^{l} h_j z^{j-1} - A_0 \mathscr{L}[y_{\varphi}(t)](z,t_0) \\
&= A_0 \sum_{j=1}^{l} d_j \mathscr{L}(h_j(t))(z,t_0) + \Big(z^{\alpha} - \lambda z^{\beta} - \sum_{k=1}^{m-2} A_k z^{\alpha_k}\Big)\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha} - \lambda z^{\beta} - \sum\limits_{k=0}^{m-2} A_k z^{\alpha_k}} \\
&\quad - A_0 \mathscr{L}[y_{\varphi}(t)](z,t_0) \\
&= \Big(z^{\alpha} - \lambda z^{\beta} - \sum_{k=1}^{m-2} A_k z^{\alpha_k}\Big)\frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha} - \lambda z^{\beta} - \sum\limits_{k=0}^{m-2} A_k z^{\alpha_k}} - A_0 \frac{\mathscr{L}(f(t))(z,t_0)}{z^{\alpha} - \lambda z^{\beta} - \sum\limits_{k=0}^{m-2} A_k z^{\alpha_k}} \\
&= \mathscr{L}(f(t))(z,t_0).
\end{aligned}
$$

Since the Laplace transform operator $\mathscr{L}$ is one-to-one on the isolated time scale, the previous equality implies that $y_{\varphi}(t)$ is a solution of the fractional dynamic equation (33). From (35) and (39), we can infer that

$$
\begin{aligned}
\mathscr{L}(y(t))(z,t_0) - \mathscr{L}(y_{\varphi}(t))(z,t_0) &= \frac{\mathscr{L}(g(t))(z,t_0)}{z^{\alpha} - \lambda z^{\beta} - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} \\
&= \mathscr{L}(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}} \star g)(t)(z,t_0).
\end{aligned}
$$

Using the linearity of the Laplace transform operator and taking the inverse Laplace transform to both sides of the preceding equality, we get

$$
y(t) - y_{\varphi}(t) = (G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}} \star g)(t), \qquad t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}.
$$

Furthermore, using the condition (32), we can obtain

$$
\begin{aligned}
|y(t) - y_{\varphi}(t)| &= |(G_{\alpha,\beta,\alpha_1,\cdots,\alpha_{m-2}} \star g)(t)| \\
&= \Big| \int_{t_0}^{t} G_{\alpha,\widehat{\beta,\alpha_1,\cdots,}\alpha_{m-2}}(t,\sigma(s))g(s)\Delta s \Big| \\
&\leqslant \int_{t_0}^{t} |G_{\alpha,\widehat{\beta,\alpha_1,\cdots,}\alpha_{m-2}}(t,\sigma(s))g(s)|\Delta s \\
&\leqslant \int_{t_0}^{t} |G_{\alpha,\widehat{\beta,\alpha_1,\cdots,}\alpha_{m-2}}(t,\sigma(s))|\varphi(s)\Delta s
\end{aligned}
$$

for all $t \in [t_0,+\infty)_{\widetilde{\mathbb{T}}}$. This completes the proof. □

As a direct consequence of Theorem 4.7, we can obtain the Hyers-Ulam stability of the linear fractional dynamic equation (33) with general form on the isolated time scale.

COROLLARY 4.8. *Let $m \in \mathbb{N} \setminus \{1,2\}$, $\lambda, A_0, \cdots, A_{m-2} \in \mathbb{R}$, and let $\alpha > \beta > \alpha_{m-2} > \cdots > \alpha_1 > \alpha_0 = 0$ with $l-1 < \alpha \leqslant l$, $l \in \mathbb{N}$, $l_{m-1} - 1 < \beta \leqslant l_{m-1}$, $l_k - 1 < \alpha_k \leqslant l_k$, $l_k \in \mathbb{N}$, $k \in \{1, \cdots, m-2\}$. Let $f(t)$ be a function defined on the isolated time scale $[t_0, +\infty)_{\widetilde{\mathbb{T}}}$. Assume that the integral $\int_{t_0}^{t} |G_{\alpha,\widehat{\beta,\alpha_1,\cdots,\alpha_{m-2}}}(t,\sigma(s))|\Delta s$ exists for all $t \in [t_0, +\infty)_{\widetilde{\mathbb{T}}}$. For a given $\varepsilon > 0$, if a function $y : [t_0, +\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ satisfies the inequality*

$$\left| D_{\Delta,t_0}^{\alpha} y(t) - \lambda D_{\Delta,t_0}^{\beta} y(t) - \sum_{k=0}^{m-2} A_k D_{\Delta,t_0}^{\alpha_k} y(t) - f(t) \right| \leqslant \varepsilon$$

for all $t \in [t_0, +\infty)_{\widetilde{\mathbb{T}}}$, then there exists a solution $y_\varepsilon : [t_0, +\infty)_{\widetilde{\mathbb{T}}} \to \mathbb{R}$ of the fractional dynamic equation (35) such that

$$|y(t) - y_\varphi(t)| \leqslant K_4 \varepsilon$$

for all $t \in [t_0, +\infty)_{\widetilde{\mathbb{T}}}$, where

$$K_4 = \sup_{t \in [t_0, +\infty)_{\widetilde{\mathbb{T}}}} \int_{t_0}^{t} |G_{\alpha,\widehat{\beta,\alpha_1,\cdots,\alpha_{m-2}}}(t,\sigma(s))|\Delta s.$$

Acknowledgements. This work was supported by the National Natural Science Foundation of China (Grant No. 11701425 and 62063031).

REFERENCES

[1] S. ABBAS, M. BENCHOHRA, N. LALEDJ et al., *Existence and Ulam stability for implicit fractional q-difference equations*, Adv. Differ. Equ. (2019) 2019:480.
[2] M. R. ABDOLLAHPOUR, A. NAJATI, *Stability of linear differential equaitons of third order*, Appl. Math. Lett. 24 (2011) 1827–1830.
[3] C. ALSINA, R. GER, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. 2 (1998) 373–380.
[4] M. BOHNER, A. PETERSON, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston, 2001.
[5] M. BOHNER, A. PETERSON, *Laplace transform and Z-transform unification and extension*, Methods Appl. Anal. 9 (2002) 151–158.
[6] M. BOHNER, A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
[7] M. BOHNER, G. SH. GUSEINOV, *The Laplace transform on isolated time scales*, Comput. Math. Appl. 60 (2010) 1536–1547.
[8] R. I. BUTT, T. ABEDLJAWAD, M. A. ALQUDAH et al., *Ulam stability of Caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach*, J. Inequal. Appl. (2019) 2019:305.
[9] D. X. CUONG, *On the Hyers-Ulam stability of Riemann-Liouville multi-order fractional differential equations*, Afr. Mat. 30 (2019) 1041–1047.
[10] S. CZERWIK, *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore, 2002.
[11] P. GĂVRUTA, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184 (1994) 431–436.
[12] S. G. GEORGIEV, *Fractional Dynamic Calculus and Fractional Dynamic Equations on Time Scales*, Springer, Switzerland, 2018.

[13] S. HILGER, *Ein Makettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. thesis, Universität Würzburg, 1988.

[14] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA 27 (1941) 222–224.

[15] J. F. JIANG, D. Q. CAO, H. T. CHEN, *The fixed point approach to the stability of fractional differential equations with Causal operators*, Qual. Theory Dyn. Syst. 15 (2016) 3–18.

[16] S. M. JUNG, *On the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 204 (1996) 221–226.

[17] S. M. JUNG, *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett. 17 (2004) 1135–1140.

[18] S. M. JUNG, *Hyers-Ulam stability of linear differential equations of first order (III)*, J. Math. Anal. Appl. 311 (2005) 139–146.

[19] S. M. JUNG, *Hyers-Ulam stability of linear differential equations of first order (II)*, Appl. Math. Lett. 19 (2006) 854–858.

[20] S. M. JUNG, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[21] Y. H. LEE, K. W. JUN, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. 238 (1999) 305–315.

[22] Y. H. LEE, K. W. JUN, *On the stability of approximately additive mappings*, Proc. Amer. Math. Soc. 128 (2000) 1361–1369.

[23] T. MIURA, *On the Hyers-Ulam stability of a differentiable map*, Sci. Math. Japan, 55 (2002) 17–24.

[24] T. MIURA, S. MIYAJIMA, S. E. TAKAHASI, *A characterization of Hyers-Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. 286 (2003) 136–146.

[25] T. MIURA, S. E. TAKAHASI, H. CHODA, *On the Hyers-Ulam stability of real continuous function valued differentiable map*, Tokyo J. Math., 24 (2001) 467–476.

[26] C. MORTICI, T. M. RASSIAS, S. M. JUNG, *The inhomogeneous Euler equation and its Hyers-Ulam stability*, Appl. Math. Lett. 40 (2015) 23–28.

[27] M. OBLOZA, *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Mat. 13 (1993) 259–270.

[28] T. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. 72 (1978) 297–300.

[29] P. K. SAHOO, P. KANNAPPAN, *Introduction to Functional Equations*, CRC Press, Boca Raton, 2011.

[30] H. REZAEI, S. M. JUNG, T. M. RASSIAS, *Laplace transform and Hyers-Ulam stability of linear differential equations*, J. Math. Anal. Appl. 403 (2013) 244–251.

[31] Y. H. SHEN, W. CHEN, *Laplace transform method for the Ulam stability of linear fractional differential equations with constant coefficients*, Mediterr. J. Math. (2017) 14: 25.

[32] Y. H. SHEN, Y. J. LI, *The z-transform method for the Ulam stability of linear difference equations with constant coefficients*, Adv. Differ. Eqn. (2018) 2018:396.

[33] S. E. TAKAHASI, T. MIURA AND S. MIYAJIMA, *On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$*, Bull. Korean Math. Soc. 39 (2002) 309–315.

[34] S. E. TAKAHASI, H. TAKAGI, T. MIURA AND S. MIYAJIMA, *The Hyers-Ulam stability constants of first order linear differential operators*, J. Math. Anal. Appl. 296 (2004) 403–409.

[35] S. M. ULAM, *Problems in Modern Mathematics*, Wiley, New York, 1960. 1024–1028.

[36] J. R. WANG, M. FEČKAN, M. ZHOU, *Presentation of solutions of impulsive fractional Langevin equations and existence results*, Eur. Phys. J. Special Top. 222 (2013) 1855–1872.

[37] J. R. X. WANG, Z. LI, *E_α-Ulam type stability of fractional order ordinary differential equations*, J. Appl. Math. Comput. 45 (2014) 449–459.

[38] J. R. X. WANG, Z. LI, *Ulam-Hyers stability of fractional Langevin equations*, Appl. Math. Comput. 258 (2015) 72–83.

[39] J. R. X. WANG, Z. LI, *A uniform method to Ulam-Hyers stability for some linear fractional equations*, Mediterr. J. Math. 13 (2016) 625–635.

[40] J. R. WANG, L. L. LV, Y. ZHOU, *New concepts and results in stability of fractional differential equations*, Commun. Nonlinear Sci. Numer. Simulat. 17 (2012) 2530–2538.
[41] J. R. WANG, M. ZHOU, M. FEČKAN, *Nonlinear impulsive problems for fractional differential equations and Ulam stability*, Comput. Math. Appl. 64 (2012) 3389–3405.

(Received April 17, 2020)

Yonghong Shen
School of Mathematics and Statistics
Tianshui Normal University
Tianshui 741001, P. R. China
e-mail: shenyonghong2008@hotmail.com

Yongjin Li
Department of Mathematics
Sun Yat-Sen University
Guangzhou, 510275, P. R. China