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INEQUALITIES FOR GAUSSIAN HYPERGEOMETRIC FUNCTIONS

XIAO-YAN MA* AND TI-REN HUANG

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Abstract. In this paper, the authors present several hypergeometric transformation inequalities for the Gaussian hypergeometric function $F(a,b;c;x)$, which are the extensions of the known hypergeometric transformation identities such as Ramanujan's cubic transformation identities, by showing the monotonicity properties of certain quotients of $F(a,b;c;x)$ and its special cases. By these results, some related known results are considerably improved.

1. Introduction

Throughout this paper, we let $\mathbb{N} = \{1,2,3\cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a,b,c \in \mathbb{R}$ with $c \neq 0,-1,-2,\cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} \text{ for } |x| < 1, \tag{1}$$

where (a,n) denotes the shifted factorial function

$$(a,n) = a(a+1)(a+2)(a+3)\cdots(a+n-1) \tag{2}$$

for $n \in \mathbb{N}$, and $(a,0) = 1$ for $a \neq 0$. $F(a,b;c;x)$ is said to be zero-balanced if $c = a+b$.

It is well known that $F(a,b;c;x)$ has many important applications in various fields of the mathematical and natural sciences, and many other special functions in mathematical physics are particular cases of this function (cf. [10, 11, 12, 16]).

One of the important special cases of $F(a,b;c;x)$ is as follows

$$F_s(x) = F\left(\frac{1}{2}-s, \frac{1}{2}+s; 1; x\right) = \sum_{n=0}^{\infty} \frac{(1/2-s,n)(1/2+s,n)}{(n!)^2} x^n, \tag{3}$$

for $s \in (-1/2, 1/2)$ and $x \in (0,1)$. It is interesting that some properties of the function F_s can be directly applied to obtain properties of the compound means, and the elliptic series for $1/\pi$ (see [6]).

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* Corresponding author.

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There have been many studies of the properties of F_s for the special cases when $s = 0, 1/6, 1/4$. The main results of these studies are as follows: In [4], Baricz presented some Landen inequalities (namely, in the case $s = 0$) for the Gaussian hypergeometric function; In [15] ([13, 14]), Wang, Chu and Jiang extended some quadratic transformation identities in the case when $s = 1/4$ (cubic transformation identities for $F(1/3, 2/3; 1; x)$ in the case when $s = 1/6$, respectively) to hypergeometric transformation inequalities for zero-balanced Gaussian hypergeometric function. However, there are few similar studies for the case when $s = 1/3$. Naturally, it would be more significant for us to extend a general hypergeometric identity satisfied by $F(1/2 - s, 1/2 + s; 1; x)$ to hypergeometric transformations inequalities for the Gaussian hypergeometric function.

On the other hand, many other beautiful hypergeometric transformation identities have been known to us, among which are the followings: In [7, (2.31)], Gavrvan presented the following transformation identity

$$F\left(a, a+\frac{1}{2}; \frac{4}{3}a+\frac{2}{3}; \frac{8r(1+r)}{(1+3r)^2}\right) = (1+3r)^{2a} F\left(a, a+\frac{1}{2}; \frac{2}{3}a+\frac{5}{6}; r^2\right), \tag{4}$$

and in [5], Berndt, Bhavgave and Garvan proved the following identity

$$F\left(a, a+\frac{1}{3}; \frac{a}{2}+\frac{5}{6}; \left(\frac{1-r}{1+2r}\right)^3\right) = \left(\frac{1+2r}{3}\right)^{3a} F\left(a, a+\frac{1}{3}; \frac{3}{2}a+\frac{1}{2}; 1-r^3\right). \tag{5}$$

Taking $a = 1/4$ in (4) and $a = 1/3$ in (5), we obtain the following well-known quadratic transformation identity

$$F\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{8r(1+r)}{(1+3r)^2}\right) = \sqrt{1+3r} F\left(\frac{1}{4}, \frac{3}{4}; 1; r^2\right) \tag{6}$$

and the Ramanujan's cubic transformation identity

$$F\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-r}{1+2r}\right)^3\right) = \frac{1+2r}{3} F\left(\frac{1}{3}, \frac{2}{3}; 1; 1-r^3\right), \tag{7}$$

respectively, which were stated by Ramanujan in his unpublish notebooks.

Changing r to $(1-r)/(1+3r)$ in (4) and r to $(1-r)/(1+2r)$ in (5), we obtain the following identities

$$F\left(a, a+\frac{1}{2}; \frac{2}{3}a+\frac{5}{6}; \left(\frac{1-r}{1+3r}\right)^2\right) = \left(\frac{1+3r}{4}\right)^{2a} F\left(a, a+\frac{1}{2}; \frac{4}{3}a+\frac{2}{3}; r'^2\right), \tag{8}$$

$$F\left(a, a+\frac{1}{3}; \frac{3}{2}a+\frac{1}{2}; \frac{9r(1+r+r^2)}{(1+2r)^3}\right) = (1+2r)^{3a} F\left(a, a+\frac{1}{3}; \frac{a}{2}+\frac{5}{6}; r^3\right), \tag{9}$$

where $r' = \sqrt{1-r^2}$.

The main purpose of this paper is to extend the hypergeometric transformation identities satisfied by $F(1/2 - s, 1/2 + s; 1; x)$ for $s \in (-1/2, 1/2)$ to the hypergeometric transformation inequalities for $F(a, b; c; x)$ by showing the monotonicity properties of the ratio of $F(a,b;c;x)$ and $F(1/2-s,1/2+s;1;x)$, and to extend the identities (4)–(9) with certain conditions fulfilled by the parameter a to transformation inequalities for the Gaussian hypergeometric functions, the Kummer hypergeometric function and the Bessel function. Some of these results improve the related known results.

At the end of this section, we recall the following lemma needed in the proofs of our main results.

LEMMA 1. (See [9, Lemma 2.1]). *Suppose that the power series* $f(x)=\sum_{n=0}^{\infty}a_nx^n$ *and* $g(x)=\sum_{n=0}^{\infty}b_nx^n$ *with* $b_n>0$ *for all* $n\in\mathbb{N}_0$ *have the common radius of convergence* $r>0$. *If the non-constant sequence* $\{a_n/b_n\}_{n=0}^{\infty}$ *is increasing (decreasing), then the function* $h(x)=f(x)/g(x)$ *is strictly increasing (decreasing, respectively) on* $(0,r)$.

2. Main results

THEOREM 1. *Let* $a,b>0$, *and* $c\in\mathbb{R}$ *such that* c *is not a negative integer or zero. Then we have the following conclusions:*

(1) If $b\geqslant\max\{c-a/3-1/6,3c/4\}$ *and* $a\geqslant 1/4$, *then for* $r\in(0,1)$,

$$F\left(a,b;c;\frac{8r(1+r)}{(1+3r)^2}\right)\geqslant(1+3r)^{2a}F\left(a,b;c;r^2\right), \tag{10}$$

$$F\left(a,b;c;\left(\frac{1-r}{1+3r}\right)^2\right)\leqslant\left(\frac{1+3r}{4}\right)^{2a}F\left(a,b;c;1-r^2\right). \tag{11}$$

Moreover, if $b\leqslant\min\{c-a/3-1/6,3c/4\}$ *and* $a\leqslant 1/4$, *then the inequalities* (10) *and* (11) *are both reversed.*

(2) If $b\geqslant\max\{c-a/2-1/6,2c/3\}$ *and* $a\geqslant 1/3$, *then for* $r\in(0,1)$,

$$F\left(a,b;c;\frac{9r(1+r+r^2)}{(1+2r)^3}\right)\geqslant(1+2r)^{3a}F\left(a,b;c;r^3\right), \tag{12}$$

$$F\left(a,b;c;\left(\frac{1-r}{1+2r}\right)^3\right)\leqslant\left(\frac{1+2r}{3}\right)^{3a}F\left(a,b;c;1-r^3\right). \tag{13}$$

Moreover, if $b\leqslant\min\{c-a/2-1/6,2c/3\}$ *and* $a\leqslant 1/3$, *then the inequalities* (12) *and* (13) *are both reversed.*

Proof. (1) Let $T_1:(0,1)\to(0,\infty)$ be the function defined by

$$T_1(x)=\frac{F(a,b;c;x)}{F(a,a+1/2;4a/3+2/3;x)}=\frac{\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}}{\sum_{n=0}^{\infty}\frac{(a,n)(a+1/2,n)}{(4a/3+2/3,n)}\frac{x^n}{n!}},$$

and for $n\in\mathbb{N}_0$, let

$$A_n=\frac{(b,n)(4a/3+2/3,n)}{(c,n)(a+1/2,n)}\text{ and }\alpha_n=\left(b+\frac{a}{3}+\frac{1}{6}-c\right)n+\left(\frac{4}{3}b-c\right)\left(a+\frac{1}{2}\right).$$

Then the sequence $\{A_n\}$ is increasing (decreasing) if and only for all $n \in \mathbb{N}_0$,

$$\frac{A_{n+1}}{A_n} = \frac{(b+n)(\frac{4}{3}a+\frac{2}{3}+n)}{(c+n)(a+\frac{1}{2}+n)} \geqslant (\leqslant) 1 \Leftrightarrow \alpha_n \geqslant 0 \ (\alpha_n \leqslant 0, \text{ respectively}).$$

Obviously, if $b \geqslant \max\{c-a/3-1/6, 3c/4\}$ $(b \leqslant \min\{c-a/3-1/6, 3c/4\})$, then $\alpha_n \geqslant 0$ ($\alpha_n \leqslant 0$) for all $n \in \mathbb{N}_0$, that is, the sequence $\{A_n\}$ is increasing (decreasing), and consequently by Lemma 1, the function T_1 is increasing (decreasing, respectively) on $(0,1)$.

Suppose that $b \geqslant \max\{c-a/3-1/6, 3c/4\}$ and $a \geqslant 1/4$. Put $x = x(r) = r^2$ and $y = y(r) = 8r(1+r)/(1+3r)^2$ for $r \in (0,1)$. Then $0 < x < y < 1$,

$$\frac{F(a,b;c;x)}{F(a,a+1/2;4a/3+2/3;x)} \leqslant \frac{F(a,b;c;y)}{F(a,a+1/2;4a/3+2/3;y)},$$

so that by the formula (4),

$$\begin{aligned} F(a,b;c;x) &\leqslant F(a,b;c;y)\frac{F(a,a+1/2;4a/3+2/3;x)}{F(a,a+1/2;4a/3+2/3;y)} \\ &= F(a,b;c;y)\frac{F(a,a+1/2;4a/3+2/3;x)}{(1+3r)^{2a}F(a,a+1/2;2a/3+5/6;x)}. \end{aligned} \tag{14}$$

Since $a \geqslant 1/4$ implies that $4a/3+2/3 \geqslant 2a/3+5/6$, one can easily see that

$$F(a,a+1/2;4a/3+2/3;x) \leqslant F(a,a+1/2;2a/3+5/6;x).$$

Hence the inequality (10) follows from (14).

Similarly, if $b \geqslant \max\{c-a/3-1/6, 3c/4\}$ and $a \geqslant 1/4$, and if we let $u = u(r) = [(1-r)/(1+3r)]^2$ and $v = v(r) = 1-r^2$, then $0 < u < v < 1$, and we have

$$\frac{F(a,b;c;u)}{F(a,a+1/2;4a/3+2/3;u)} \leqslant \frac{F(a,b;c;v)}{F(a,a+1/2;4a/3+2/3;v)},$$

so that by the formula (8),

$$\begin{aligned} F(a,b;c;u) &\leqslant \left(\frac{1+3r}{4}\right)^{2a} F(a,b;c;v)\frac{F(a,a+1/2;4a/3+2/3;u)}{F(a,a+1/2;2a/3+5/6;u)} \\ &\leqslant \left(\frac{1+3r}{4}\right)^{2a} F(a,b;c;v) \end{aligned}$$

which yields the inequality (11).

The proof of the second conclusion in part (1) is similar, and we omit the details.

(2) For $n \in \mathbb{N}_0$, let $a_n = (a,n)(b,n)/[(c,n)n!]$, $b_n = (a,n)(a+1/3,n)/[(3a/2+1/2,n)n!]$ and $B_n = a_n/b_n = (b,n)(3a/2+1/2,n)/[(c,n)(a+1/3,n)]$. Then

$$T_2(x) \equiv \frac{F(a,b;c;x)}{F(a,a+1/3;3a/2+1/2;x)} = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}.$$

With the same method as that used in the proof of part (1), it is easy to verify that the sequence $\{B_n\}$ is increasing (decreasing), which implies that T_2 is increasing (decreasing) on $(0,1)$ by Lemma 1, provided that $b \geqslant \max\{c-a/2-1/6, 2c/3\}$ ($b \leqslant \min\{c-a/2-1/6, 2c/3\}$, respectively).

First, we consider the case when $b \geqslant \max\{c-a/2-1/6, 2c/3\}$ and $a \geqslant 1/3$, and let $s = r^3$ and $t = 9r(1+r+r^2)/(1+2r)^3$ for $r \in (0,1)$. Then $0 < s < t < 1$ and

$$\frac{F(a,b;c;s)}{F(a,a+1/3;3a/2+1/2;s)} \leqslant \frac{F(a,b;c;t)}{F(a,a+1/3;3a/2+1/2;t)},$$

and hence by (9),

$$\begin{aligned} F(a,b;c;s) &\leqslant F(a,b;c;t)\frac{F(a,a+1/3;3a/2+1/2;s)}{F(a,a+1/3;3a/2+1/2;t)} \\ &= F(a,b;c;t)\frac{F(a,a+1/3;3a/2+1/2;s)}{(1+2r)^{3a}F(a,a+1/3;a/2+5/6;s)}. \end{aligned} \tag{15}$$

Observe that $F(a,a+1/3;3a/2+1/2;s) \leqslant F(a,a+1/3;a/2+5/6;s)$, since $a \geqslant 1/3$. Hence the inequality (12) follows from (15).

Second, substituting r in the inequality (12) by $(1-r)/(1+2r)$, we obtain the inequality (13). □

REMARK 1. (1) It is easy to see that the equalities (10) and (11) hold for $a = 1/4, b = 3/4$ and $c = 1$, in which case if we take $r = k^2$ in (10) and $r = k'^2$ in (11), respectively, then we obtain [12, Proposition 2.4 (i) & (ii)]. The inequalities (10) and (11) considerably improve the related known results such as [15, Theorem 2.1].

(2) It is clear that the inequalities (12) and (13) become equalities for $a = 1/3, b = 2/3$ and $c = 1$, in which case if we take $r = k$ in (12) and $r = k'$ in (13), respectively, then we obtain [11, Proposition 3.2 (i) & (ii)]. The inequalities (12) and (13) are the generalizations of [13, Theorem 2.3].

In the sequel, for $|s| < 1/2$, we let

$$\begin{aligned} D_1 &= \left\{(a,b) | a,b>0, ab \leqslant 1/4-s^2, ab-(1/4-s^2)(a+b) \leqslant 0\right\}, \\ D_2 &= \left\{(a,b) | a,b>0, ab \geqslant 1/4-s^2, ab-(1/4-s^2)(a+b) \geqslant 0\right\}. \end{aligned}$$

THEOREM 2. *For $a,b,c \in \mathbb{R}$, c is not a negative integer or zero, and for $|s| < 1/2$, define the function Q on $(0,1)$ by*

$$Q(x) = F(a,b;c;x)/F(1/2-s, 1/2+s; 1; x).$$

If $a+b \geqslant c$ ($a+b \leqslant c$), and if $4ab/(1-4s^2) \geqslant \max\{1,c\}$ ($4ab/(1-4s^2) \leqslant \min\{1,c\}$), then Q is increasing (decreasing, respectively) on $(0,1)$. In particular, if $(a,b) \in D_1$ ($(a,b) \in D_2$) with $c = a+b$, then Q is decreasing (increasing, respectively) on $(0,1)$.

Proof. For $n\in\mathbb{N}_0$, let $\tilde{a}_n=(a,n)(b,n)/[(c,n)n!]$, $\tilde{b}_n=(1/2-s,n)(1/2+s,n)/(n!)^2$,

$$C_n=\frac{\tilde{a}_n}{\tilde{b}_n}=\frac{(a,n)(b,n)}{(c,n)}\cdot\frac{n!}{(1/2-s,n)(1/2+s,n)}$$

and $\beta_n=(a+b-c)n^2+\left(ab+a+b-c-1/4+s^2\right)n+\left[ab-(1/4-s^2)c\right]$. Then $Q(x)$ is of the following expression

$$Q(x)=\left(\sum_{n=0}^{\infty}\tilde{a}_n x^n\right)\left(\sum_{n=0}^{\infty}\tilde{b}_n x^n\right)^{-1},$$

and the sequence $\{C_n\}$ is increasing, which implies that Q is increasing on $(0,1)$ by Lemma 1, if and only if

$$\frac{C_{n+1}}{C_n}=\frac{(a+n)(b+n)(n+1)}{(c+n)(1/2-s+n)(1/2+s+n)}\geqslant 1\Leftrightarrow\beta_n\geqslant 0.$$

Clearly, if $a+b\geqslant c$ and $4ab/(1-4s^2)\geqslant\max\{1,c\}$, then $\beta_n\geqslant 0$ for all $n\in\mathbb{N}_0$, and the sequence C_n is increasing, and hence Q is increasing on $(0,1)$.

The proof of the remaining conclusions are similar, and we omit the details. □

REMARK 2. Theorem 2 extends [4, Theorem 1], [13, Theorem 2.3], [14, Theorem 2.1] and [8, Lemma 2.4] to the hypergeometric transformation inequalities for the Gaussian hypergeometric functions. The details are as follows:

(1) Set $s=0, x=r^2, y=4r/(1+r^2)$ for $r\in(0,1)$, then $0<x<y<1$ and $Q(x)\leqslant Q(y)$ or $Q(x)\geqslant Q(y)$ by the monotonicity property of Q, and hence [4, Theorem 1] follows from Theorem 2 and [1, (13)].

(2) Set $s=1/6, x=r^3, y=9r(1+r+r^2)/(1+2r)^3$ for $r\in(0,1)$, then $0<x<y<1$ and $Q(x)\leqslant Q(y)$ or $Q(x)\geqslant Q(y)$ by the monotonicity property of Q, and hence we obtain [13, Theorem 2.3] by Theorem 2 and [5, Corollary 2.4]. In particular, if $c=a+b$, then we obtain [14, Theorem 2.1].

(3) Set $s=1/4, x=r^2, y=8r(1+r)/(1+3r)^2$ for $r\in(0,1)$, then $0<x<y<1$ and $Q(x)\leqslant Q(y)$ or $Q(x)\geqslant Q(y)$ by the monotonicity property of Q, and hence we obtain [8, Lemma 2.4] by Theorem 2 and [5, Theorem 9.4].

In the sequel, we let

$$\omega_{n,s}=\frac{(n!)^2}{(1/2-s,n)(1/2+s,n)} \tag{16}$$

for $n\in\mathbb{N}_0$ and $|s|<1/2$. It is easy to verify that the sequence $\{\omega_{n,s}\}$ is increasing. Using this sequence, one can easily extend Theorem 2 to the following theorem, whose proof is similar to that of Theorem 2 so that we omit the details.

THEOREM 3. *Suppose that the power series $f(x)=\sum_{n=0}^{\infty}a_n x^n$ is convergent for all $x\in(0,1)$, and the sequence $\{a_n\omega_{n,s}\}$ is increasing, where $a_n\in\mathbb{R}$ for $n\in\mathbb{N}_0$.*

Let $\lambda_f(x) = f(x^2)$. *Then the function* $S(x) \equiv f(x)/F_s(x)$ *is increasing on* $(0,1)$. *In particular, for all* $r \in (0,1)$ *and* $s = 1/4$,

$$\lambda_f\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right) \geqslant \sqrt{1+3r}\lambda_f(r). \tag{17}$$

Moreover, if the sequence $\{a_n\omega_{n,s}\}$ *is decreasing, then the function* S *is decreasing on* $(0,1)$ *and* (17) *is reversed.*

At the end of this paper, we shall extend Theorem 3 to the generalized Bessel function [2] u_ν and the Kummer hypergeometric function [3] $\Phi(p,q;\cdot)$, which are defined by

$$u_\nu(x) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(\kappa,n)} \cdot \frac{x^n}{n!}, \ \Phi(p,q;x) = \sum_{n=0}^{\infty} \frac{(p,n)}{(q,n)} \cdot \frac{x^n}{n!}, \tag{18}$$

respectively, where $\nu, b, c, p, q \in \mathbb{R}, \kappa = \nu + (b+1)/2$ and $q \notin \mathbb{N}_0$. For $n \in \mathbb{N}_0$ and $|s| < 1/2$, it is easy to show that the sequences

$$\{\omega_{n,s}(-c/4)^n/[(\kappa,n)n!]\} \text{ and } \{\omega_{n,s}(p,n)/[(q,n)n!]\}$$

are decreasing provided that $\kappa \geqslant \max\{-1, s^2 - c/4 - 1/4, -c/(1-4s^2)\}$ and $q \geqslant \max\{0, s^2 + p + 3/4, 4p/(1-4s^2)\}$. Putting $\lambda_{u_\nu}(r) = u_\nu(r^2)$ and $\lambda_\Phi(r) = \Phi(p,q;r^2)$, then by Theorem 3, one can immediately obtain the following corollary.

COROLLARY 1. *Let* $\nu, b, c, p, q \in \mathbb{R}$ *and* $|s| < 1/2$ *such that* $\kappa \geqslant \max\{-1, s^2 - c/4 - 1/4, -c/(1-4s^2)\}$ *and* $q \geqslant \max\{0, s^2 + p + 3/4, 4p/(1-4s^2)\}$. *Then* $F_1(x) \equiv u_\nu(x)/F_s(x)$ *and* $F_2(x) \equiv \Phi(p,q;x)/F_s(x)$ *are decreasing on (0,1), and in particular, for all* $r \in (0,1)$ *and* $s = 1/4$,

$$\lambda_{u_\nu}\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right) \leqslant \sqrt{1+3r}\lambda_{u_\nu}(r), \tag{19}$$

$$\lambda_\Phi\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right) \leqslant \sqrt{1+3r}\lambda_\Phi(r). \tag{20}$$

REMARK 3. If we take $s = 0$, $x = r^2$ and $y = 4r/(1+r^2)$ ($s = 1/6$, $x = r^3$ and $y = 9r(1+r+r^2)/(1+2r)^3$) in Theorem 3 and in Corollary 1, then it is not difficult for us to verify that $0 < x < y < 1$, $S(x) \leqslant S(y)$, $F_1(x) \geqslant F_1(y)$ and $F_2(x) \geqslant F_2(y)$ by the monotonicity properties of S, F_1 and F_2, thus resulting in [4, Theorems 2 & 3] ([13, Theorem 2.4 & Corollary 2.5], respectively).

QUESTION. As indicated in Section 1, there have been many studies of F_s in the case when $s = 0, 1/4, 1/6$, and however, there are few similar study of the corresponding properties of F_s for $s = 1/3$. The following question is natural: *What are the analogues of the known results, such as Theorem* 1, (17) *and* (19)–(20), *for* F_s *in the case when* $s = 1/3$?

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Xiao-Yan Ma
Department of Mathematics
Zhejiang Sci-Tech University
Hangzhou, Zhejiang, 310018, P.R. China
e-mail: mxy@zstu.edu.cn

Ti-Ren Huang
Department of Mathematics
Zhejiang Sci-Tech University
Hangzhou, Zhejiang, 310018, P.R. China
e-mail: htiren@zstu.edu.cn